Solution to the Inverse Problem for Upper Asymptotic Density

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Abstract

Inverse problems study the structure of a set A when the "size" of $A+A$ is small. In the article, the structure of an infinite set A of natural numbers with positive upper asymptotic density is characterized when A is not a subset of an infinite arithmetic progression of difference greater than one and $A+A$ has the least possible upper asymptotic density. For example, if the upper asymptotic density α of A is strictly between 0 and $\frac{1}{2}$, the upper asymptotic density of $A+A$ is equal to $\frac{3}{2}\alpha$, and A is not a subset of an infinite arithmetic progression of difference greater than one, then A is either a large subset of the union of two infinite arithmetic progressions with the same common difference $k=\frac{2}{\alpha}$ or for every increasing sequence h_n of positive integers such that the relative density of A in $[0,h_n]$ approaches α, the set $A\cap[0,h_n]$ can be partitioned into two parts $A\cap[0,c_n]$ and $A\cap[b_n,h_n]$ such that c_n/h_n approaches 0, *i.e.* the size of $A\cap[0,c_n]$ is asymptotically small comparing with the size of $[0,h_n]$, and $(h_n-b_n)/h_n$ approaches α, *i.e.* the size of $A\cap[b_n,h_n]$ is asymptotically almost same as the size of the interval $[b_n,h_n]$. The results here answer a question of the author in [8].

1 Introduction

Let $\mathbb{N}$ be the set of all natural numbers, including 0. Let A and B always denote the sets of natural numbers, and let a,b,c,h,i,j,k,m,n,x,y,z always denote natural numbers. For integers m,n and set A, we write $[m,n]$ exclusively for the interval of integers $\{k\in\mathbb{N}: m\leqslant k\leqslant n\}$ and write $A[m,n]$ for the set $A\cap[m,n]$ and $A(m,n)$ for the number of elements in $A[m,n]$. Sometimes, $A(1,n)$ is written as $A(n)$. For sets A and B, we write $A\pm B$ for the set $\{a\pm b: a\in A \text{ and } b\in B\}$ and write $2A$ for $A+A$. For a set A and a number b, we write $A\pm b$ for $A\pm\{b\}$ and write $b\pm A$ for $\{b\}\pm A$. In this paper we often use $(2A)(m,n)$ for $|\{x\in 2A: m\leqslant x\leqslant n\}|$ and $2A(m,n)$ for 2 times $A(m,n)$. We write *a.p.* as an abbreviation for "arithmetic progression", which can be either finite or infinite. An *a.p.* of a single element has difference d for every

Mathematics Subject Classification 2000 Primary 11B05, 11B13, 11U10, 03H15

Keywords: upper asymptotic density, inverse problem, nonstandard analysis

*The author was supported in part by the NSF grant DMS–#0070407.

non-negative integer d. We call a set $J = I_0 \cup I_1$ a bi-arithmetic progression, or *b.p.* for abbreviation, if I_0 and I_1 are two *a.p.* of the same difference d and the sets $2I_0$, $2I_1$, and $I_0 + I_1$ are pairwise disjoint. The number d is called the difference of J. When we say that A is a subset of a *b.p.* $I_0 \cup I_1$ we often implicitly assume that $A \cap I_0 \neq \emptyset$ and $A \cap I_1 \neq \emptyset$ unless we specifies otherwise. For a set A, the upper asymptotic density of A is defined by

$$\bar{d}(A) = \limsup_{n\to\infty} \frac{A(n)}{n}.$$

The main results of this paper characterize the structural properties of A when A is not a subset of an *a.p.* of difference > 1 and

$$\bar{d}(2A) = \inf\left\{\bar{d}(2B) : \begin{array}{c} B \text{ is not a subset of an } a.p. \text{ of} \\ \text{difference } > 1 \text{ and } \bar{d}(B) \geqslant \bar{d}(A) \end{array}\right\}.$$

For the motivation of the main results, we would like to quote a few sentences from the preface of the book [11]: "The classical problems in additive number theory are *direct problems*, in which we start with a set A of integers and proceed to describe the h–fold sumset hA, that is, the set of all sums of h elements of A. In an *inverse problems*, we begin with the sumset hA and try to deduce information about the underlying set A. In the last few years, there has been remarkable progress in the study of inverse problems for finite sets in additive number theory. There are important inverse theorems due to Freiman, Kneser, Plünnecke, Vosper, and others. In particular, Ruzsa recently discovered a new method to prove a generalization of Freiman's theorem." Although the results in this paper are not directly related to the Freiman's Theorem mentioned above, they share the same pattern, which says that if $2A$ is small, then A must have some structure.

In fact, the idea of inverse problem occurs also in some of the theorems involving densities. The theorems about Shnirel'man's pairs and Mann's pairs in [5] deduce information about the Shnirel'man density of A and Shnirel'man density of B when the Shnirel'man density of $A + B$ is small. Kneser's Theorem (cf. [1, 4] deduces information about the arithmetic structure of A and B when the lower asymptotic density of $A + B$ is small. In [8], the inverse problems for upper asymptotic density are considered. In [8] the structural properties of A are characterized when the upper asymptotic density of $2A + \{0, 1\}$ is small. However, adding $\{0, 1\}$ to $2A$ seems to be a non–traditional requirement. The results should be more interesting if the requirement can be dropped. However, if A itself is an *a.p.*, then the upper asymptotic density of $A+A$ is exactly the same as the upper asymptotic density of A. Hence it is natural to assume that "A is not a subset of an *a.p.* I of difference > 1" because otherwise we can replace $\mathbb{N}$ by I and discuss the properties of A in I instead.

Why do we need to add $\{0,1\}$ to $2A$ in the first place in [8]? Let $\alpha = \bar{d}(A)$ and $a_0 = \min A$. By Lemma 1.5 below, one can prove the following: If $\alpha \leqslant \frac{1}{2}$ and $\gcd(A - a_0) = 1$, i.e. the greatest common divisor of all numbers in $A - a_0$ is one, then $\bar{d}(2A) \geqslant \frac{3}{2}\alpha$. If $\alpha > \frac{1}{2}$, then $\bar{d}(2A) \geqslant \frac{\alpha+1}{2}$. Note that the two inequalities above are optimal. The equalities hold for the following two examples.

Example 1.1 *For every real number* $0 \leqslant \alpha \leqslant 1$*, let*

$$A = \bigcup_{n=1}^{\infty} [\lceil (1-\alpha)2^{2^n} \rceil, 2^{2^n}].$$

Then $\bar{d}(A) = \alpha$*,* $\bar{d}(2A) = \frac{1+\alpha}{2}$ *if* $\alpha \geqslant \frac{1}{2}$*, and* $\bar{d}(2A) = \frac{3}{2}\alpha$ *if* $\alpha \leqslant \frac{1}{2}$*.*

Example 1.2 *Let* $k, m, n \in \mathbb{N}$ *be such that* $k \geqslant 4$ *and* $2m, 2n, m+n$ *are pairwise distinct modulo* k*. Let*

$$A = \{m + ik : i \in \mathbb{N}\} \cup \{n + ik : i \in \mathbb{N}\}.$$

Then $\bar{d}(A) = \frac{2}{k} = \alpha \leqslant \frac{1}{2}$ *and* $\bar{d}(2A) = \frac{3}{k} = \frac{3}{2}\alpha$*. It is easy to choose* k, m, n *such that* $\gcd(A - a_0) = 1$.

At the time when [8] was written, we believed that if A is a set with positive upper asymptotic density $\alpha < \frac{1}{2}$ such that $\gcd(A - \min A) = 1$ and the upper asymptotic density of $2A$ reaches its smallest possible value, then A should be a set similar to the one in Example 1.1 or to the one in Example 1.2. If we require that $\bar{d}(2A+\{0,1\}) = \frac{3}{2}\alpha$ when $\bar{d}(A) = \alpha \leqslant \frac{1}{2}$, then A cannot be the set similar to the one in Example 1.2. Hence we need only to show that A is a set similar to the one in Example 1.1 as done in [8]. This greatly simplifies the proof. Besides, adding $\{0,1\}$ to the set $2A$ makes it possible to apply Besicovitch's Theorem [4, page 6] to the proof of [8, Lemma 2.1]. Without adding $\{0,1\}$, we need not only to consider that A can be a set similar to the one in Example 1.2, but also to find a new way of proofs by-passing Besicovitch's Theorem. These difficulties were overcome recently. We prove a key lemma in §2, which is inspired by Kneser's Theorem mentioned above. The key lemma allows us by–passing Besicovitch's Theorem in the proofs.

Next, we state the main theorem of this paper. Without loss of generality we always assume $0 \in A$.

Theorem 1.3 *Let A be a set of natural numbers, $0 \in A$, and $\bar{d}(A) = \alpha > 0$.*

Part I: Assume $\alpha > \frac{1}{2}$. Then $\bar{d}(2A) = \frac{1+\alpha}{2}$ implies that for every increasing sequence $\langle h_n : n \in \mathbb{N}\rangle$ with $\lim_{n\to\infty} \frac{A(0,h_n)}{h_n+1} = \alpha$, we have

$$\lim_{n\to\infty} \frac{(2A)(0,h_n)}{h_n+1} = \alpha.$$

Part II: Assume $\alpha < \frac{1}{2}$ and $\gcd(A) = 1$. Then $\bar{d}(2A) = \frac{3}{2}\alpha$ implies that either (a) there exist $k > 4$ and $c \in [1, k-1]$ such that $\alpha = \frac{2}{k}$ and

$$A \subseteq \{ik : i \in \mathbb{N}\} \cup \{c + ik : i \in \mathbb{N}\}$$

or (b) for every increasing sequence $\langle h_n : n \in \mathbb{N}\rangle$ with $\lim_{n\to\infty} \frac{A(0,h_n)}{h_n+1} = \alpha$, there exist two sequences $0 \leqslant c_n \leqslant b_n \leqslant h_n$ such that

$$\lim_{n\to\infty} \frac{A(b_n,h_n)}{h_n - b_n + 1} = 1,$$

$$\lim_{n\to\infty} \frac{c_n}{h_n} = 0,$$

and $[c_n + 1, b_n - 1] \cap A = \emptyset$ for every $n \in \mathbb{N}$.

Part III: Assume $\alpha = \frac{1}{2}$ and $\gcd(A) = 1$. Then $\bar{d}(2A) = \frac{3}{2}\alpha$ implies that either (a) there exists $c \in \{1, 3\}$ such that

$$A \subseteq \{4i : i \in \mathbb{N}\} \cup \{c + 4i : i \in \mathbb{N}\}$$

or (b) for every increasing sequence $\langle h_n : n \in \mathbb{N}\rangle$ with $\lim_{n\to\infty} \frac{A(0,h_n)}{h_n+1} = \alpha$, we have

$$\lim_{n\to\infty} \frac{(2A)(0,h_n)}{h_n+1} = \alpha.$$

Remark 1.4 *(1) The proof of Part I of Theorem 1.3 is easy and will be omitted. (See [8, (2) of the remarks in Section 4] for an explanation.)*

(2) Part I and (b) of Part III cannot be improved so that the set A has the structure similar to the structure described in (b) of Part II. See [8, (1) of the remarks in Section 4] for the argument.

(3) The value of $\bar{d}(2A)$ in each of the conditions of all three parts is the least possible value $\bar{d}(2A)$ can have.

(4) In Part II (a) and Part III (a), A is a large subset of the union of the two a.p. *in terms of upper asymptotic density because both have the same upper asymptotic density.*

(5) Recently G. Bordes [3] generalizes Part II of Theorem 1.3 for sets A with small upper asymptotic density. He characterizes the structure of A when $\bar{d}(A) \leqslant \alpha_0$ for some small positive number α_0 and $\bar{d}(2A) < \frac{5}{3}\bar{d}(A)$.

In the next section §2, we will prove several lemmas including a key lemma necessary for the proof of Theorem 1.3. In §3, the proof of Theorem 1.3 is split into two nonstandard theorems, one for Part II and the other for Part III of Theorem 1.3. At the end of §3, a corollary is given. In both sections, the techniques from nonstandard analysis are used.

Before getting into nonstandard analysis, we would like to list a few existing theorems as lemmas, which are needed in §2 and §3.

Two of Freiman's theorems (cf.[11, Theorem 1.15 and Theorem 1.16, page 28] or cf.[2, Proposition 1.1]) will be frequently cited.

Lemma 1.5 (G. Freiman) *Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be such that $0 = a_0 < a_1 < \cdots < a_{k-1} = n$ and $\gcd(A) = 1$. If $k \leqslant \frac{n+3}{2}$, then $(2A)(0, 2n) \geqslant 3k - 3$. If $k \geqslant \frac{n+3}{2}$, then $(2A)(0, 2n) \geqslant k + n$.*

Lemma 1.6 (G. Freiman) *Let $A \subseteq \mathbb{N}$ be such that $|A| = k > 2$. If $|2A| = 2k - 1 + b \leqslant 3k - 4$, then A is a subset of an* a.p. *of the length $k + b \leqslant 2k - 3$.*

The next lemma is due to Lev and Smeliansky (see [9] and [11, p.118]).

Lemma 1.7 (V. Lev & P. Y. Smeliansky) *Let A and B be two finite sets of non-negative integers. Suppose $0 \in A \cap B$, both A and B contain more than one element, $\gcd(A) = 1$, $m = \max A$, and $n = \max B \leqslant m$. If $m = n$, then $|A + B| \geqslant \min\{m + |B|, |A| + 2|B| - 3\}$. If $m > n$, then $|A+B| \geqslant \min\{m + |B|, |A| + 2|B| - 2\}$.*

Note that Lemma 1.7 can be easily modified to the following form: Let A and B both be finite subsets of $\{dn : n \in \mathbb{N}\}$. Suppose $0 \in A \cap B$, A and B contain more than one element, $\gcd(A) = d$, $m = \max A$, and $n = \max B \leqslant m$. If

$m = n$, then $|A+B| \geqslant \min\left\{\frac{m}{d} + |B|, |A| + 2|B| - 3\right\}$. If $m > n$, then $|A+B| \geqslant \min\left\{\frac{m}{d} + |B|, |A| + 2|B| - 2\right\}$.

The next lemma is due to van der Corput (see [4, p.18]).

Lemma 1.8 (van der Corput) *Let A, B be subsets of non-negative integers with $0 \in A \cap B$ and let γ be a real number between 0 and 1. If for each $n > 0$,*

$$\inf\left\{\frac{1 + A(1,m) + B(1,m)}{m+1} : m = 1, 2, \ldots, n\right\} \geqslant \gamma,$$

then

$$\inf\left\{\frac{1 + (A+B)(1,m)}{m+1} : m = 1, 2, \ldots, n\right\} \geqslant \gamma.$$

2 Lemmas in Nonstandard Analysis

As mentioned in §1, the techniques from nonstandard analysis are needed in §2 and §3. One of the advantages of nonstandard methods is that an asymptotic argument such as upper asymptotic density in the standard world can be translated into one *finite argument in a nonstandard world so that instead of dealing with a sequence of intervals in an upper asymptotic density argument, we can deal with only one interval of *finite length in the nonstandard world. For basic knowledge of nonstandard analysis, the reader is recommended to consult [10], [6], or [7].

We work within a fixed $\aleph_1$–saturated nonstandard universe $^*V$ throughout this article. For each standard set A, we write $^*A$ for the nonstandard version of A in $^*V$. For example, $^*\mathbb{N}$ is the set of all natural numbers in $^*V$, and if A is the set of all even numbers in $\mathbb{N}$, then $^*A$ is the set of all even numbers in $^*\mathbb{N}$. If we do not specify that A, B are sets of standard natural numbers, A, B are always assumed to be *internal* sets of (standard and nonstandard) natural numbers. Now $a, b, c, h, i, j, k, m, n, x, y, z$ can take values in $^*\mathbb{N}$. The integers in $^*\mathbb{N} \smallsetminus \mathbb{N}$ are called hyperfinite integers. The letters H, K and N are exclusively used for hyperfinite integers. The Greek letters α, β, γ, δ, and ϵ are reserved exclusively for *standard* real numbers.

For the convenience of handling nonstandard arguments, we would like to introduce some notation of comparisons. For real numbers r, s in $^*V$, by $r \approx s$ we mean that $r - s$ is an infinitesimal, by $r \ll s$ ($r \gg s$) we mean that $r < s$ ($r > s$) and $r \not\approx s$, and by $r \lessapprox s$ ($r \gtrapprox s$) we mean $r < s$ ($r > s$) or $r \approx s$. Given a hyperfinite integer H and two real numbers r, s, by $r \sim_H s$ we mean that $\frac{s-r}{H} \approx 0$, by $r \prec_H s$ ($r \succ_H s$) we mean that $r < s$ ($r > s$) and $r \not\sim_H s$, and by $r \preceq_H s$ ($r \succeq_H s$) we mean that $r \prec_H s$ ($r \succ_H s$) or $r \sim_H s$. It is often said that a is insignificant with respect to H if $a \sim_H 0$. In the most of the cases the subscript H is clearly given so that it will be dropped as a subscript for convenience. Note that the comparison relations $\ll$, $\gg$, $\approx$, etc. can be interpreted in terms of $\prec$, $\succ$, $\sim$, etc. or vice versa when H is clearly given. For example, $\frac{a}{H} \lessapprox \frac{b}{H}$ iff $a \preceq b$. We use, for example, $\preceq$ more often than the use of $\lessapprox$ because fractions can be avoided. When using $\sim$, $\prec$, $\preceq$, etc. insignificant quantities can often be neglected. For example, instead of using $A(0,H) \sim \alpha(H+1)$, we can use its equivalent form $A(0,H) \sim \alpha H$. For another example, when $a \leqslant c \leqslant b$, we often write $A(a,c) \sim A(a,b) + A(b,c)$ instead of writing $A(a,c) = A(a,b) + A(b+1,c)$. For a real number $r \in {}^*\mathbb{R}$ bounded by a standard real number, let $st(r)$, the standard part of r, be the unique standard real number α such that $r \approx \alpha$.

Next lemma is stated for showing how upper asymptotic density can be translated into a nonstandard version.

Lemma 2.1 *Let $A \subseteq \mathbb{N}$ and let $0 \leqslant \alpha \leqslant 1$. Then $\bar{d}(A) \geqslant \alpha$ if and only if there is a hyperfinite integer H such that ${}^*A(0,H) \succeq \alpha H$.*

Proof: "$\Rightarrow$" Let $h_1 < h_2 < \cdots$ be an increasing sequence in $\mathbb{N}$ such that $\frac{A(0,h_n)}{h_n+1} > \alpha - \frac{1}{n}$. Then the internal set $X = \{n \in {}^*\mathbb{N} : \frac{{}^*A(0,h_n)}{h_n+1} > \alpha - \frac{1}{n}\}$ contains all standard natural numbers. Hence X must contain a hyperfinite integer N. This implies that $\frac{{}^*A(0,h_N)}{h_N+1} > \alpha - \frac{1}{N} \approx \alpha$.

"$\Leftarrow$" Let $\epsilon > 0$ and $k \in \mathbb{N}$. We have $\frac{{}^*A(0,H)}{H+1} > \alpha - \epsilon$. This means that the sentence "there exists $x \in {}^*\mathbb{N}$ with $x > k$ such that $\frac{{}^*A(0,x)}{x+1} > \alpha - \epsilon$" is true in ${}^*V$. By the transfer principle the sentence is also true in V. Hence there exists $x \in \mathbb{N}$ with $x > k$ such that $\frac{A(0,x)}{x+1} > \alpha - \epsilon$. Since $\epsilon > 0$ can be arbitrarily small and $k \in \mathbb{N}$ can be arbitrarily large, we have $\bar{d}(A) \geqslant \alpha$. $\square$(Lemma 2.1)

Lemma 2.2 *Let $A \subseteq [0,H]$ and $0 \leqslant \alpha \leqslant 1$. If $A(0,x) \preceq \alpha(x+1)$ and $A(0,H) \succeq \alpha(H+1)$, then there exists a $y \succeq x$ such that $A(0,y) \sim \alpha(y+1)$ and either $y \sim H$ or for every $y \prec z \leqslant H$, $A(0,z) \succ \alpha(z+1)$.*

Proof: Let

$$\beta = \sup\left\{ st\left(\frac{z}{H+1}\right) : z \in [0,H] \text{ and } A(0,z) \preceq \alpha(z+1)\right\}.$$

By the completeness of the standard real line, β is well defined. Let $y \in [0, H]$ be such that $\frac{y}{H+1} \approx \beta$. Clearly $y \succeq x$ by the definition of β.

It is easy to see that if $y \prec H$, then $A(0,z) \succ \alpha(z+1)$ for every $y \prec z \leqslant H$ by the supremality of β.

Suppose $A(0,y) \succ \alpha(y+1)$. For every positive $n \in \mathbb{N}$ there is a $z < y$ with $z > y - \frac{H}{n}$ such that $A(0,z) < (\alpha + \frac{1}{n})(z+1)$. Let X_n be the internal set containing all $z \in [y - \frac{H}{n}, y]$ such that $A(0,z) < (\alpha + \frac{1}{n})(z+1)$. Then $X_1 \supseteq X_2 \supseteq \cdots$ are all non-empty. By $\aleph_1$–saturation[1] there is a $z \sim y$ such that $A(0,z) \preceq \alpha(z+1)$. Hence $A(0,y) \sim A(0,z) \preceq \alpha(z+1) \sim \alpha(y+1)$, a contradiction. By a similar argument we can also show that $A(0,y) \prec \alpha(y+1)$ is impossible. □(Lemma 2.2)

Remark 2.3 *In Lemma 2.2 one can get various versions by reversing the direction of the inequalities or replacing the set A by $H - A$.*

Lemma 2.4 *Let $A \subseteq [0, H]$. Suppose $0, H \in A$. If $0 \leqslant x_1 \prec x_2 \leqslant H$ satisfy the following*

(1) $(2A)(2x_1, 2x_2) \succ 3A(x_1, x_2)$,

(2) if $0 \prec x_1$, then $A[0,x]$ is not a subset of an a.p. *of difference $d \geqslant 2$ and $A(0,x) \preceq \frac{1}{2}(x+1)$ for some $x \sim x_1$,*

(3) if $x_2 \prec H$, then $A[x,H]$ is not a subset of an a.p. *of difference $d \geqslant 2$ and $A(x,H) \preceq \frac{1}{2}(H-x+1)$ for some $x \sim x_2$,*

then $|2A| \succ 3|A|$.

Proof: By Lemma 1.5, we have $(2A)(0, 2x_1) \succeq 3A(0, x_1)$ and $(2A)(2x_2, 2H) \succeq 3A(x_2, H)$. Hence

$$\begin{aligned}
&(2A)(0, 2H) \\
&\quad \sim (2A)(0, 2x_1) + (2A)(2x_1, 2x_2) + (2A)(2x_2, 2H) \\
&\quad \succ 3A(0, x_1) + 3A(x_1, x_2) + 3A(x_2, H) \\
&\quad \sim 3A(0, H).
\end{aligned}$$

□(Lemma 2.4)

[1] One form of $\aleph_1$–saturation is that the intersection of any countable sequence of non-empty internal sets $X_1 \supseteq X_2 \supseteq \cdots$ is non-empty.

The next lemma appears but is not clearly stated in [8]. The full proof can be found there. Since the proof in [8] is not well organized, we would like to include it here just for the reader's convenience.

Lemma 2.5 *Let $A \subseteq [0, H]$ for a hyperfinite integer H and $0 \prec a \prec H$. If $0 \in A$, $A(0, x) \succ \frac{1}{2}x$ for every $0 \prec x \prec a$, $A(0, a) \sim \frac{1}{2}a$, and $(2A)(a, c) \sim 0$ for some $c \succ a$, then there is $b \sim \frac{1}{2}a$ such that $A[0, a] \subseteq [0, b]$.*

Proof: Let $b = \max A[0, a]$. Choose two standard natural numbers $p < q$ such that $\gcd(p, q) = 1$ and

$$a + \frac{2a}{p} + 2 < c.$$

For each $i = 0, 1, \ldots, 2p$, let $x_i = [\frac{a}{2p} i]$. Then

$$A[x_{p-i}, x_{p-i+1}] + A[x_{p+i}, x_{p+i+1}] \subseteq (2A)[a - 2, c].$$

Hence for each of $i = 0, 1, \ldots, p - 1$, $A(x_{p-i}, x_{p-i+1}) \succ 0$ implies $A(x_{p+i}, x_{p+i+1}) = \emptyset$ and $A(x_{p+i}, x_{p+i+1}) \succ 0$ implies $A(x_{p-i}, x_{p-i+1}) = \emptyset$. Note that when $i = 0$, one has $A(x_p, x_{p+1}) \sim 0$. Since $A(0, a) \sim \frac{1}{2}a$, then there are exactly half of the i's in $\{1, 2, \ldots, p - 1\} \cup \{p + 1, p + 2, \ldots, 2p - 1\}$ such that $A(x_i, x_{i+1}) \sim x_{i+1} - x_i$ and for the rest of the i's, $A[x_i, x_{i+1}] = \emptyset$. Since $A(x_p, x_{p+1}) \sim 0$, then $A(x_0, x_1) \sim x_1 - x_0$.

By the same procedure, one can define $y_j = [\frac{a}{2q} j]$ for $j = 0, 1, \ldots, 2q$ so that there are exactly half of the j's in $\{1, 2, \ldots, q - 1\} \cup \{q + 1, q + 2, \ldots, 2q - 1\}$ such that $A(y_j, y_{j+1}) \sim y_{j+1} - y_j$ and for the rest of the j's, $A[y_j, y_{j+1}] = \emptyset$. Since $A(y_q, y_{q+1}) \sim 0$, then $A(y_0, y_1) \sim y_1 - y_0$.

Since p and q are relatively prime, for $i \in \{1, 2, \ldots, p - 1\}$ and $j \in \{1, 2, \ldots, q - 1\}$, one has $x_i \not\sim y_j$. Hence there is no $i \in \{1, 2, \ldots, p - 1\}$ such that $A(x_{i-1}, x_i) \not\sim A(x_i, x_{i+1})$ because otherwise, one can take a $j \in \{1, 2, \ldots, q - 1\}$ such that $x_i \in [y_j, y_{j+1}]$, which would make $A(y_j, y_{j+1}) \not\sim 0$ and $A(y_j, y_{j+1}) \not\sim y_{j+1} - y_j$ at the same time.

Since $A(x_0, x_1) \sim x_1 - x_0$, then $A(0, x_p) \sim x_p$ and $A[x_{p+1}, a] = \emptyset$. So $x_p \leqslant b < x_{p+1}$. By the fact that p and q can be chosen arbitrarily large in $\mathbb{N}$, we have $b \sim \frac{1}{2}a$. This ends the proof of the lemma. $\square$(Lemma 2.5)

The next lemma is trivial and will be frequently referred as the pigeonhole principle.

Lemma 2.6 *Let* $A \subseteq {}^*\mathbb{N}$ *and* $x, y, t \in {}^*\mathbb{N}$. *If* $A(x, x+t) + A(y-t, y) > t+1$, *then* $x + y \in (2A)$.

The next three lemmas are needed in the proof of the key lemma.

Lemma 2.7 *Let* $A \subseteq [0, H]$ *be a hyperfinite set such that* $A[0, H-1]$ *is a subset of a* b.p. $I_0 \cup I_1$ *of difference* $d > 3$, $A[0, H-1]$ *is not a subset of an* a.p. *of difference* > 1, *and* A *is not a subset of a* b.p. *of difference* d. *If* $|A \cap I_0| < \frac{1}{4}|A|$ *or* $|A \cap I_1| < \frac{1}{4}|A|$, *then* $\frac{|2A|}{|A|} \gtrapprox 3 + \frac{1}{4}$.

Proof: Let $A_i = I_i \cap A[0, H-1]$ and $a_i = \min A_i$ for $i = 0, 1$. Suppose $H \equiv a_0 \pmod d$. Then $2a_0, 2a_1, a_0+a_1$ are not pairwise distinct modulo d because otherwise A is a subset of a *b.p.* $(I_0 \cup \{H\}) \cup I_1$. Clearly $a_0 \not\equiv a_1 \pmod d$ because otherwise $A[0, H-1]$ is a subset of an *a.p.* of difference $d > 1$. Hence $2a_0 \not\equiv a_0 + a_1 \pmod d$. By the same reason $2a_1 \not\equiv a_0 + a_1 \pmod d$. So we have $2a_0 \equiv 2a_1 \pmod d$. If d is an odd number, then $a_0 \equiv a_1 \pmod d$. If d is an even number, then $a_0 \equiv a_1 \pmod{\frac{d}{2}}$. But each of them contradicts that $A[0, H-1]$ is not a subset of an *a.p.* of difference > 1. So we can assume that $H \not\equiv a_0 \pmod d$. By the same reason we can also assume $H \not\equiv a_1 \pmod d$.

Suppose $|A_0| < \frac{1}{4}|A|$. Since $(H + A_1) \cap (2A_1 \cup (A_0 + A_1)) = \emptyset$, then

$$\begin{aligned}|2A| &\geqslant |2A_1| + |A_0 + A_1| + |H + A_1| \\ &\geqslant 3|A_1| + 3|A_0| + |A_1| - 2|A_0| - 2 \\ &\geqslant 3|A| - 5 + \frac{1}{4}|A| - 1,\end{aligned}$$

which implies the lemma. The proof of the case when $|A_1| < \frac{1}{4}|A|$ is identical. □(Lemma 2.7)

Lemma 2.8 *Let* $A \subseteq [0, H]$ *be a hyperfinite set such that* $A[0, H-1]$ *is a subset of a* b.p. $I_0 \cup I_1$ *of difference* $d > 3$, $A[0, H-1]$ *is not a subset of an* a.p. *of difference* > 1, *and* A *is not a subset of a* b.p. *of difference* d. *If* $|A \cap I_i| \geqslant \frac{1}{4}|A|$ *for* $i = 0, 1$, *then* $\frac{|2A|}{|A|} \gtrapprox 3 + \frac{1}{4}$.

Proof: We use the same notation as in Lemma 2.7. Again we can assume that H is distinct from a_0, a_1 modulo d. If $H + a_0 \equiv 2a_1 \pmod d$ and $H + a_1 \equiv 2a_0 \pmod d$, then $3(a_1 - a_0) \equiv 0 \pmod d$. If d is a multiple of 3, then $a_1 - a_0 \equiv 0 \pmod{\frac{d}{3}}$, which implies that $A[0,H-1]$ is a subset of an *a.p.* of difference $\frac{d}{3}>1$. If d is not a multiple of 3, then $a_1-a_0\equiv 0 \,(\mathrm{mod}\, d)$, which implies that $A[0,H-1]$ is a subset of an *a.p.* of difference of $d>1$. So we can assume that either $H+a_0\not\equiv 2a_1 \,(\mathrm{mod}\, d)$ or $H+a_1\not\equiv 2a_0 \,(\mathrm{mod}\, d)$. Suppose $H+a_0\not\equiv 2a_1 \,(\mathrm{mod}\, d)$. Then $(H+A_0)\cap((2A_0)\cup(2A_1)\cup(A_0+A_1))=\emptyset$. Hence

$$\begin{aligned}|2A| &\geqslant |2A_0|+|2A_1|+|A_0+A_1|+|H+A_0|\\ &\geqslant 4|A_0|+3|A_1|-3\geqslant 3|A|-6+\frac{1}{4}|A|.\end{aligned}$$

The proof of the case $H+a_1\not\equiv 2a_0 \,(\mathrm{mod}\, d)$ is symmetric. □(Lemma 2.8)

Lemma 2.9 *Let $A\subseteq[0,H]$ be a hyperfinite subset of a* b.p. $I_0\cup I_1$ *of difference $d>1$. Let $a_i=\min(A\cap I_i)$. Suppose* $\gcd((A_0-a_0)\cup(A_1-a_1))=d$. *If* $\gcd(A_i-a_i)=d_i>d$ *and $|A_i|\geqslant\frac{1}{4}|A|$ for either $i=0$ or $i=1$, then $\frac{|2A|}{|A|}\gtrapprox 3+\frac{1}{4}$.*

Proof: Assume $\gcd(A_0-a_0)=d_0>d$ and $|A_0|\geqslant\frac{1}{4}|A|$. The lemma is true because $|A_0+A_1|\geqslant 2|A_0|+|A_1|-2$. Same for the case $\gcd(A_1-a_1)=d_1>d$ and $|A_1|\geqslant\frac{1}{4}|A|$. □(Lemma 2.9)

Next we introduce more notation and definitions, which are needed in the key lemma.

An infinite initial segment U of ${}^*\mathbb{N}$ is called a cut if $U+U\subseteq U$. Clearly $U=\mathbb{N}$ and $U={}^*\mathbb{N}$ are cuts. A cut $U\neq{}^*\mathbb{N}$ is an external set. For example, $\mathbb{N}$ is external. For a hyperfinite integer H, the set

$$U_H=\bigcap_{n\in\mathbb{N}}[0,[H/n]]$$

is an external cut. From now on, the only cut we need is U_H for a given H. Hence when H is clearly given, the letter U is always used for U_H. We often write $x>U$ for $x\in{}^*\mathbb{N}\smallsetminus U$ and write $x<U$ for $x\in U$. When we say that "for all sufficiently large $x<U$... " we mean the statement "there is a $y<U$ such that for every $x>y$ in U..." and when we say "there is a sufficiently large $x<U$..." we mean the statement "for every $y<U$, there is an $x<U$, $y\leqslant x$, such that ...". Note that if $x<U$ and $y>U$, then $\frac{x}{y}\approx 0$.

Suppose $U\subseteq[0,H]$ is a cut. Given a function $f:[0,H]\mapsto{}^*\mathbb{R}$ (not necessarily internal) bounded by a standard real number, the lower U–density of f is defined by the following:

$$\underline{d}_U(f) = \sup\left\{\inf\left\{st(f(n)) : n \in U \setminus [0,m]\right\} : m \in U\right\}.$$

Given a set $A \subseteq [0,H]$, let $f_A(x) = \frac{A(0,x)}{x+1}$ for every $x \in [0,H]$. The lower U–density of A is defined by

$$\underline{d}_U(A) = \underline{d}_U(f_A).$$

For each $x \in {}^*\mathbb{N}$, we define the lower $(x+U)$–density of A by the following:

$$\underline{d}_{x+U}(A) = \underline{d}_U((A-x) \cap {}^*\mathbb{N}).$$

Remark 2.10 *(1) Suppose we replace U by $\mathbb{N}$ in the definition of $\underline{d}_U$. Then for every $A \subseteq \mathbb{N}$, $\underline{d}(A) = \underline{d}_{\mathbb{N}}({}^*A)$, where $\underline{d}(A) = \liminf_{n\to\infty} \frac{A(n)}{n}$ is the standard definition of the lower asymptotic density of A.*

(2) It is easy to check that for every $a \in U$,

$$\underline{d}_U(A+a) = \underline{d}_U(A)$$

and

$$\underline{d}_U(A \setminus [0,a]) = \underline{d}_U(A).$$

(3) Let H be hyperfinite and $A \subseteq [0,H]$. If $\underline{d}_U(A) > \gamma$, then there are $x \in U$ and $y \in [0,H] \setminus U$ such that for every $x \leqslant z \leqslant y$, $\frac{A(0,z)}{z+1} > \gamma$. This is true because the following reason: Let $\epsilon > 0$ be such that $\underline{d}_U(A) > \gamma + \epsilon$. Then we can find an $x \in U$ such that for every $z \geqslant x$ in U, $\frac{A(0,z)}{z+1} > \gamma$. Now the internal set $X = \{y \geqslant x : \forall z \in [x,y]\left(\frac{A(0,z)}{z+1} > \gamma\right)\}$ contains all elements in $U \setminus [0, x-1]$. Hence X must contain an element $y > U$, which gives the desired result.

(4) If $\underline{d}_U(A) = \alpha$, then there is an $x \in U$ such that for every $x < y < U$, one has $\frac{A(0,y)}{y+1} \gtrapprox \alpha$. This is true because the following reason: Let $X_n = \{y \in U : \frac{A(0,y)}{y+1} > \alpha - \frac{1}{n}\}$ for each $n \in \mathbb{N}$. Then there exists an $a_n \in U$ such that $U \setminus [0, a_n - 1] \subseteq X_n$. By $\aleph_1$–saturation we have that $\bigcap_{n=1}^{\infty}[a_n, \frac{H}{n}] \neq \emptyset$. Let x be in the intersection. It is easy to see that $x \in U$ and x is greater than every a_n for $n \in \mathbb{N}$. Hence for every $y \geqslant x$ in U, we have $\frac{A(0,y)}{y+1} \gtrapprox \alpha$. Note that the reason above also shows that the cofinality of U is uncountable, i.e. every countable increasing sequence in U is upper bounded in U.

Another important notation needed is called e–transform (cf.[11, p.42]). It is also called τ–transformation (cf.[4, p.58]). Let $A, B \subseteq {}^*\mathbb{N}$ and $a \in A$. An e_a–transform of

(A,B) is the pair $(A',B') = e_a(A,B)$ such that

$$A' = A \cup (B+a) \text{ and } B' = B \cap (A-a).$$

Since in the key lemma we mainly concern the part of a set in U, we always assume $a \in A \cap U$ when an e_a–transform is applied to the pair (A,B).

Remark 2.11 *The following are important properties of the e_a–transform. Let $(A',B') = e_a(A,B)$:*

(1) $A' \supseteq A$ and $B' \subseteq B$.

(2) $A'+B' \subseteq A+B$. Hence $\underline{d}_U(A'+B') \leqslant \underline{d}_U(A+B)$.

(3) If $x \in U$ and $\frac{a}{x} \approx 0$, then for every y with $x < y < U$ we have

$$\frac{A(0,y)+B(0,y)}{y+1} \approx \frac{A'(0,y)+B'(0,y)}{y+1}.$$

Hence $\underline{d}_U(f_{A'}+f_{B'}) = \underline{d}_U(f_A+f_B)$ when $a \in U$.

(4) If $0 \in A \cap B$, then $0 \in A' \cap B'$ because $a \in A$.

We often write $E(A,B)$ for a finite sequence E of e–transforms $e_{a_n} \circ e_{a_{n-1}} \circ \cdots \circ e_{a_1}$ successively applied to (A,B) such that each step of the application is well defined. We say that e_a occurs in E if e_a is in the sequence. If $(A',B') = E(A,B)$, then (1), (2), and (4) of Remark 2.11 are still true and (3) of Remark 2.11 is true for all sufficiently large $x \in U$.

Let U be a cut. A *b.p.* $J = I_0 \cup I_1$ is called U–unbounded if both $I_0 \cap U$ and $I_1 \cap U$ are upper unbounded in U. Note that a U–unbounded *b.p.* has its difference at least 3.

We are now ready to prove the key lemma. The key lemma is inspired by Kneser's Theorem (cf [4]). In the proof of key lemma and the proofs in the next section, we number the claims, subclaims, cases, subcases, subsubcases, etc., so that the reader can see how they are nested.

Lemma 2.12 *Let H be hyperfinite, $U = U_H$, and $A \subseteq [0,H]$ be such that $0 < \underline{d}_U(A) = \alpha < \frac{2}{3}$. If $A \cap U$ is neither a subset of an* a.p. *of difference greater than 1 nor a subset of a U–unbounded* b.p.*, then there is a $\gamma > 0$ such that for every $N > U$, there is a $K \in A$, $U < K < N$, such that*

$$\frac{(2A)(0,2K)}{2K+1} \geqslant \frac{3A(0,K)}{2(K+1)} + \gamma.$$

Proof: We prove the lemma by proving a sequence of claims and cases. Without loss of generality, we assume $0 \in A$. In the proof we only deal with the numbers in U. Hence the comparison relations $\prec, \succ, \preceq, \succeq$ with respect to H are no longer useful because for every $x \in U$ we have $x \sim 0$.

Claim 2.12.1: If $\underline{d}_U(2A) > \frac{3}{2}\alpha$, then the lemma is true.

Proof of Claim 2.12.1: There are $x < U$ and $y > U$ in A such that for every $x < z < 2y$, $\frac{(2A)(0,z)}{z+1} \geqslant \frac{3}{2}\alpha + 2\epsilon$, where $\epsilon = (\underline{d}_U(2A) - \frac{3}{2}\alpha)/3$. Let $\gamma = \frac{1}{2}\epsilon$. For every $N > U$ choose a $K > U$ such that $K < N$, $K < y$, and $\frac{A(0,K)}{K+1} < \alpha + \epsilon$. This implies that

$$\frac{(2A)(0,2K)}{2K+1} \geqslant \frac{3}{2}\alpha + 2\epsilon = \frac{3}{2}(\alpha + \epsilon) + \frac{1}{2}\epsilon \geqslant \frac{3A(0,K)}{2(K+1)} + \gamma.$$

Note that K can be chosen from A because otherwise one can replace K by the smallest integer in A which is greater than K. This ends the proof. □(Claim 2.12.1)

If $\frac{1}{2} < \alpha < \frac{2}{3}$, then there is an $x \in U$ such that $U \setminus [0, x-1] \subseteq (2A)$ by the pigeonhole principle. Hence $\underline{d}_U(2A) = 1 > \frac{3}{2}\alpha$. By Claim 2.12.1 we can assume that $0 < \alpha \leqslant \frac{1}{2}$.

Claim 2.12.2: If for every $x < U$, there is a $y \in A$ with $x < y < U$ such that

$$\frac{(2A)(0,2y)}{2y+1} \gg \frac{3A(0,y)}{2(y+1)},$$

then the lemma is true.

Proof of Claim 2.12.2: By the fact that the cofinality of U is uncountable, one can find $\gamma > 0$ such that the set $Y \cap U$ is unbounded in U, where

$$Y = \left\{ y \in A : \frac{(2A)(0,2y)}{2y+1} \geqslant \frac{3A(0,y)}{2(y+1)} + \gamma \right\}$$

is clearly internal. For every $N > U$ let K be the largest element in $Y \cap [0, N-1]$. Then $K \in Y$ and $U < K < N$. This shows the lemma is true. □(Claim 2.12.2)

Claim 2.12.3: For every finite sequence E of e–transforms with $(A', B') = E(A, A)$, there is an $x \in U$ such that for all $y \in B'$, $x < y < U$, we have

$$\frac{B'(0,y)}{y+1} \gtrapprox \frac{A(0,y)}{2(y+1)}.$$

Proof of Claim 2.12.3: Suppose not. Then there is a finite sequence E of e–transforms such that $(A', B') = E(A, A)$ and for every $x \in U$, there is a $y > x$ in $B' \cap U$ such that $\frac{B'(0,y)}{y+1} \ll \frac{A(0,y)}{2(y+1)}$. Then

$$\begin{aligned}
\frac{(2A)(0,2y)}{2y+1} &\gtrapprox \frac{(A'+B')(0,2y)}{2y+1} \\
&\gtrapprox \frac{A'(0,y) + |A'[0,y]+y|}{2y+1} \approx 2\frac{A'(0,y)}{2y+1} \\
&\approx 2\frac{A'(0,y)+B'(0,y)}{2y+1} - \frac{B'(0,y)}{y+1} \\
&\approx 2\frac{2A(0,y)}{2y+1} - \frac{B'(0,y)}{y+1} \\
&\gg 2\frac{A(0,y)}{y+1} - \frac{A(0,y)}{2(y+1)} \approx \frac{3A(0,y)}{2(y+1)}.
\end{aligned}$$

Hence by Claim 2.12.2, the lemma is true. $\square$(Claim 2.12.3)

Next we need to define two functions f and g on subsets of $^*\mathbb{N}$. For every $X \subseteq {}^*\mathbb{N}$ let

$$f(X) = \begin{cases} \min\{|x-y|-1 : x, y \in X \cap U \text{ and } x \neq y\}, & \begin{array}{l}\text{if there are } x, y \in X \cap U \\ \text{such that } x \neq y \text{ and} \\ |x-y| \text{ is finite;}\end{array} \\ \infty, & \text{otherwise.} \end{cases}$$

$$g(X) = \begin{cases} \gcd(X \cap U), & \text{if } \gcd(X \cap U) \text{ is finite;} \\ \infty, & \text{otherwise.} \end{cases}$$

Clearly $g(C) \leqslant f(C) + 1$ for every C. Note that if $e_a(C, D) = (C', D')$, then $f(D') \geqslant f(D)$, $f(C') \leqslant f(C)$, $g(D') \geqslant g(D)$, and $g(C') \leqslant g(C)$. Let

$$S = \left\{ f(B') : \begin{array}{l} \text{there is a finite sequence } E \text{ of } e\text{–transforms} \\ \text{such that } E(A, A) = (A', B') \end{array} \right\}.$$

If S contains ∞ or S is unbounded in $\mathbb{N}$, then there is a sequence E of e–transforms such that $(A', B') = E(A, A)$ and $f(B') > \frac{3}{\alpha}$. By Remark 2.11 there is an $a \in U$ such that for every $x > a$ in U,

$$
\begin{aligned}
&\frac{(2A)(0,x)}{x+1}\\
&\quad \gtrapprox \frac{(A'+B')(0,x)}{x+1} \gtrapprox \frac{A'(0,x)}{x+1}\\
&\quad \gtrapprox \frac{A'(0,x)+B'(0,x)}{x+1} - \frac{B'(0,x)}{x+1}\\
&\quad \approx 2\frac{A(0,x)}{x+1} - \frac{B'(0,x)}{x+1}\\
&\quad \gtrapprox 2\alpha - \frac{\alpha}{3} = \frac{3}{2}\alpha + \frac{1}{6}\alpha.
\end{aligned}
$$

Hence $\underline{d}_U(2A) \geqslant \frac{3}{2}\alpha + \frac{1}{6}\alpha$, which, together with Claim 2.12.1, implies the lemma. By the argument above we can now assume that the set S is bounded in $\mathbb{N}$. Let

$$
t = \min\left\{ f(A') : \begin{array}{c} (A',B') = E(A,B) \text{ for some} \\ \text{finite sequence } E \text{ of } e\text{–transforms} \end{array} \right\}.
$$

Since S is upper bounded in $\mathbb{N}$, we can fix a finite sequence E_0 of e–transforms with $(A',B') = E_0(A,A)$ such that

(1) $f(A') = t$;

(2) $f(B') = l$ and $f(B'') = l$ for each E with $E(A',B') = (A'',B'')$;

(3) $g(B') = m$ and $g(B'') = m$ for each E with $E(A',B') = (A'',B'')$;

(4) let $F = \{k \in [0,m-1] : \exists a \in A' \cap U\, (a \equiv k \,(\mathrm{mod}\, m))\}$ such that for each E with $E(A',B') = (A'',B'')$ and $F'' = \{k \in [0,m-1] : \exists a \in A'' \cap U\, (a \equiv k \,(\mathrm{mod}\, m))\}$ we have $F = F''$.

Let $\bar{a} = \max\{a \in U : e_a \text{ occurs in } E_0\}$. The rest of the proof is divided into four cases in terms of the size of F. From now on let's assume that the key lemma is not true. We will derive a contradiction in each of the following cases. Let $|F| = n$ and let $F = \{0 = k_0 < k_1 < \cdots < k_{n-1}\}$. For each $i = 0,1,\ldots,n-1$ let $I_i = \{k_i + xm : x \in [0,H]\}$, $A_i = A \cap I_i$, $A'_i = A' \cap I_i$, and $a_i = \min A_i$. Let $T = \{a_i : i = 0,1,\ldots,n-1\}$. Without loss of generality we assume $T + B' \subseteq A'$ because otherwise we can first perform n successive e–transforms on (A',B') to get (A'',B'') so that $T + B'' \subseteq A''$ and then replace (A',B') by (A'',B'').

Case 2.12.1: $|F| \geqslant 4$.

For all sufficiently large $x \in B' \cap U$,

$$
\frac{(2A)(0,2x)}{2x+1} \gtrapprox \frac{(A'+B')(0,2x)}{2x+1}
$$

$$\gtrapprox \frac{|T + B'[0,x]| + |T + x + B'[0,x]|}{2x+1}$$
$$\gtrapprox \frac{8B'(0,x)}{2x+1} \gtrapprox \frac{4B'(0,x)}{x+1}$$
$$\gtrapprox 2\frac{A(0,x)}{x+1} \gg \frac{3A(0,x)}{2(x+1)}.$$

The second last inequality is a consequence of Claim 2.12.3. By Claim 2.12.2, the lemma is true, which contradicts the assumption. $\square$(Case 2.12.1)

Case 2.12.2: $|F| = 3$.

By Claim 2.12.1 and $\gcd(A \cap U) = 1$ together with Lemma 1.5, we have $\underline{d}_U(2A) = \frac{3}{2}\alpha$. Since $|F| = 3$, for every sufficiently large $x \in B' \cap U$ with $\gcd(B'[0,x]) = m$, we have that

$$\frac{(2A)(0,2x)}{2x+1} \gtrapprox \frac{(A'+B')(0,2x)}{2x+1}$$
$$\gtrapprox \frac{|T + B'[0,x]| + |T + x + B'[0,x]|}{2x+1}$$
$$\gtrapprox \frac{6B'(0,x)}{2(x+1)} \gtrapprox \frac{3A(0,x)}{2(x+1)}.$$

If there exist sufficiently large $x \in U$ such that $\frac{(2A)(0,2x)}{2x+1} \gg \frac{3A(0,x)}{2(x+1)}$, then the lemma follows from Claim 2.12.2. So we can assume that for all sufficiently large $x \in U$, $\frac{2B'(0,x)}{x+1} \approx \frac{A(0,x)}{x+1}$.

Let

$$F_0 = \{k \in [0, m-1] : \exists a \in A \cap U \, (a \equiv k \pmod m)\}.$$

Since $0 \in A$, then $0 \in F_0$. If $F_0 = \{0\}$, then A is a subset of an *a.p.* of difference $m \geqslant 3$. If $|F_0| = 2$, then $A \cap U$ is either a subset of an *a.p.* of difference $\frac{m}{2} > 1$ or a subset of a U–unbounded *b.p.* (this is because if $A \cap U$ is a subset of a *b.p.* of difference m, which is not U–unbounded, then there is an e–transform e_a with $(A'', B'') = e_a(A', B')$ such that B'' is U–bounded, which contradicts Claim 2.12.3). Hence we can assume that $|F_0| = 3$ and so $F_0 = F$. Let $F = \{0 = k_0 < k_1 < k_2\}$.

Since $T + B' \subseteq A'$, then for every sufficiently large $x \in B' \cap U$,

$$\frac{B'(0,x)}{x+1} \lessapprox \frac{A'_i(0,x)}{x+1}$$

for $i = 0, 1, 2$. Fix such an $x \in U$ with $\frac{\bar{a}}{x} \approx 0$. By Claim 2.12.3,

$$\frac{B'(0,x)}{x+1} \approx \frac{A'_i(0,x)}{x+1}$$

for $i=0,1,2$. Since $B'[0,x] \subseteq A_0[0,x] \subseteq A'_0[0,x]$, then $\frac{A_0(0,x)}{x+1} \approx \frac{A(0,x)}{2(x+1)}$. This implies $\frac{A_1(0,x)+A_2(0,x)}{x+1} \approx \frac{A_0(0,x)}{x+1}$.

Let $C, D \subseteq [0,H]$, $a \in C \cap U$, and $D_0 \subseteq D$. Let $(C', D') = e_a(C, D)$. We call that a number $c \in C' \setminus C$ is obtained from D_0 via e_a if there is a $d \in D_0$ such that $c = a + d$. Let E be a finite sequence of e–transforms such that $(C', D') = E(C, D)$. Let $D_0 \subseteq D$ and $c \in C' \setminus C$. We call that c is obtained from D_0 via E if c is obtained from D_0 via an e–transform e_a, which occurs in E.

For $i = 0,1,2$ let $z_i = \max A_i[0,x]$ and $z'_i = \max A'_i[0,x]$. Since $T + x \subseteq A'$, then we have that

$$\begin{aligned}\frac{(2A)(0,2x)}{2x+1} &\gtrapprox \frac{(A'+B')(0,2x)}{2x+1}\\ &\gtrapprox \frac{|B'[0,x]+B'[0,x]| + |\{a_1, a_2, x+a_1, x+a_2\} + B'[0,x]|}{2x+1}\\ &\gtrapprox \frac{6B'(0,x)}{2x+1} \gtrapprox \frac{3A(0,x)}{2(x+1)}.\end{aligned}$$

If the lemma is not true, then by Claim 2.12.2 we have that

$$\frac{|B'[0,x]+B'[0,x]|}{2x+1} \approx \frac{2B'(0,x)}{2x+1}.$$

This implies that $\frac{B'(0,x)}{x+1} \approx \frac{1}{m}$ by Lemma 1.6. Hence $\frac{A'_i(0,x)}{x+1} \approx \frac{1}{m}$ for $i = 0,1,2$. As a consequence we have $\frac{z'_i}{x} \approx 1$ for $i = 1,2$.

We now show that either $\frac{z_1}{x} \approx 1$ or $\frac{z_2}{x} \approx 1$. Suppose not and let $z = \max\{z_1, z_2\}$. Then for $i = 1,2$, $A_i[z+1,x] = \emptyset$. Since $\frac{A'_1(z+1,x)}{x-z} \approx \frac{1}{m}$, then $\frac{A'_1(z+1+\bar{a},x)}{x-z} \approx \frac{1}{m}$. Since also $\frac{A_0(z+1,x)-B'(z+1,x)}{x-z} \approx 0$ because $\frac{B'(0,x)}{x+1} \approx \frac{1}{m}$, then there must be some elements in $A'[z+1+\bar{a},x]$ that are obtained from $A_1[z+1,x] \cup A_2[z+1,x]$ via E_0. This contradicts that $A_1[z+1,x] \cup A_2[z+1,x] = \emptyset$

Claim 2.12.2.1 $F_0 \cup \{m\}$ is an *a.p.* of difference k_1.

Proof of Claim 2.12.2.1 Suppose not. Then either $2k_1 \notin \{k_2, m\}$ or $k_1 + k_2 \neq m$. We want to show that $\frac{(2A)(0,2x)}{2x+1} \gg \frac{3A(0,x)}{2(x+1)}$, which contradicts the assumption that the lemma is not true by Lemma 2.12.2.

Case 2.12.2.1.1: $2k_1 \notin \{k_2, m\}$.

Hence $(2A_1) \cap (A_0 + A) = \emptyset$. We want to show that this case is impossible.

Subcase 2.12.2.1.1.1: $\frac{A_1(0,x)}{x+1} \gg 0$ and $\frac{z_1}{x} \ll 1$.

Hence $\frac{z_2}{x} \approx 1$. This implies that $A_1'[z_1, x]$ obtains numbers from A_2 via E_0. So there is a $c \in \{0, k_1, k_2\}$ such that $k_1 \equiv c + k_2 \pmod{m}$. Clearly there is only one possible value for c, i.e. $c = k_2$. So we have $2k_2 = m + k_1$.

If $k_1 + k_2 \neq m$, then $2A_1$, $A_1 + A_2$, and $A_0 + A$ are pairwise disjoint. Hence

$$\frac{(2A)(0,2x)}{2x+1} \gtrapprox \frac{4A_0(0,x) + 4A_1(0,x) + 2A_2(0,x)}{2x+1} \gg \frac{3A(0,x)}{2(x+1)}.$$

So we can assume $k_1 + k_2 = m$. But that, together with $2k_2 = m + k_1$, implies that $F \cup \{m\}$ is an *a.p.* of difference k_1. □(Subcase 2.12.2.1.1.1)

Subcase 2.12.2.1.1.2: $\frac{A_1(0,x)}{x+1} \gg 0$ and $\frac{z_1}{x} \approx 1$.

We now have

$$\frac{(A_0 + A_1)(0,2x)}{2x+1} \gtrapprox \frac{|A_0[0,x] + \{a_1, z_1\}|}{2x+1} \approx \frac{2A_0(0,x)}{2x+1}.$$

Hence

$$\frac{(2A)(0,2x)}{2x+1} \gtrapprox \frac{5A_0(0,x) + A_2(0,x) + 2A_1(0,x)}{2x+1} \gg \frac{3A(0,x)}{2(x+1)}.$$

□(Subcase 2.12.2.1.1.2)

Subcase 2.12.2.1.1.3: $\frac{A_1(0,x)}{x+1} \approx 0$.

Clearly $A_1'[0,x]$ obtains numbers from A_2. Hence $2k_2 = m + k_1$. This implies $2k_2 \neq m$. If $k_1 + k_2 \neq m$, then $(A_1 + A_2)$, $2A_2$, $2A_0$, and $A_0 + A_2$ are pairwise disjoint. Hence

$$\frac{(2A)(0,2x)}{2x+1} \gtrapprox \frac{3A_0(0,x) + 4A_2(0,x)}{2x+1} \gg \frac{3A(0,x)}{2(x+1)}.$$

If $k_1 + k_2 = m$, then $F \cup \{m\}$ is an *a.p.* of difference k_1. □(Case 2.12.2.1.1)

Case 2.12.2.1.2: $2k_1 \in \{k_2, m\}$.

If $2k_1 = m$, then $k_1 + k_2 > m$. If $2k_1 = k_2$, then $k_1 + k_2 = m$ implies that $F \cup \{m\}$ is an *a.p.* of difference k_1. Hence we can assume that $k_1 + k_2 \neq m$.

Subcase 2.12.2.1.2.1: $\frac{A_2(0,x)}{x+1} \gg 0$ and $\frac{z_1}{x} \approx 1$.

We have

$$\frac{(A_0 + A_1)(0,2x)}{2x+1} \gtrapprox \frac{|A_0[0,x] + \{a_1, z_1\}|}{2x+1} \gtrapprox \frac{2A_0(0,x)}{2x+1}.$$

Hence

$$\frac{(2A)(0,2x)}{2x+1} \gtrapprox \frac{(A_0 + A)(0,2x) + (A_1 + A_2)(0,2x)}{2x+1}$$
$$\gtrapprox \frac{5A_0(0,x) + 2A_2(0,x) + A_1(0,x)}{2x+1} \gg \frac{3A(0,x)}{2(x+1)}.$$

□(Subcase 2.12.2.1.2.1)

Subcase 2.12.2.1.2.2: $\frac{A_2(0,x)}{x+1} \gg 0$ and $\frac{z_1}{x} \ll 1$.

Then $\frac{z_2}{x} \approx 1$. Clearly, $A_1'[z_1, x]$ obtains numbers from A_2 via E_0. Hence we have $2k_2 = m + k_1$. If $2k_1 = k_2$, then $F \cup \{m\}$ is an *a.p.* of difference k_1. Hence we can assume that $2k_1 = m$.

If $\frac{A_1(0,x)}{x+1} \gg 0$, then

$$\begin{aligned} \frac{(2A)(0,2x)}{2x+1} &\gtrapprox \frac{(A_0 + A)(0,2x) + (A_1 + A_2)(0,2x)}{2x+1} \\ &\gtrapprox \frac{5A_0(0,x) + 2A_1(0,x) + A_2(0,x)}{2x+1} \gg \frac{3A(0,x)}{2(x+1)}. \end{aligned}$$

If $\frac{A_1(0,x)}{x+1} \approx 0$, then $\frac{A_2(0,x)}{x+1} \approx \frac{1}{m}$. Hence

$$\begin{aligned} \frac{(2A)(0,2x)}{2x+1} &\gtrapprox \frac{(2A_0)(0,2x) + (A_0 + A_2)(0,2x)}{2x+1} \\ &+ \frac{(A_1 + A_2)(0,2x) + (2A_2)(0,2x)}{2x+1} \\ &\gtrapprox \frac{4A_0(0,x) + 3A_2(0,x)}{2x+1} \gg \frac{3A(0,x)}{2(x+1)}. \end{aligned}$$

□(Subcase 2.12.2.1.2.2)

Subcase 2.12.2.1.2.3: $\frac{A_2(0,x)}{x+1} \approx 0$.

Then $\frac{A_1(0,x)}{x+1} \approx \frac{1}{m}$ and $A_2'[0,x]$ obtains numbers from A_1 via E_0. This implies $k_2 = 2k_1$. Now we have that $2A_0$, $A_0 + A_1$, $2A_1$, and $A_1 + A_2$ are pairwise disjoint. Hence

$$\frac{(2A)(0,2x)}{2x+1} \gtrapprox \frac{7A_0(0,x)}{2x+1} \gg \frac{3A(0,x)}{2(x+1)}.$$

This ends the proof of the claim. □(Claim 2.12.2.1)

If $k_1 > 1$, then A is a subset of an *a.p.* with the difference $k_1 > 1$ by Claim 2.12.2.1. So we can assume $m = 3$ and $F = \{0, 1, 2\}$.

Let $z < U$ be sufficiently large with $\frac{A(0,z)}{z+1} \approx \alpha$ and let $x \in B'$ be such that $x \geqslant z$ and $[z, x-1] \cap B' = \emptyset$. By Claim 2.12.3,

$$\begin{aligned} \frac{(2A)(0,2x)}{2x+1} &\gtrapprox \frac{(A' + B')(0,2x)}{2x+1} \\ &\gtrapprox \frac{|B'[0,x] + B'[0,x]| + |a_1 + B'[0,x] + B'[0,x]|}{2x+1} \end{aligned}$$

$$+\frac{|a_2 + B'[0,x] + B'[0,x]|}{2x+1}$$
$$\gtrapprox \frac{6B'(0,x)}{2x+1} \gtrapprox \frac{3A(0,x)}{2(x+1)}.$$

Hence we have

$$\frac{(2A)(0,2x)}{2x+1} \gg \frac{3A(0,x)}{2(x+1)}$$

unless

$$\frac{|B'[0,x]+B'[0,x]|}{2x+1} \approx \frac{2B'(0,x)}{2x+1}.$$

But the latter implies, by Lemma 1.6, that $B'[0,x]$ is a subset of an *a.p.* of length $B'(0,x)+b$ with $\frac{b}{x} \approx 0$. This implies $\frac{x-z}{x} \approx 0$. Hence $\frac{A(0,x)}{x+1} \approx \frac{A(0,z)}{z+1} \approx \alpha$. Note that $\gcd(B') = m = 3$. So we have $\frac{B'(0,x)}{x+1} \approx \frac{1}{3}$. This implies

$$\frac{(2A)(0,2x)}{2x+1} \gtrapprox \frac{6B'(0,x)}{2x+1} \approx 1 \gg \frac{3}{2}\alpha \approx \frac{3A(0,x)}{2(x+1)}.$$

By Claim 2.12.2, the lemma is true, which contradicts the assumption. $\square$(Case 2.12.2)

Case 2.12.3: $|F| = 2$.

Since $A \cap U$ is neither a subset of an *a.p.* with the difference greater than 1 nor a subset of a U–unbounded *b.p.*, then we have $m = 2$ and $F = \{0,1\}$. Let a be the least odd number in A. We can again assume $B' \subseteq A'$ and $a + B' \subseteq A'$.

Let $z \in U$ be sufficiently large with $\gcd(B'[0,z]) = 2$ such that $\frac{A(0,z)}{z+1} \approx \alpha$ and $\frac{A(0,y)}{y+1} \gtrapprox \alpha$ for every $y \geqslant z$ in U. Let $x \in B' \cap U$ be such that $x \geqslant z$ and $[z, x-1] \cap B' = \emptyset$. Let $\frac{A(0,x)}{x+1} \approx \beta$. Since $B' \subseteq A'$, $a + B' \subseteq A'$ and $B' \cap (a + B') = \emptyset$, then $\frac{A'(0,x)}{x+1} \gtrapprox 2\frac{B'(0,x)}{x+1}$. Hence there is an $\epsilon > 0$ such that $\beta - \frac{B'(0,x)}{x+1} \approx \epsilon$. This implies $\frac{A'(0,x)}{x+1} \approx \beta + \epsilon$, $\beta + \epsilon \geqslant 2(\beta - \epsilon)$, and, by Claim 2.12.3, $\beta - \epsilon \geqslant \frac{1}{2}\beta$. Hence $2\epsilon \leqslant \beta \leqslant 3\epsilon$. Note that $\beta \leqslant \frac{1}{2}$ because $\frac{x-z}{x} \approx 0$ implies $\beta = \alpha \leqslant \frac{1}{2}$ and $\frac{x-z}{x} \gg 0$ implies that $\frac{A'(z,x)}{x-z} \approx 2\frac{A(z,x)}{x-z}$, which implies $\frac{A(z,x)}{x-z} \lessapprox \frac{1}{2}$, hence again $\beta \approx \frac{A(0,x)}{x+1} \lessapprox \max\{\alpha, \frac{1}{2}\} = \frac{1}{2}$.

Let A'_0 be the set of all even numbers in A' and A'_1 be the set of all odd numbers in A'. Let $\frac{A'_0(0,x)}{x+1} \approx \beta_0 > 0$ and $\frac{A'_1(0,x)}{x+1} \approx \beta_1 > 0$. Then $\beta_0 + \beta_1 = \beta + \epsilon$. By Lemma 1.7 in which let $B'[0,x]$ be A and $A'_i[0,x]$ be B, we have

$$\frac{(2A)(0,2x)}{2x+1} \gtrapprox \frac{(A'+B')(0,2x)}{2x+1}$$

$$
\begin{aligned}
&\approx \frac{(A_0' + B')(0,2x) + (A_1' + B')(0,2x)}{2x+1}\\
&\gtrapprox \frac{|A_0'[0,x] + B'[0,x]| + |A_1'[0,x] + B'[0,x]|}{2x+1}\\
&\gtrapprox \min\left\{\frac{2\beta_0+1}{4}, \beta_0 + \frac{\beta-\epsilon}{2}\right\} + \min\left\{\frac{2\beta_1+1}{4}, \beta_1 + \frac{\beta-\epsilon}{2}\right\}.
\end{aligned}
$$

Since

$$\frac{2\beta_0+1}{4} + \frac{2\beta_1+1}{4} = \frac{\beta+\epsilon+1}{2} \geqslant \frac{3\beta+\epsilon}{2} \gg \frac{3}{2}\beta,$$

$$
\begin{aligned}
&\frac{2\beta_0+1}{4} + \beta_1 + \frac{\beta-\epsilon}{2}\\
&\quad = \frac{2\beta_0 + 1 + 4\beta_1 + 2\beta - 2\epsilon}{4} = \frac{4\beta + 1 + 2\beta_1}{4} \geqslant \frac{3\beta+\beta_1}{2} \gg \frac{3\beta}{2},
\end{aligned}
$$

$$
\begin{aligned}
&\beta_0 + \frac{\beta-\epsilon}{2} + \frac{2\beta_1+1}{4}\\
&\quad = \frac{4\beta_0 + 2\beta - 2\epsilon + 2\beta_1 + 1}{4} = \frac{4\beta + 2\beta_0 + 1}{4} \gg \frac{3\beta}{2},
\end{aligned}
$$

$$\beta_0 + \frac{\beta-\epsilon}{2} + \beta_1 + \frac{\beta-\epsilon}{2} = \beta + \epsilon + \beta - \epsilon = 2\beta \gg \frac{3}{2}\beta,$$

then

$$\frac{(2A)(0,2x)}{2x+1} \gg \frac{3}{2}\beta \approx \frac{3A(0,x)}{2(x+1)},$$

which, by Claim 2.12.2, contradicts the assumption that the key lemma is not true. □(Case 2.12.3)

Case 2.12.4: $|F| = 1$.

This is the last case and is the most difficult one. We will prove several claims in this case. Since A is not a subset of a *a.p.* with the difference greater than 1, then we have $m = g(B') = \gcd(B') = 1$.

Claim 2.12.4.1: For every sufficiently large $x < U$ with $\gcd(B'[0,x]) = 1$ such that $\frac{A(0,x)}{x+1} \approx \alpha$, we have $\frac{B'(0,x)}{x+1} \approx \alpha$.

Proof of Claim 2.12.4.1: Suppose not. For every $y < U$, there is an $x > y$ in U with $\frac{A(0,x)}{x+1} \approx \alpha$ and $\frac{B'(0,x)}{x+1} \ll \alpha$. Then we have $\frac{A'(0,x)}{x+1} \gg \alpha$. Let $z \geqslant x$ be such that $[x, z-1] \cap B' = \emptyset$. Let $\frac{A(0,z)}{z+1} \approx \beta$ and let $\epsilon > 0$ be such that $\frac{B'(0,z)}{z+1} \approx \alpha - \epsilon$.

Note that $\frac{x}{z} < \frac{1}{2}$ implies

$$\frac{B'(0,z)}{z+1} \lessapprox \frac{B'(0,x)}{2(x+1)} \ll \frac{\alpha}{2} \lessapprox \frac{A(0,z)}{2(z+1)},$$

which contradicts Claim 2.12.3. Hence we can assume $\frac{x}{z} \geqslant \frac{1}{2}$.

Suppose $\frac{x}{z} \approx 1$. Then by Lemma 1.7

$$\begin{aligned}\frac{(2A)(0,2x)}{2x+1} &\approx \frac{(2A)(0,2z)}{2z+1} \\ &\gtrapprox \frac{|A'[0,z]+B'[0,z]|}{2z+1} \\ &\gtrapprox \min\left\{\alpha+\epsilon+\frac{\alpha-\epsilon}{2}, \frac{\alpha+\epsilon+1}{2}\right\} \\ &\geqslant \frac{3\alpha+\epsilon}{2} \gg \frac{3A(0,x)}{2(x+1)}.\end{aligned}$$

This contradicts the assumption that Lemma 2.12 is false.

Suppose $\frac{1}{2} \leqslant \frac{x}{z} \ll 1$.

Note that

$$\frac{A'(x,z)}{z+1} \approx \frac{A'(x,z)+B'(x,z)}{z+1} \approx 2\frac{A(x,z)}{z+1}.$$

Since $\frac{B'(0,x)}{x+1} \ll \alpha$, then $\frac{A'(0,x)}{x+1} \gg \alpha \approx \frac{A(0,x)}{x+1}$. Hence

$$\begin{aligned}\frac{(2A)(0,2z)}{2z+1} &\gtrapprox \frac{(A'+B')(0,2z)}{2z+1} \\ &\gtrapprox \frac{|A'[0,z]+B'[0,z]|}{2z+1} \\ &\gtrapprox \frac{\min\{2A'(0,z)+B'(0,z), A'(0,z)+z\}}{2z+1} \\ &\approx \min\left\{\frac{2A'(0,x)+2A'(x,z)+B'(0,x)}{2z+1}, \frac{A'(0,x)+A'(x,z)+x+(z-x)}{2z+1}\right\} \\ &\gtrapprox \min\left\{\frac{2A(0,x)+4A(x,z)+A'(0,x)}{2z+1}, \frac{A'(0,x)+2A(x,z)+2A(0,x)+A(x,z)}{2z+1}\right\} \\ &\gg \min\left\{\frac{3A(0,z)}{2(z+1)}, \frac{3A(0,z)}{2(z+1)}\right\} = \frac{3A(0,z)}{2(z+1)}.\end{aligned}$$

By Claim 2.12.2, we have a contradiction. □(Claim 2.12.4.1)

Recall that

$$t = f(A') = \min\left\{f(A'') : \begin{array}{l}\text{there is a finite sequence of } e\text{–transforms} \\ E \text{ with } (A'',B'') = E(A,A)\end{array}\right\}.$$

Claim 2.12.4.2: If $t = 0$, then the lemma is true. Hence we can assume $t > 0$.

Proof of Claim 2.12.4.2: Suppose $t = 0$. Then $A' \cap U$ contains two consecutive numbers. We first show by induction that for every $k \in \mathbb{N}$, there is a finite sequence

E' of e–transforms such that $(A'', B'') = E'(A', B')$ and $A'' \cap U$ contains k consecutive numbers.

It is clearly true for $k = 2$. Suppose there is a finite sequence E_1 of e–transforms with $(A_1, B_1) = E_1(A', B')$ and $A_1 \cap U$ contains $k - 1$ consecutive numbers $a, a + 1, \ldots, a + k - 2$. Now we need the next subclaim to show that $B_1 \cap U$ contains two consecutive numbers. Without loss of generality we can assume $[a, a+k-2]+B_1 \subseteq A_1$.

Subclaim 2.12.4.2.1: For every finite set $T \subseteq A_1 \cap U$ and every number $x < U$, there is a $y \in B_1 \cap U$, $y > x$, such that $T + y \subseteq B_1$.

Proof of Subclaim 2.12.4.2.1: Suppose the subclaim is not true. Without loss of generality we assume $T + B_1 \subseteq A_1$. Let $|T| = s$. For each $b \in B_1$, let $h(b) = \min((T + b) \smallsetminus B_1)$ when the set $(T + b) \smallsetminus B_1$ is not empty and let $h(b) = H$ otherwise. Since the subclaim is not true, then for every $b \in B_1 \cap U$, $h(b) \in (A_1 \smallsetminus B_1) \cap U$ is well defined. So for every $i \in (A_1 \smallsetminus B_1) \cap U$, $|h^{-1}(i)| \leqslant s$. Since $B_1 \subseteq A_1$, we have

$$\frac{A_1(0, x)}{x + 1} \gtrapprox \frac{B_1(0, x)}{x + 1}(1 + \frac{1}{s}) \gg \frac{B_1(0, x)}{x + 1}$$

for all sufficiently large $x < U$ with $\frac{A(0,x)}{x+1} \approx \alpha$, which contradicts Claim 2.12.4.1. Note that Subclaim 2.12.4.2.1 is also true if A_1 is replaced by A'' where $(A'', B'') = \bar{E}(A', B')$ for some sequence $\bar{E}$ of e–transforms. □(Subclaim 2.12.4.2.1)

Since $A_1 \cap U$ contains $k - 1$ consecutive numbers $[a, a_k - 2]$, then by the subclaim above, we can assume that $B_1 \cap U$ contains $k - 1$ consecutive numbers $b + [a, a + k - 2]$ for some $b \in B_1$. Hence $[a, a + k - 2] + b + [a, a + k - 2] \subseteq [a, a + k - 2] + B_1 \subseteq A_1$. Hence A_1 contains $2k - 3 \geqslant k$ consecutive numbers.

Note that using the same idea we can prove that if $A' \cap U$ contains two numbers of difference d, then for every $k \in \mathbb{N}$ there is a finite sequence E of e–transforms with $(A'', B'') = E(A', B')$ such that $A'' \cap U$ contains an *a.p.* of difference d and length k.

Next we conduct the second half of the proof of Claim 2.12.4.2.

Subclaim 2.12.4.2.2: If $t = 0$, then $\underline{d}_U(2A) \geqslant 2\alpha$.

Proof of Subclaim 2.12.4.2.2: The idea of the proof can be found in [4, p.61]. We include the proof here because nonstandard setting is involved.

Suppose $A' \cap U$ contains k consecutive numbers. Without loss of generality, we assume $[0, k - 1] \subseteq A'$ and $[0, k - 1] + B' \subseteq A'$. It suffices now to show

$$\underline{d}_U(A' + B') \geqslant \frac{k}{k + 1}\gamma$$

for every $\gamma < 2\alpha$.

Since

$$\underline{d}_U(f_{A'} + f_{B'}) = \underline{d}_U(2f_A) = 2\underline{d}_U(A) = 2\alpha > \gamma,$$

then there exists an $x_0 \in U$ such that for every $x \in U$ with $x \geqslant x_0$,

$$\frac{A'(0,x) + B'(0,x)}{x+1} > \gamma.$$

Let x_0 be the least such number. Then

$$\frac{A'(x_0, x_0+x) + B'(x_0, x_0+x)}{x+1} > \gamma$$

for every $x \in U$. It is easy to see that $x_0 \in (A' \cup B') \cap U \subseteq A'$. Let $x_1 = \min\{z \in B' : z \geqslant x_0\}$ and let

$$\bar{A} = (A' - x_0) \text{ and } \bar{B} = (B' - x_1).$$

It is easy to check that $0 \in \bar{A} \cap \bar{B}$ and

$$\underline{d}_U(\bar{A} + \bar{B}) = \underline{d}_U(A' + B').$$

By Lemma 1.8 we need only to show that for every $x \in U$, $1 + \bar{A}(1,x) + \bar{B}(1,x) \geqslant \frac{k}{k+1}\gamma(x+1)$.

Suppose $x \geqslant \frac{k+1}{\gamma} - 1$.

Then $\gamma(x+1) \geqslant k+1$ and hence

$$(k+1)\gamma(x+1) - (k+1) \geqslant k\gamma(x+1).$$

This implies

$$\gamma(x+1) - 1 \geqslant \frac{k}{k+1}\gamma(x+1).$$

Since

$$\bar{A}(1,x) = A'(x_0, x_0+x) - 1$$

and

$$\bar{B}(1,x) = B'(x_1+1, x_1+x) \geqslant B'(x_0, x_1+x) - 1 \geqslant B'(x_0, x_0+x) - 1,$$

then

$$\begin{aligned} &1 + \bar{A}(1,x) + \bar{B}(1,x) \\ &\quad\geqslant A'(x_0, x_0+x) + B'(x_0, x_0+x) - 1 \\ &\quad> \gamma(x+1) - 1 \geqslant \frac{k}{k+1}\gamma(x+1). \end{aligned}$$

Suppose $x < x_1 - x_0$.

Since $B'(x_0, x_0 + x) = 0$, then

$$\begin{aligned}
&1 + \bar{A}(1, x) + \bar{B}(1, x) \\
&\quad \geqslant A'(x_0, x_0 + x) = A'(x_0, x_0 + x) + B'(x_0, x_0 + x) \\
&\quad > \gamma(x + 1) > \frac{k}{k+1}\gamma(x + 1).
\end{aligned}$$

Suppose $x_1 - x_0 \leqslant x \leqslant x_1 - x_0 + k - 1$.

Since $x_1 + [0, k-1] = [x_1, x_1 + k - 1] \subseteq [0, k-1] + B' \subseteq A'$, then

$$\begin{aligned}
&1 + \bar{A}(1, x) + \bar{B}(1, x) \\
&\quad \geqslant A'(x_0, x_1 - 1) + x - (x_1 - x_0) + B'(x_0, x_1 - 1) + 1 \\
&\quad > \gamma(x_1 - x_0) + x - (x_1 - x_0) + 1 \\
&\quad > \gamma(x_1 - x_0) + \gamma(x - (x_1 - x_0) + 1) \\
&\quad = \gamma(x + 1) > \frac{k}{k+1}\gamma(x + 1).
\end{aligned}$$

Suppose $x_1 - x_0 + k \leqslant x < \frac{k+1}{\gamma} - 1$.

Then we have $\gamma(x + 1) < k + 1$, which implies $\frac{k}{k+1}\gamma(x + 1) < k$. Hence

$$\begin{aligned}
&1 + \bar{A}(1, x) + \bar{B}(1, x) \\
&\quad \geqslant A'(x_1, x_1 + k - 1) \\
&\quad = k > \frac{k}{k+1}\gamma(x + 1).
\end{aligned}$$

Now we have that for all $x < U$

$$\frac{1 + \bar{A}(1, x) + \bar{B}(1, x)}{x + 1} \geqslant \frac{k}{k+1}\gamma.$$

By Lemma 1.8, we have $\underline{d}_U(\bar{A} + \bar{B}) \geqslant \frac{k}{k+1}\gamma$. Since $k \in \mathbb{N}$ and $\gamma < 2\alpha$ are arbitrary, we have $\underline{d}_U(2A) = \underline{d}_U(\bar{A} + \bar{B}) \geqslant 2\alpha$. Hence by Claim 2.12.1 the lemma is true. This finishes the proof of Claim 2.12.4.2. □(Claim 2.12.4.2)

We can now assume $t > 0$ and fix c such that both c and $c + t + 1$ lie in A'.

Claim 2.12.4.3: If $T \subseteq (2A) \cap U$ is a finite set, then there is an $a \in A \cap U$ and a sequence E of e–transforms such that $(A'', B'') = E(A', B')$ and $a + T \subseteq A''$.

Proof of 2.12.4.3: Let $T = \{a_1 + b_1, a_2 + b_2, \ldots, a_k + b_k\}$, where $a_i, b_i \in A$ for $i = 1, 2, \ldots, k$. Without loss of generality we can assume $a_i + B' \subseteq A'$ for $i = 1, 2, \ldots, k$.

By Subclaim 2.12.4.2.1 we can assume that there is an $a \in B' \cap U \subseteq A$ such that $a + \{b_1, b_2, \ldots, b_k\} \subseteq B'$. Since $a_i + B' \subseteq A'$, then

$$a + T \subseteq \{a_1 + a + b_1, a_2 + a + b_2, \ldots, a_k + a + b_k\} \subseteq \bigcup_{i=1}^{k} (a_i + B') \subseteq A'.$$

$\square$(Subclaim 2.12.4.3)

Claim 2.12.4.4: For all $a_1 < a_2$ in $A' \cap U$ such that $a_2 - a_1$ is in $\mathbb{N}$, i.e. a_1 and a_2 are finite units apart, we have $(t+1) \mid (a_2 - a_1)$, i.e. $a_2 - a_1$ is a multiple of $t+1$.

Proof of Claim 2.12.4.4: Suppose the claim is not true. Without loss of generality we can assume that $\{a_1, a_2\} + B' \subseteq A'$. By the same argument as in the proof of Claim 2.12.4.2, for $k \in \mathbb{N}$ and $(t+1)k > a_2 - a_1$, we can assume that $A' \cap U$ contains an *a.p.* of length $k+1$ with difference $t+1$. Let $T = \{a, a+(t+1), \ldots, a+k(t+1)\} \subseteq A' \cap U$. By Subclaim 2.12.4.2.1, we can assume $T + b \subseteq B'$ for some $b \in B'$. We now have $(a_1 + b + T) \cup (a_2 + b + T) \subseteq A'$. Note that $a_1 + b + a < a_2 + b + a < a_1 + b + a + k(t+1)$, i.e. the element $a_2 + b + a$ is between the first and last elements of the *a.p.* $a_1 + b + T$. Since $(t+1) \nmid (a_2 - a_1)$, then $a_2 + b + a \notin a_1 + b + T$. This gives two elements in $A' \cap U$ with the difference $< t+1$, which contradicts the minimality of t. $\square$(Claim 2.12.4.4)

From Claim 2.12.4.4 we can see that for all $a \in A'$, $(A' - a) \cap \mathbb{N}$ is a subset of $\{k(t+1) : k \in \mathbb{N}\}$.

Claim 2.12.4.5: For each $x < U$, the set $(A \cap U) \smallsetminus [0, x]$ is not a subset of an *a.p.* of difference $k > 1$.

Proof of Claim 2.12.4.5: Since $A \cap U$ is not a subset of an *a.p.* of difference > 1, then there is $a \in A \cap U$ such that $A \cap (a + A) \cap U$ is bounded in U. Hence if $(A'', B'') = e_a(A, A)$, then $B'' \cap U$ is bounded in U, which contradicts Claim 2.12.3. $\square$(Claim 2.12.4.5)

Claim 2.12.4.6: For each $x < U$, the set $(A \cap U) \smallsetminus [0, x]$ is not a subset of a U–unbounded *b.p.*.

Proof of Claim 2.12.4.6: Suppose the claim is not true. Let $(A \cap U) \smallsetminus [0, x] \subseteq I_1 \cup I_2$, where $I_1 \cap U = \{a + uk : u \in U\}$ and $I_2 \cap U = \{b + uk : u \in U\}$ with $a, b \in [0, k-1]$ and $\{2a, 2b, a+b\}$ are, modulo k, pairwise distinct. By Claim 2.12.4.5 we can assume $A \cap I_1$ and $A \cap I_2$ are unbounded in U. Since $A \cap U$ is not a subset of a U–unbounded *b.p.*, there is a $c \leqslant x$ in A such that $c \not\equiv a, b \pmod{k}$. For $i = 1, 2$ let $A_i = A \cap I_i$ and

$l_i = \min A_i$. Without loss of generality we assume $\{l_1, l_2, c\} + B' \subseteq A'$. Since $B' \subseteq A'$, then $B' \setminus [0, x] \subseteq I_1 \cup I_2$. Let $B'_i = B' \cap I_i$. Given sufficiently large $y \in U$ such that $\frac{A(0,y)}{y+1} \approx \alpha$, suppose we have $B'_1(0, y) \leqslant B'_2(0, y)$. Then we can find an $s \in \{l_1, l_2, c\}$ such that $(s + B'_2) \cap U \cap (I_1 \cup I_2) = \emptyset$. Hence

$$\frac{A'(0,y)}{y+1} \gtrapprox \frac{B'(0,y) + B'_2(0,y)}{y+1} \gtrapprox \frac{3B'(0,y)}{2(y+1)}.$$

This implies $\frac{B'(0,y)}{y+1} \ll \alpha$, which contradicts Claim 2.12.4.1 □(Claim 2.12.4.6)

In order to finish the proof of Case 2.12.4 we construct two internal increasing sequences of integers $\{x_j : j \in J\}$ and $\{y_j : j \in J\}$ in A for some interval $J = [0, K]$ such that the following are true:

(1) $x_j < y_j < x_{j+1} < y_{j+1}$,
(2) $A(y_j, x_{j+1}) = 2$,
(3) either $\frac{A(x_j,y_j)}{y_j - x_j + 1} \leqslant \epsilon$, where $\epsilon = \frac{\alpha}{400}$, or $\frac{(2A)(2x_j,2y_j)}{A(x_j,y_j)} \geqslant 3 + \frac{1}{100}$ for all $x_j, y_j \in U$,
(4) $\{x_j, y_j : j \in J\} \cap U$ is upper unbounded in U.

Claim 2.12.4.7: The existence of the sequences described above leads to a contradiction to the assumption that the key lemma is not true.

Proof of Claim 2.12.4.7: Let $y \in U$ be sufficiently large such that $y = y_{j_0}$. By Claim 2.12.2, we need only to show that $\frac{(2A)(0,2y)}{2y+1} \gg \frac{3A(0,y)}{2(y+1)}$.

In the sum $\sum_{j=0}^{j_0}$ below we write $\sum'$ for the sum over all j's such that $\frac{A(x_j,y_j)}{y_j - x_j + 1} \leqslant \epsilon$ and write $\sum''$ for the sum over the rest of j's.

Let $\beta = \frac{A(0,y)}{y+1}$. Then $\beta \geqslant \alpha$. Also

$$\beta \approx \frac{\sum' A(x_j, y_j) + \sum'' A(x_j, y_j)}{y+1} \leqslant \epsilon + \frac{\sum'' A(x_j, y_j)}{y+1}$$

implies $\frac{\sum'' A(x_j,y_j)}{y+1} \gtrapprox \beta - \epsilon$. Hence we have

$$\begin{aligned} \frac{(2A)(0,2y)}{2y+1} &\gtrapprox \frac{\sum_{j=0}^{j_0} (2A)(2x_j, 2y_j)}{2y+1} \\ &\gtrapprox \frac{(3 + \frac{1}{100}) \sum'' A(x_j, y_j)}{2y+1} \gtrapprox (3 + \frac{1}{100}) \frac{\beta - \epsilon}{2} \\ &\gtrapprox (3 + \frac{1}{100}) \frac{\beta - \frac{\alpha}{400}}{2} \geqslant \frac{301}{100} \cdot \frac{399}{800} \beta \gg \frac{3}{2} \beta \approx \frac{3A(0,y)}{2(y+1)}. \end{aligned}$$

□(Claim 2.12.4.7)

Now we construct the sequences x_j and y_j.

Let $x_0 = 0$. If we have found y_j in A, then $x_{j+1} > y_j$ is just the least element in A, which is greater than y_j. Suppose we have found x_j. We need to find y_j.

Let

$$z = \min \left\{ x \in A : \begin{array}{l} x > x_j \text{ and } A[x_j, x] \text{ is not a subset} \\ \text{of a } b.p. \text{ with the difference } > 1 \end{array} \right\}.$$

when z is well defined. Otherwise the process stops. By Claim 2.12.4.5 and Claim 2.12.4.6, if $x_j \in U$, then $z \in U$. Also by Claim 2.12.4.4 and the assumption that $t > 0$, we have that if $x_j \in U$, then $z - x_j$ must be a hyperfinite integer. If there exists a $z' \in A[x_j + 100, z]$ such that $\frac{A(x_j, z')}{z' - x_j + 1} < \epsilon$, then let y_j be the least such z'. Otherwise let $y_j = z$.

We now check that y_j is what we want. We need only to check when $y_j \in U$. If $\frac{A(x_j, y_j)}{y_j - x_j + 1} < \epsilon$, then we are done. Suppose $\frac{A(x_j, y_j)}{y_j - x_j + 1} \geqslant \epsilon$. Let $A[x_j, y_j - 1]$ be a subset of a $b.p.\, I_1 \cup I_2$ of difference k. Here we allow $I_2 \cap A[x_j, y_j - 1] = \emptyset$. For that case $A[x_j, y_j - 1]$ is a subset of an $a.p.$ of difference > 1. Let $A_i = A[x_j, y_j - 1] \cap I_i$, $l_i = \min A_i = \min I_i$, and $u_i = \max A_i = \max I_i$ for $i = 1, 2$ when they are well defined. If $k > 3$, then by the minimality of z we have $A_2 \neq \emptyset$. By Lemma 2.7 and Lemma 2.8 we have $\frac{(2A)(2x_j, 2y_j)}{A(x_j, y_j)} \geqslant 3 + \frac{1}{100}$.

Subcase 2.12.4.1 $k = 3$.

Again $A_2 \neq \emptyset$ by the minimality of z. Without loss of generality, we assume $l_1 \equiv 0 \pmod 3$, $l_2 \equiv 1 \pmod 3$, $y_j \equiv 2 \pmod 3$, and $l_1 < l_2$. Let $A' = A[x_j, y_j]$.

If $|A_2| < \frac{1}{4}|A'|$, then

$$\begin{aligned} \frac{(2A)(2x_j, 2y_j)}{|A'|} &\gtrapprox \frac{|2A_1| + |A_1 + A_2| + |y_j + A_1|}{|A'|} \\ &\gtrapprox \frac{4|A_1| + |A_2|}{|A'|} \approx 3 + \frac{|A_1| - 2|A_2|}{|A'|} \gtrapprox 3 + \frac{1}{4}. \end{aligned}$$

So we can assume $|A_2| \geqslant \frac{1}{4}|A'|$. By the same reason we can assume $|A_1| \geqslant \frac{1}{4}|A'|$. We now want to show that

$$\frac{(2A)(2x_j, 2y_j)}{|A'|} \gtrapprox 3 + \frac{1}{100}.$$

Suppose $\frac{(2A)(2x_j, 2y_j)}{|A'|} \ll 3 + \frac{1}{100}$. Then there exist non-negative integers b_1, b_2, and b_3 such that

$$|2A_1| + |2A_2| + |A_1 + A_2|$$

$$= (2|A_1| - 1 + b_1) + (2|A_2| - 1 + b_2) + (|A_1| + |A_2| - 1 + b_3)$$
$$\leqslant |2A'| < 3(|A_1| + |A_2|) + \frac{1}{100}|A'| - 3.$$

This implies $b_i \leqslant \frac{|A'|}{100} \leqslant \frac{|A_i|}{25}$ for $i = 1, 2$. By Lemma 1.6, A_i is a subset of an *a.p.* with the length at most $\frac{26}{25}|A_i|$, which means $\frac{u_i - l_i}{3} + 1 \leqslant \frac{26}{25}|A_i|$. Note that

$$|A_2| \geqslant \frac{1}{4}|A'| \geqslant \frac{1}{3}|A_1| \geqslant \frac{25}{78}(\frac{u_1 - l_1}{3} + 1).$$

If $l_1 < l_2 < u_2 < u_1$, then $u_1 - l_1$ is hyperfinite and

$$\begin{aligned} |2A_1| + |1 + 2A_2| &\geqslant 2|A_1| - 1 + 2|A_2| - 1 \\ &\geqslant \frac{50}{26}(\frac{u_1 - l_1}{3} + \frac{u_1 - l_1}{9}) - 2 \\ &\geqslant \frac{50}{26}(\frac{4(u_1 - l_1)}{9}) - 2 \geqslant \frac{200}{78}(\frac{u_1 - l_1}{3} + 1) - 5 \\ &> \frac{2u_1 - 2l_1}{3} + 1. \end{aligned}$$

Since $(2A_1) \cup (1 + 2A_2) \subseteq [2l_1, 2u_1] \cap \{3u : u \in U\}$, then $(2A_1) \cap (1 + (2A_2)) \neq \emptyset$. This implies that $(2A_1) \cup (2A_2)$ contains two consecutive numbers. Hence $2A$ contains two consecutive numbers, which contradicts $t > 0$ by Claim 2.12.4.3.

If $l_1 < u_1 < l_2 < u_2$, then $(y_j + A_2) \cap (2A_1) = \emptyset$. Hence

$$\frac{(2A)(2x_j, 2y_j)}{|A'|} \geqslant \frac{|2A_1| + |2A_2| + |A_1 + A_2| + |y_j + A_2|}{|A'|} \gtrapprox 3 + \frac{1}{4}.$$

Finally we assume $l_1 < l_2 < u_1 < u_2$. If $u_1 - l_2 \geqslant \frac{1}{20}(u_2 - l_1)$, then

$$\begin{aligned} |2A_1| + |1 + 2A_2| &\geqslant 2|A_1| - 1 + 2|A_2| - 1 \\ &\geqslant \frac{50}{26}(\frac{u_1 - l_1}{3} + \frac{u_2 - l_2}{3} + 2) - 2 \\ &\geqslant \frac{50}{26}(\frac{u_1 - l_2}{3} + \frac{u_2 - l_1}{3} + 2) - 2 \\ &\geqslant \frac{50}{26}(\frac{21(u_2 - l_1)}{60} + 2) - 2 \\ &> \frac{2u_2 - 2l_1}{3} + 2. \end{aligned}$$

The last inequality uses the fact that $u_2 - l_1$ is hyperfinite. Hence again $(2A_1) \cap (1 + (2A_2)) \neq \emptyset$. This implies $2A$ contains two consecutive numbers, which is again a contradiction.

Suppose $u_1 - l_2 < \frac{1}{20}(u_2 - l_1)$. Then

$$\begin{aligned} l_2 &> u_1 - \frac{1}{20}(u_2 - l_1) \\ &\geqslant u_1 - \frac{1}{20}(u_2 - l_2 + u_1 - l_1) \\ &\geqslant u_1 - \frac{1}{20}(3 \cdot \frac{26}{25}|A'|) \\ &\geqslant u_1 - (\frac{39}{250} \cdot 4|A_2|) \\ &\geqslant u_1 - \frac{78}{125}(\frac{u_2 - l_2}{3} + 1) \\ &\geqslant u_1 - \frac{1}{3}(u_2 - l_2). \end{aligned}$$

This shows $\frac{2}{3}(u_2 - l_2) < u_2 - u_1$, which implies $2u_1 < l_2 + u_2$. Hence $(y_j + A_2) \cap (2A_1) = \emptyset$. So again we have

$$\frac{(2A)(2x_j, 2y_j)}{A(x_j, y_j)} \gtrapprox 3 + \frac{1}{4}.$$

□(Subcase 2.12.4.1)

Subcase 2.12.4.2 $k = 2$.

Suppose $A_2 \neq \emptyset$. Without loss of generality, we assume $l_1 < l_2$ and A_1 is a set of even numbers. Since $2I_1$ and $2I_2$ are disjoint, then $u_1 < l_2$. Clearly y_j must be an even number.

If $|A_2| \geqslant \frac{1}{100}A(x_j, y_j) - 1$, then

$$\frac{(2A)(2x_j, 2y_j)}{A(x_j, y_j)} \gtrapprox \frac{|2A_1| + |2A_2| + |A_1 + A_2| + |y_j + A_2|}{A(x_j, y_j)} \gtrapprox 3 + \frac{1}{100}.$$

Hence we can assume $|A_2| < \frac{1}{100}A(x_j, y_j) - 1$, which implies $|A_1| > \frac{99}{100}A(x_j, y_j)$.

If $|2A_1| \geqslant 2|A_1| + \frac{1}{100}A(x_j, y_j)$, then $\frac{(2A)(2x_j, 2y_j)}{A(x_j, y_j)} \gtrapprox 3 + \frac{1}{100}$.

If $|2A_1| \leqslant 2|A_1| - 1 + \frac{1}{100}A(x_j, y_j)$, then by Lemma 1.6, A_1 is a subset of an *a.p.* of length at most $|A_1| + \frac{1}{100}A(x_j, y_j) < \frac{100}{99}|A_1|$.

If $\gcd(A_1 - l_1) > 2$, then $|A_1 + A_2| \geqslant 2|A_1|$. Hence $\frac{(2A)(2x_j, 2y_j)}{A(x_j, y_j)} \gtrapprox 3 + \frac{98}{100}$. So we can assume $\gcd(A_1 - l_1) = 2$. This implies $\frac{u_1 - l_1}{2} + 1 \leqslant \frac{100}{99}|A_1|$. We want to show that the distance between y_j and u_1 is large.

Let $v = \min\left\{x \in [l_1, u_1] : x \text{ is even and } \frac{A_1(x, u_1)}{u_1 - x + 2} \leqslant \frac{1}{4}\right\}$. Then

$$\frac{99}{100}(\frac{u_1 - l_1}{2} + 1) \leqslant |A_1|$$

$$
\begin{aligned}
&= A_1(l_1, v-2) + A_1(v, u_1) \\
&\leqslant \frac{v-l_1}{2} + \frac{u_1-v+2}{4} \\
&= \frac{u_1-l_1}{2} + 1 - \frac{u_1-v+2}{4}.
\end{aligned}
$$

Hence $u_1 - v + 2 \leqslant \frac{1}{25}(\frac{u_1-l_1}{2}+1)$ and $v \geqslant \frac{49u_1+l_1+98}{50}$. By the pigeonhole principle and by the definition of v, all even numbers in $[l_1+u_1, v+u_1-2]$ are in $2A_1$.

If $l_2 < u_1+v-l_1$, then $2A$ contains two consecutive numbers l_1+l_2 and l_1+l_2-1, which contradicts $t>0$ by Claim 2.12.4.3. So we can assume that $l_2 \geqslant u_1+v-l_1$. Hence $y_j > u_1+v-l_1$. This shows $y_j - u_1 > v - l_1$. So

$$
\begin{aligned}
A_1(2u_1-y_j, u_1) &\geqslant A_1(u_1-v+l_1, u_1) \\
&\geqslant A_1(l_1 + [\frac{1}{25}(\frac{u_1-l_1}{2}+1)-2], u_1) \\
&\geqslant |A_1| - \frac{1}{50}(\frac{u_1-l_1}{2}+1) \\
&\geqslant \frac{99}{100}(\frac{u_1-l_1}{2}+1) - \frac{1}{50}(\frac{u_1-l_1}{2}+1) \\
&= \frac{97}{100}(\frac{u_1-l_1}{2}+1) \\
&\geqslant \frac{97}{100}|A_1| \geqslant \frac{97\times 99}{100\times 100}A(x_j,y_j) \\
&> \frac{95}{100}A(x_j,y_j).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\frac{(2A)(2x_j,2y_j)}{A(x_j,y_j)} \\
&\quad \gtrapprox \frac{|2A_1| + |A_1+A_2| + |y_j + A_1[2u_1-y_j, u_1]|}{A(x_j,y_j)} \\
&\quad \gtrapprox 3 + \frac{\frac{95}{100}A(x_j,y_j) - 2|A_2|}{A(x_j,y_j)} \gtrapprox 3 + \frac{93}{100}.
\end{aligned}
$$

Suppose $A_2 = \emptyset$ and again assume A_1 is a set of even numbers. Then y_j is an odd number and $l_1 + y_j < 2u_1$ because otherwise $A[x_j, y_j]$ is a subset of a *b.p.* of difference 2, which contradicts the minimality of z. If $|2A_1| \geqslant 2|A_1| + \frac{1}{100}|A_1|$, then

$$
\frac{(2A)(2x_j,2y_j)}{A(x_j,y_j)} \gtrapprox \frac{3|A_1| + \frac{1}{100}|A_1|}{A(x_j,y_j)} \gtrapprox 3 + \frac{1}{100}.
$$

So we can assume $|2A_1| < 2|A_1| + \frac{1}{100}|A_1|$. By Lemma 1.6 we have

$$\frac{u_1 - l_1}{2} + 1 \leqslant |A_1| + \frac{1}{100}|A_1| = \frac{101}{100}|A_1|.$$

Let

$$v = \max\left\{x \in [l_1, u_1] : x \text{ is an even number and } \frac{A_1(l_1, x)}{x - l_1 + 2} \leqslant \frac{1}{4}\right\}$$

and

$$v' = \min\left\{x \in [l_1, u_1] : x \text{ is an even number and } \frac{A_1(x, u_1)}{u_1 - x + 2} \leqslant \frac{1}{4}\right\}.$$

Then $v - l_1 \leqslant \frac{4}{101}(\frac{u_1 - l_1}{2} + 1)$ and $u_1 - v' \leqslant \frac{4}{101}(\frac{u_1 - l_1}{2} + 1)$. Hence $v' - v$ is hyperfinite. By the pigeonhole principle we have $\{x \in [l_1 + v + 2, u_1 + v' - 2] : x \text{ is even }\} \subseteq 2A_1$. Hence all even numbers in $[2l_1 + v + 2, 2u_1 + v' - 2]$ are elements of $(3A_1) = A_1 + A_1 + A_1$. Since all odd numbers in $y_j + [l_1 + v + 2, u_1 + v' - 2]$ are elements of $(3A)$ and since $2l_1 + v + 2 < y_j + u_1 + v' - 2$ and $y_j + l_1 + v + 2 < 2u_1 + v' - 2$, then there are $x_1, x_2, x_3, z_1, z_2 \in A_1$ such that

$$x_1 + x_2 + x_3 = z_1 + z_2 + y_j + 1.$$

Next we want to show that there is sequence E of e–transforms with $(A'', B'') = E(A, A)$ such that $A'' \cap U$ contains two consecutive numbers, which contradicts the assumption $t > 0$.

Let $R = \{x_1 + x_2, x_3, z_1 + z_2, y_j\}$. Then $R \subseteq 2A$. By Claim 2.12.4.3 there is $c_1 \in A \cap U$ and a sequence E_1 of e–transforms such that $(A_1, B_1) = E_1(A', B')$ and $c_1 + R \subseteq A_1$. By Subclaim 2.12.4.2.1 there is $c_2 \in B_1 \cap U$ such that $c_2 + c_1 + R \subseteq B_1$. Hence $(c_2 + c_1 + R) + (c_2 + c_1 + R) \subseteq 2B_1 \subseteq A_1 + B_1 \subseteq 2A$. Again by Claim 2.12.4.3 there is $c_3 \in A \cap U$ and a sequence E_2 of e–transforms with $(A'', B'') = E_2(A_1, B_1)$ such that

$$c_3 + (c_2 + c_1 + R) + (c_2 + c_1 + R) \subseteq A''.$$

This shows that A'' contains two consecutive numbers $c_3 + 2c_2 + 2c_1 + x_1 + x_2 + x_3$ and $c_3 + 2c_2 + 2c_1 + z_1 + z_2 + y_j$. This ends the proof of Case 2.12.4 for $|F| = 1$. Hence the lemma is proven. $\square$(Lemma 2.12)

3 Proofs of the Main Theorem

In order to use nonstandard techniques, we first translate Theorem 1.3 into nonstandard forms. We translate the part II of Theorem 1.3 into Theorem 3.1 and the part

III of Theorem 1.3 into Theorem 3.2. Then we prove Theorem 3.1 and Theorem 3.2.

Theorem 3.1 *Let $A \subseteq [0, H]$, $0, H \in A$, and $0 < \alpha < \frac{1}{2}$ be such that*

(1) $\gcd(A[0, x]) = 1$ *for every hyperfinite x,*

(2) $A(0, H) \sim \alpha H$,

(3) for every hyperfinite $x \leqslant H$, $A(0, x) \preceq \alpha x$,

(4) for every hyperfinite $x \leqslant H$, $(2A)(0, 2x) \preceq 3\alpha x$.

Then

(a) either there are $a, d \in \mathbb{N}$ such that $\alpha = \frac{2}{d}$ and

$$A \subseteq \left\{ dn : 0 \leqslant n \leqslant \frac{H}{d} \right\} \cup \left\{ a + dn : 0 \leqslant n \leqslant \frac{H}{d} \right\}$$

(b) or there are $0 \leqslant c \leqslant b \leqslant H$ such that $c \sim 0$, $[c + 1, b - 1] \cap A = \emptyset$, and $A(b, H) \sim H - b$.

Proof of Part II of Theorem 1.3 from Theorem 3.1: Let $A \subseteq \mathbb{N}$ be such that $0 \in A$ and $\gcd(A) = 1$. Let $0 < \alpha < \frac{1}{2}$ be such that $\bar{d}(A) = \alpha$ and $\bar{d}(2A) = \frac{3}{2}\alpha$. Take an increasing sequence $h_n \in \mathbb{N}$ such that $\lim_{n\to\infty} \frac{A(h_n)}{h_n} = \alpha$. Without loss of generality we can assume $h_n \in A$ for every $n \in \mathbb{N}$. Let's fix an arbitrary hyperfinite integer K. Let $H = h_K$ and let $B = {}^*\!A \cap [0, H]$. We now check that (1)–(4) of Theorem 3.1 are true for B in the place of A there. Clearly $0, H \in B$.

(1) is trivially true because $A \subseteq B[0, x]$ for every hyperfinite x. (2) is true because $\frac{A(h_n)}{h_n} \to \alpha$ implies $\frac{B(0,h_K)}{h_K} \approx \alpha$. (3) is true because otherwise $\bar{d}(A) > \alpha$ by Lemma 2.1. (4) is true because otherwise $\bar{d}(2A) > \frac{3}{2}\alpha$ by Lemma 2.1.

Now for every hyperfinite integer K either (a) or (b) of Theorem 3.1 is true with $H = h_K$ and A replaced by $B = {}^*\!A \cap [0, H]$. If there is a hyperfinite integer K such that (a) of Theorem 3.1 is true for $H = h_K$ and $B = {}^*\!A \cap [0, h_K]$, then there are $a, d \in \mathbb{N}$ such that $\alpha = \frac{2}{d}$ and

$$A \subseteq B \subseteq \left\{ dn : 0 \leqslant n \leqslant \frac{H}{d} \right\} \cup \left\{ a + dn : 0 \leqslant n \leqslant \frac{H}{d} \right\}.$$

So clearly (a) of Part II of Theorem 1.3 is true for A. Otherwise for every hyperfinite integer K, (b) of Theorem 3.1 is true for $H = h_K$ and $B = {}^*\!A \cap [0, H]$. Given a standard positive integer k and let X_k be the set of all non-negative integers n such that there are $0 \leqslant c \leqslant b \leqslant h_n$ satisfying $\frac{c}{h_n} < \frac{1}{k}$, $\frac{B(b,h_n)}{h_n - b} > 1 - \frac{1}{k}$, and $[c+1, b-1] \cap B =$

$\emptyset$, where $B = {}^*A \cap [0, h_n]$. Then X_k is internal and contains every hyperfinite integer. Hence there is a standard positive integer m_k such that X_k contains all positive integers greater than or equal to m_k. Without loss of generality we can assume that $m_1 < m_2 < \cdots$. For each $k = 1, 2, \ldots$ and for each $n \in [m_k, m_{k+1} - 1]$ choose $c_n = c$ and $b_n = b$ so that $0 \leqslant c \leqslant b \leqslant h_n$, $\frac{c}{h_n} > \frac{1}{k}$, $\frac{A(b,h_n)}{h_n - b} > 1 - \frac{1}{k}$, and $[c+1, b-1] \cap A = \emptyset$. Choose $c_n = b_n = 0$ for every $n \in [0, m_1 - 1]$. It is easy to see that $\lim_{n\to\infty} \frac{c_n}{h_n} = 0$, $[c_n + 1, b_n - 1] \cap A = \emptyset$, and $\lim_{n\to\infty} \frac{A(b_n, h_n)}{h_n - b_n + 1} = 1$. □(Part II of Theorem 1.3)

Theorem 3.2 *Let $A \subseteq [0, H]$ and $0, H \in A$ be such that*

(1) $\gcd(A[0, x]) = 1$ *for every hyperfinite x,*

(2) $A(0, H) \sim \frac{1}{2}H$,

(3) for every hyperfinite $x \leqslant H$, $A(0, x) \preceq \frac{1}{2}x$,

(4) for every hyperfinite $x \leqslant H$, $(2A)(0, 2x) \preceq \frac{3}{2}x$.

Then

(a) either there is an $a \in \{0, 3\}$ such that

$$A \subseteq \left\{4n : 0 \leqslant n \leqslant \frac{H}{4}\right\} \cup \left\{a + 4n : 0 \leqslant n \leqslant \frac{H}{4}\right\}$$

(b) or $(2A)(0, H) \sim A(0, H)$.

Proof of Part III of Theorem 1.3 from Theorem 3.2: Let $A \subseteq \mathbb{N}$ be such that $0 \in A$ and $\gcd(A) = 1$ such that $\bar{d}(A) = \frac{1}{2}$ and $\bar{d}(2A) = \frac{3}{4}$. Take an increasing sequence $h_n \in \mathbb{N}$ such that $\lim_{n\to\infty} \frac{A(h_n)}{h_n} = \frac{1}{2}$. Again we can assume $h_n \in A$ for every $n \in \mathbb{N}$. Let's fix an arbitrary hyperfinite integer K. Let $H = h_K$ and let $B = {}^*A \cap [0, H]$. By the same idea as in the proof above, we have $0, H \in B$ and can check that (1)–(4) of Theorem 3.2 are true for B in the place of A.

Now it is easy to see that (a) of Theorem 3.2 for B implies (a) of Part III of Theorem 1.3 for A and (b) of Theorem 3.2 for B implies (b) of part III of Theorem 1.3 for A. □(Part III of Theorem 1.3)

Now we are ready to prove Theorem 3.1 and Theorem 3.2.

Proof of Theorem 3.1: Note that (4) of Theorem 3.1 implies $|2A| \sim 3|A|$. Suppose A is a subset of a *b.p.* $I_0 \cup I_1$ of difference $d > 1$. Let $A_i = A \cap I_i$, $l_i = \min A_i$, and $u_i = \max A_i$ for $i = 0, 1$. Without loss of generality we assume $0 = l_0$. By (1) of Theorem 3.1 we have $l_1 \in \mathbb{N}$. Since

$$|2A| \sim |2A_0| + |2A_1| + |A_0 + A_1| \succeq 2|A_0| + 2|A_1| + |A_0| + |A_1| \sim 3|A|,$$

then $|2A_i| \sim 2|A_i|$. Hence by Lemma 1.6 we have $|A_i| \sim I_i(l_i, u_i)$.

Suppose $d = 2$. Then $u_0 < l_1$. Hence

$$|A| \sim |A_1| \sim \frac{1}{2}(u_1 - l_1) \sim \frac{1}{2}H,$$

which contradicts $\alpha < \frac{1}{2}$.

Suppose $d = 3$. If $u_0 \sim 0$, then by the argument above we have $\alpha = \frac{1}{3}$. Now

$$(2A)(0, H) \sim |0 + A_1| + |l_1 + A_1| \sim 2|A_1| \sim \frac{2}{3}H$$

contradicts (4) of Theorem 3.1 for $x = [\frac{H}{2}]$. By the same idea we can show that $u_1 \sim 0$ is also impossible. Let $u = \min\{u_0, u_1\}$. Then $A(0, u) \sim A_0(0, u) + A_1(0, u) \sim \frac{2}{3}u$, which contradicts (3) of Theorem 3.1.

Suppose $d \geqslant 4$. If $u_0 \sim 0$, then $\alpha \leqslant \frac{1}{d}$. This implies

$$\frac{2}{d}H \sim (2A)(0, H) \preceq \frac{3}{2}\alpha H.$$

Hence $\frac{1}{d} \geqslant \alpha \geqslant \frac{4}{3d} > \frac{1}{d}$, which is absurd. If $0 \prec u_0 \prec H$, then $u_1 = H$ and

$$\frac{1}{d}(u_0 + H) \sim |A_0| + |A_1| = |A| \sim \alpha H$$

implies $\alpha < \frac{2}{d}$. Now we have $A(0, u_0) \sim \frac{2}{d}u_0 \succ \alpha u_0$, which contradicts (3) of Theorem 3.1. By symmetry, $u_1 \prec H$ is impossible. Hence $u_0 \sim u_1 \sim H$, which implies $\alpha = \frac{2}{d}$. Hence (a) of Theorem 3.1 is true.

From the arguments above we can assume that A is not a subset of a *b.p.* of difference > 1.

For notational convenience we deal with the set $B = H - A$ in the rest of the proof.

Case 3.1.1: $\underline{d}_U(B) < \alpha$.

Then there is an $x \succ 0$ such that $B(0, x) \prec \alpha x$. This implies $A(0, H - x) \succ \alpha(H - x)$, which contradicts (3) of Theorem 3.1. □(Case 3.1.1)

Case 3.1.2: $\underline{d}_U(B) > \frac{1}{2}$.

Let $0 \prec b \prec H$ be such that $B(0, b) \sim \frac{1}{2}b$ and for every $0 \prec x \prec b$, $B(0, x) \succ \frac{1}{2}x$. Then $(2B)(0, b) \sim b$.

Subcase 3.1.2.1 $B(b, H) \sim 0$.

Since

$$(2B)(0,2H) \succeq (2B)(0,b) + (2B)(b,H) + (2B)(H,H+b) \sim 3|B| + (2B)(b,H),$$

then $(2B)(b,H) \sim 0$. By Lemma 2.5 we have $\max B[0,b] = \bar{b} \sim \frac{1}{2}b$. If $c = \min B[b,H] \sim H$, then (b) of Theorem 3.1 is true. If $c \prec H$, then

$$|2B| \succeq b + |c + B[0,H-c]| + |H + B[0,b]| \sim 3|B| + B(0,H-c) \succ 3|B|,$$

which contradicts (4) of Theorem 3.1. □(Subcase 3.1.2.1)

Subcase 3.1.2.2 $B(b,H) \succ 0$.

Let $c \in B[b,H]$ be such that $B(b,c) \sim 0$ and for every $c \prec x \leqslant H$, $B(c,x) \succ 0$. By (1) and (3) of Theorem 3.1 and Lemma 1.5 we have $(2B)(2c,2H) \succeq 3B(c,H)$. Choose an $x \in B$ with $c \prec x \prec c+b$. Then

$$\begin{aligned}
|2B| \succeq\ & (2B)(0,c) + |c + B[0,x-c]| \\
& + |x + B[0,2c-x]| + (2B)(2c,2H) \\
\succeq\ & 2B(0,c) + B(0,x-c) + B(0,2c-x) + 3B(c,H) \\
\sim\ & 3|B| + B(0,x-c) - B(2c-x,c).
\end{aligned}$$

Note that $B(0,x-c) \succ \frac{1}{2}(x-c)$ and

$$B(2c-x,c) \sim B(2c-x,b) \preceq B(b+c-x,b) \prec \frac{1}{2}(x-c).$$

Then we have $|2B| \succ 3|B|$, a contradiction. □(Case 3.1.2)

Case 3.1.3: $\alpha \leqslant \underline{d}_U(B) \leqslant \frac{1}{2}$.

We divide the proof into three subcases according the arithmetic structure of $B \cap U$.

Subcase 3.1.3.1: $B \cap U$ is neither a subset of an *a.p.* of difference > 1 nor a subset of a U–unbounded *b.p.*.

By Lemma 2.12 there is a $y \in B$ with $0 \prec y \prec H$ such that $(2B)(0,2y) \succ 3B(0,y)$. By (1) and (3) of Theorem 3.1 we have $B(y,H) \preceq \alpha(H-y)$ and $\gcd(B[y,H]-y) = 1$. By Lemma 2.4 we have $|2B| \succ 3|B|$. □(Subcase 3.1.3.1)

Subcase 3.1.3.2 $B \cap U$ is a subset of an *a.p.* of difference $d > 1$.

Let $a = \min\{x \in B : \gcd(B[0,x]) = 1\}$ and $c = \max B[0, a-1]$. Since $0 \in B$, then $a > U$. If $c \in U$, then $B[c+1, a-1] = \emptyset$, which implies $\frac{B(0,a)}{a+1} \approx 0$. Hence $\frac{B(a,H)}{H-a+1} \gg \alpha$, which contradicts (3) of Theorem 3.1. So we have $c > U$ or equivalently $c \succ 0$. Let $d = \gcd(B[0,c])$. Note that $d > 1$ is a standard integer because otherwise $\frac{B(0,c)}{c+1} \lessapprox \frac{1}{d} \approx 0$, which implies $\frac{B(c,H)}{H-c+1} \gg \alpha$. The proof is easy if $\frac{1}{d} < \frac{3}{2}\alpha$ because $(2B)(0,c) \preceq \frac{1}{d}c \prec \frac{3}{2}\alpha c$ implies $(2B)(c, 2H) \succ \frac{3}{2}\alpha(2H - c)$, which contradicts (4) of Theorem 3.1. So we assume $\frac{1}{d} \geqslant \frac{3}{2}\alpha$. The proof of this case is much more tedious than the other cases.

Firstly, we can assume that $a \prec H$ by the following reason: Suppose $a \sim H$. Then

$$3|B| \sim |2B| \succeq |B[0,c] + B[0,c]| + |a + B[0,c]| \sim 3|B|$$

implies $B(0,c) \sim \frac{1}{d}c$ by Lemma 1.6. If $c \sim H$, then $|B| \sim \frac{1}{d}H \succeq \frac{3}{2}\alpha H$, which contradicts (3) of Theorem 3.1. If $c \prec H$, then by (1) of Theorem 3.1 there exists a $b \sim H$ such that $b \not\equiv a \pmod{d}$. Hence

$$\begin{aligned}
|2B| &\succeq (2B)(0,H) + |\{a,b\} + B| \\
&\sim B(0,c) + |c + B[0, H-c]| + 2|B| \\
&\sim 3|B| + B(0, H-c) \succ 3|B|.
\end{aligned}$$

Secondly, we can assume that $B(a,H) \succ 0$ by the following reason: Suppose $B(a,H) \sim 0$. If $|B[0,c] + B[0,c]| \succ 2B(0,c)$, then

$$|2B| \succeq |B[0,c] + B[0,c]| + |a + B[0,c]| \succ 3|B|.$$

So we can assume $|B[0,c] + B[0,c]| \sim 2B(0,c)$. By Lemma 1.6 we have $B(0,c) \sim \frac{1}{d}c$. This implies $B(a+c-H, c) \succ 0$. Hence we have

$$\begin{aligned}
|2B| &\succeq |B[0,c] + B[0,c]| \\
&\quad + |a + B[0,c]| + |H + B[a+c-H, c]| \\
&\succeq 3B(0,c) + B(a+c-H, c) \\
&\sim 3|B| + B(a+c-H, c) \succ 3|B|.
\end{aligned}$$

Now we are ready to prove the subcase. The proof of Subcase 3.1.3.2 is divided into three subsubcases.

Subsubcase 3.1.3.2.1 $d = 2$.

Since we have $|B[a,H]+B[a,H]| \succeq 3B(a,H)$, then

$$3|B| \sim |2B| \succeq (2B)(0,2c) + |a+B[0,c]| + (2B)(2a,2H) \succeq 3|B|$$

implies that $B(0,c) \sim \frac{1}{2}c$ by Lemma 1.6. Without loss of generality we can assume $c, c-2, c-4 \in B$. Suppose $c \prec a$. Choose an $x \succ a$ in B such that either $B(a,x) \sim 0$ or $x-a < a-c$. Then we have

$$\begin{aligned} |2B| &\succeq 2B(0,c) + |a+B[0,c]| \\ &\quad +|x+B[a+c-x,c]| + |B[a,H]+B[a,H]| \\ &\succeq 3B(0,c) + 3B(a,H) + B(a+c-x,c) \succ 3|B|. \end{aligned}$$

Hence we can assume $c \sim a$. Recall that we have $B(0,c) \succ 0$, $B(a,H) \succ 0$, $\gcd(B[0,c]) = 2$, $\gcd(B[a,H]-a) = 1$, and $\frac{B(a,H)}{H-a+1} \lessapprox \alpha$.

Subsubsubcase 3.1.3.2.1.1 $\underline{d}_{a+U}(B) = 0$.

Choose an $x \in B$ with $x \succ a$ such that $B(a,x) < \frac{1}{8}(x-a+1)$. Then $|B[x,H]+B[x,H]| \succeq 3B(x,H)$ by (3) of Theorem 3.1 and Lemma 1.5. Hence

$$\begin{aligned} |2B| &\succeq |B[0,c]+B[0,c]| + |a+B[0,c]| \\ &\quad +|x+B[a+c-x,c]| + |B[x,H]+B[x,H]| \\ &\succeq 3B(0,c) + \frac{1}{2}(x-a+1) + 3B(x,H) \\ &\succeq 3|B| + \frac{1}{2}(x-a+1) - 3B(a,x) \succ 3|B|. \end{aligned}$$

□(Subsubsubcase 3.1.3.2.1.1)

Subsubsubcase 3.1.3.2.1.2 $\underline{d}_{a+U}(B) > \frac{1}{2}$.

By the same proof as in Case 3.1.2 we have that either $(2B)(2a,2H) \succ 3B(a,H)$ or there are $\bar{b} < c'$ in $B[a,H]$ such that $B[a,H] \subseteq [a,\bar{b}] \cup [c',H]$, $\bar{b}-a \sim B(a,H)$, and $c' \sim H$. The first possibility above implies $|2B| \succ 3|B|$ by Lemma 2.4 and the second also implies $|2B| \succ 3|B|$ because $2\bar{b} \prec a+H$ and

$$\begin{aligned} |2B| &\succeq (2B)(0,2a) + (2B)(2a,2\bar{b}) \\ &\quad +(2B)(2\bar{b},a+H) + |H+B[a,H]| \\ &\succeq |B[0,c]+B[0,c]| + |a+B[0,c]| \\ &\quad +2B(a,\bar{b}) + |H+B[2\bar{b}-H,a]| + B(a,H) \\ &\succeq 3B(0,c) + 3B(a,H) + B(2\bar{b}-H,a) \\ &= 3|B| + B(2\bar{b}-H,a) \succ 3|B|. \end{aligned}$$

The last inequality holds because $c \sim a \succ 2\bar{b} - H$ and $B(0,c) \sim \frac{1}{2}c$. $\Box$(Subsubsubcase 3.1.3.2.1.2)

Subsubsubcase 3.1.3.2.1.3 $0 < \underline{d}_{a+U}(B) \leqslant \frac{1}{2}$.

Since $c \sim a$, then $c-4, c-2, c, a \in (c-4+U)$ and $\underline{d}_{c-4+U}(B) = \underline{d}_{a+U}(B)$. Hence $\gcd((B[c-4,H]-c+4) \cap U) = 1$ and $(B[c-4,H]-c+4) \cap U$ is not a subset of a U–unbounded *b.p.* of difference $d > 1$. By Lemma 2.12 there exists a $y \succ a$ in B such that $(2B)(2(c-4), 2y) \succ 3B(c-4,y)$. Note that $B(y,H) \prec \frac{1}{2}(H-y)$ and $\gcd(B[y,H]-y) = 1$ when $H-y \notin \mathbb{N}$. Hence by Lemma 2.4 $|B[c-4,H]+B[c-4,H]| \succ 3B(c-4,H)$, which implies $|2B| \succ 3|B|$. This ends the proof of Subsubcase 3.1.3.2.1. $\Box$(Subsubcase 3.1.3.2.1)

Subsubcase 3.1.3.2.2 $d = 3$.

By the same reasons as in Subsubcase 3.1.3.2.1 we can assume that $B(0,c) \sim \frac{1}{3}c$ and $c \sim a$. Since B is not a subset of a *b.p.* of difference > 1, we can define

$$b = \min\left\{x \in B : x \notin \{0,a\} \pmod 3\right\}.$$

Let $B_0 = B \cap \{3n : n \in {}^*\mathbb{N}\}$, $B_a = B \cap \{a+3n : n \in {}^*\mathbb{N}\}$, and $B_b = B \cap \{b+3n : n \in {}^*\mathbb{N}\}$. Let l_0, l_a, l_b be the least element of B_0, B_a, B_b, respectively. Let u_0, u_a, u_b be the largest element of B_0, B_a, B_b, respectively.

Subsubsubcase 3.1.3.2.2.1 $b \sim H$.

We have $|B| \sim |B_0| + |B_a|$. We can also assume $|B_a| \succ 0$ because otherwise

$$|2B| \succeq |2B_0| + |a + B_0| + |b + B_0| = 4|B_0| = 4|B|.$$

Since $B_0 \cup B_a$ is a subset of a *b.p.*, then $B_0(l_0, u_0) \sim \frac{1}{3}(u_0 - l_0)$ and $B(l_a, u_a) \sim \frac{1}{3}(u_a - l_a)$. This implies $u_a \prec H$ or $u_0 \prec H$ because otherwise $B(a,H) \sim \frac{2}{3}(H-a)$, which contradicts (3) of Theorem 3.1. Suppose $u_a \prec H$ and $u_a \leqslant u_0$. Then

$$\begin{aligned}|2B| &\succeq |2B_0| + |2B_a| \\ &\quad + |B_0 + B_a| + |b + B_0[u_a + u_0 - b, u_0]| \\ &\succeq 3|B| + B_0(u_a + u_0 - b, u_0) \succ 3|B|.\end{aligned}$$

By the same reason, if $u_0 \prec H$ and $u_0 \leqslant u_a$, then $|2B| \succ 3|B|$. Note that if both $u_0 \prec H$ and $u_a \prec H$ are true, then either $u_0 \leqslant u_a$ or $u_a \leqslant u_0$. $\Box$(Subsubsubcase 3.1.3.2.2.1)

Subsubsubcase 3.1.3.2.2.2 $b \prec H$.

Note that we have $\gcd(B[b, H] - b) = 1$ and $B(b, H) \preceq \alpha(H - b)$. Let us redefine $u_0 = \max(B_0[0, b-1])$ and $u_a = \max(B_a[0, b-1])$. Then by Lemma 1.6 we have $B_0(l_0, u_0) \sim \frac{1}{3}(u_0 - l_0)$ and $B_a(l_a, u_a) \sim \frac{1}{3}(u_a - l_a)$. By the same reason as in Subsubsubcase 3.1.3.2.2.1 we can assume $u_0, u_a \sim b$ and $B_a(l_a, u_a) \succ 0$.

If $\underline{d}_{b+U}(B) = 0$, then there is an $x \in B$, $x \succ b$ such that either $x - b < u_0$ and $B(b, x) \preceq \frac{1}{10}(x - b)$, or $B(b, x) \sim 0$. Hence

$$\begin{aligned}
|2B| &\succeq |B_0[l_0, u_0] + B_0[l_0, u_0]| \\
&\quad + |B_a[l_a, u_a] + B_a[l_a, u_a]| + |B_0[l_0, u_0] + B_a[l_a, u_a]| \\
&\quad + |B[x, H] + B[x, H]| + |x + B_0[2u_0 - x, u_0]| \\
&\succeq 3B(0, b) + 3B(x, H) + B_0(2u_0 - x, u_0) \\
&\succeq 3B(0, b) + 3B(x, H) + \frac{1}{3}(x - u_0) \\
&\succeq 3|B| + \frac{1}{3}(x - u_0) - \frac{3}{10}(x - b) \succ 3|B|.
\end{aligned}$$

If $\underline{d}_{b+U}(B) > \frac{1}{2}$, then by the same reason as in the proof of Case 3.1.2 we have that either $(2B)(2b, 2H) \succ 3B(b, H)$, which implies $|2B| \succ 3|B|$, or $B[b, H] \subseteq [b, \bar{b}] \cup [c', H]$, where $\bar{b} - b \sim B(b, H)$ and $c' \sim H$. The latter again implies, by the same argument as in the proof of Subsubsubcase 3.1.3.2.1.2, $|2B| \succ 3|B|$.

Now we can assume $0 < \underline{d}_{b+U}(B) \leqslant \frac{1}{2}$. Let $u = \min\{u_0, u_a\}$. Then $u \sim b$. Since $B_0(0, u_0) \sim \frac{1}{3}(u_0 - l_0)$ and $B_a(l_a, u_a) \sim \frac{1}{3}(u_a - l_a)$, we can assume $u - 3 \in B$. Hence $(B - (u - 3)) \cap U$ is neither a subset of an *a.p.* of difference > 1 nor a subset of a U–unbounded *b.p.* of difference > 1. By Lemma 2.12 there exists a $y \in B$ with $y \succ b$ such that $(2B)(2(u - 3), 2y) \succ 3B(u - 3, y)$. Finally by Lemma 2.4, $|2B| \succ 3|B|$. $\square$(Subsubcase 3.1.3.2.2)

Subsubcase 3.1.3.2.3 $d \geqslant 4$.

Since B is not a subset of a *b.p.* of difference > 1, the number

$$b = \min\{x \in B : x \notin \{0, a\} \pmod{d}\}$$

is well defined. If $b \equiv 2a \pmod{d}$, then $a + b \not\equiv 0 \pmod{d}$ because otherwise $B[0, b]$ is a subset of an *a.p.* with difference $\frac{d}{3} > 1$. Hence we have either $b \not\equiv 2a \pmod{d}$ or $a + b \not\equiv 0 \pmod{d}$. This implies

$$(b + B_0[0, u_0]) \cap (B[0, b-1] + B[0, b-1]) = \emptyset$$

or

$$(b + B_a[l_a, u_a]) \cap (B[0, b-1] + B[0, b-1]) = \emptyset.$$

Again it is easy to check that $u_a \sim b$, $u_0 \sim b$, $B_a(l_a, u_a) \succ 0$, $B_0(0, u_0) \succ 0$, $B_0(0, u_0) \sim \frac{1}{d}u_0$ and $B_a(l_a, u_a) \sim \frac{1}{d}(u_a - l_a)$. Hence

$$\begin{aligned} |2B| &\succeq |B_0[0, u_0] + B_0[0, u_0]| + |B_a[l_a, u_a] + B_a[l_a, u_a]| \\ &\quad + |B_0[0, u_0] + B_a[l_a, u_a]| + |B[b, H] + B[b, H]| \\ &\quad + \min\{|b + B_0[0, u_0]|, |b + B_a[l_a, u_a]|\} \\ &\succeq 3B(0, b) + 3B(b, H) + \min\{B_0(0, u_0), B_a(l_a, u_a)\} \succ 3|B|. \end{aligned}$$

This ends the proof of Subcase 3.1.3.2. $\square$(Subcase 3.1.3.2)

Subcase 3.1.3.3 $B \cap U$ is a subset of a U–unbounded *b.p.* of difference d.

Let $b = \min\{x \in [0, H] : B[0, x]$ is not a subset of a *b.p.* of difference $d.\}$ and let $B[0, b-1]$ be a subset of a *b.p.* $I_1 \cup I_2$ of difference d. Let $B_i = B[0, b-1] \cap I_i$, $l_i = \min B_i$, and $u_i = \max B_i$. Note that $B(0, b-1) \succeq \alpha b$ by (3) of Theorem 3.1. By Subcase 3.1.3.2 we can assume $\gcd(B[0, b-1]) = 1$. If $d > 3$, then the subcase follows from Lemma 2.7, Lemma 2.8, and Lemma 2.4. Suppose $d = 3$. Since $|2B| \sim 3|B|$, then $|B[0, b-1] + B[0, b-1]| \sim 3B(0, b-1)$ implies $B_i(l_i, , u_i) \sim \frac{1}{3}(u_i - l_i)$ by Lemma 1.6. Since $B \cap U$ is already a subset of a U–unbounded *b.p.* of difference 3, then $l_1, l_2 \in U$ and $u_1, u_2 \notin U$. Hence $\underline{d}_U(B) = \frac{2}{3}$, which contradicts the assumption $\underline{d}_U(B) \leqslant \frac{1}{2}$ for Case 3.1.3. This completes the proof of Case 3.1.3 as well as the proof of Theorem 3.1. $\square$(Theorem 3.1)

Proof of Theorem 3.2: Suppose A is a subset of a *b.p.* $I_0 \cup I_1$ of difference $d > 1$. Let $A_i = A \cap I_i$, $l_i = \min A_i$, and $u_i = \max A_i$ for $i = 0, 1$. Without loss of generality we assume $0 = l_0$. By (1) of Theorem 3.2 we have $l_1 \in \mathbb{N}$. Since

$$|2A| \sim |2A_0| + |2A_1| + |A_0 + A_1| \succeq 2|A_0| + 2|A_1| + |A_0| + |A_1| \sim 3|A|,$$

then by Lemma 1.6 we have $|A_i| \sim I_i(l_i, u_i)$.

Suppose $d = 2$. Then $u_0 < l_1$. Hence $(2A)(0, H) \sim |0 + A_1| + |l_1 + A_1| \sim H \succ \frac{3}{4}H$, which contradicts (4) of Theorem 3.2.

Suppose $d = 3$. If $u_0 \sim 0$ or $u_1 \sim 0$, then $|A| \preceq \frac{1}{3}H$, which contradicts (2) of Theorem 3.2. So we have $u_i \succ 0$ for $i = 0, 1$. Since $|2A| \sim 3|A|$, then $A_i(l_i, u_i) \sim \frac{1}{3}(u_i - l_i)$ for $i = 0, 1$. This implies $\underline{d}_U(A) = \frac{2}{3}$, a contradiction.

Suppose $d > 4$. Then $|A| \preceq \frac{2}{d}H \prec \frac{1}{2}H$, which contradicts (3) of Theorem 3.2.

Suppose $d = 4$. If $u_0 \prec H$, then $u_1 = H$ and

$$|A| \sim |A_0| + |A_1| \preceq \frac{1}{4}(u_0 + H) \prec \frac{1}{2}H,$$

which contradict (3) of Theorem 3.2. By symmetry, $u_1 \prec H$ is also impossible. Hence $u_0 \sim u_1 \sim H$, which implies (a) of Theorem 3.2.

From the arguments above we can assume that A is not a subset of a *b.p.* of difference > 1.

For notational convenience we again deal with the set $B = H - A$.

Case 3.2.1 $\underline{d}_U(B) < \frac{1}{2}$.

Same as in Case 3.1.1. □(Case 3.2.1)

Case 3.2.2: $\underline{d}_U(B) > \frac{1}{2}$.

Suppose for every $0 \prec x \prec H$ we have $B(0, x) \succ \frac{1}{2}x$, then $(2B)(0, H) \sim H \sim 2B(0, H)$. Hence by (4) of Theorem 3.2 we have $B(0, H) \succeq (2B)(H, 2H) \succeq |H + B[0, H]| \sim B(0, H)$, which implies $(2B)(H, 2H) \sim B(0, H)$. Hence (b) of Theorem 3.2 is true. Otherwise there is $0 \prec b \prec H$ such that $B(0, b) \sim \frac{1}{2}b$ and $B(0, x) \succ \frac{1}{2}x$ for every $0 \prec x \prec b$. Now the proof is the same as the proof of Subcase 3.1.2.2. □(Case 3.2.2)

Case 3.2.3: $\underline{d}_U(B) = \frac{1}{2}$.

If $B \cap U$ is neither a subset of an *a.p.* of difference > 1 nor a subset of a U–unbounded *b.p.*, then by the same proof as in Subcase 3.1.3.1 we have $|2B| \succ 3|B|$.

Suppose $B \cap U$ is a subset of an *a.p.* of difference $d > 1$. Then $d = 2$. Since B is not a subset of an *a.p.* of difference > 1, then B contains an odd number. Let b be the least odd number in B. Then $b \succ 0$. If $B(0, b) \prec \frac{1}{2}b$, then $B(b, H) \succ \frac{1}{2}(H - b)$, which contradicts (3) of Theorem 3.2. Hence $B(0, b) \sim \frac{1}{2}b$. Since $(2B)(0, 2H) \sim \frac{3}{2}H$ and $(2B)(0, b - 1) \sim \frac{1}{2}b$, then

$$\begin{aligned}(2B)(b, 2H) &\sim \frac{3}{2}H - \frac{1}{2}b \\ &= \frac{3}{4}(2H - b) + \frac{1}{4}b \succ \frac{3}{4}(2H - b),\end{aligned}$$

which contradicts (4) of Theorem 3.2.

Suppose $B \cap U$ is a subset of a U–unbounded *b.p.* $I_0 \cup I_1$ of difference d. Then $d > 2$. Let $c = \min\{x \in B : B[0, x] \text{ is not a subset of a } b.p. \text{ of difference } d.\}$. By the argument above we can assume $\gcd(B[0, c - 1]) = 1$.

If $d \geqslant 4$, then by Lemma 2.7 and Lemma 2.8 we have $(2B)(0,2c) \succ 3B(0,c)$. By Lemma 2.4 we have $|2B| \succ 3|B|$.

Suppose $d = 3$. Without loss of generality we assume $B[0,c-1] = B_0 \cup B_1$, where $B_i = \{x \in B[0,c-1] : x \equiv i \pmod 3\}$ for $i = 0,1$ and $c \equiv 2 \pmod 3$. Let $l_i = \min B_i$ and $u_i = \max B_i$. Since we can assume that $B \cap U$ is not a subset of an *a.p.* of difference > 1 by the argument above we have $l_1 \sim 0$. Since $(2B)(0,2c-1) \sim |2B_0| + |2B_1| + |B_0 + B_1| \succeq 3B(0,c-1)$, then $|2B| \sim 3|B|$ implies $(2B)(0,2c-1) \sim 3B(0,c-1)$, which implies $|B_i| \sim \frac{1}{3}(u_i - l_i)$. If $u_0 \sim 0$ or $u_1 \sim 0$, then $\underline{d}_U(B) \leqslant \frac{1}{3}$. Otherwise $\underline{d}_U(B) \geqslant \frac{2}{3}$. But each of them contradicts $\underline{d}_U(B) = \frac{1}{2}$. □(Theorem 3.2)

Next we would like to present a corollary of Theorem 1.3. Let

$$Q = \left\{ \frac{2}{k} : k \geqslant 4 \right\}.$$

Corollary 3.3 *Suppose A is not a subset of an* a.p. *of difference > 1 and $\bar{d}(A) = \alpha < 1$. If $\alpha \notin Q$ and*

$$\bar{d}(2A) = \min \left\{ \bar{d}(2B) : \begin{array}{c} B \textit{ is not a subset of an a.p.} \\ \textit{of difference } > 1 \textit{ and } \bar{d}(B) \geqslant \alpha \end{array} \right\},$$

then $\underline{d}(A) = 0$.

Proof: The corollary is trivially true if $\alpha = 0$. So we can assume $\alpha > 0$. Since $\alpha \notin Q$, then A cannot have the structure characterized in the conclusion of Part II (a) or Part III (a) of Theorem 1.3. If $\alpha < \frac{1}{2}$, then A has the structure characterized in the conclusion of Part II (b) of Theorem 1.3. It is then easy to see that $\lim_{n\to\infty} \frac{A(b_n)}{b_n} \leqslant \lim_{n\to\infty} \frac{c_n}{b_n} = 0$. Hence $\underline{d}(A) = 0$.

Suppose $\alpha \geqslant \frac{1}{2}$ and $\underline{d}(A) = \beta > 0$. Without loss of generality we assume $0 \in A$. Let $h_n \in A$ be such that $\lim_{n\to\infty} h_n = \infty$ and $\lim_{n\to\infty} \frac{A(h_n)}{h_n} = \alpha$. Let K be an arbitrary hyperfinite integer and let $H = h_K$. Let $B = {}^*A[0,H]$. Since $\alpha \notin Q$, then $(2B)(0,H) \sim B(0,H)$. This implies that for each a with $0 \prec a \leqslant H$, $(2B)(0,a) \sim B(0,a)$. By the transfer principle we have that for every hyperfinite $x \leqslant H$, $\frac{B(0,x)}{x+1} \gtrapprox \beta$. Let

$$\gamma = \inf \left\{ st\left(\frac{B(0,x)}{x+1} \right) : 0 \prec x \leqslant H \right\}.$$

Then $\gamma \geqslant \beta$. If $\gamma > \frac{1}{2}$, then by the pigeonhole principle we have that $[0,H] \smallsetminus U \subseteq 2B$. This implies $(2B)(0,H) \sim H \succ \alpha H \sim B(0,H)$ because $\alpha < 1$. Hence we have a contradiction because $(2B)(0,H) \sim B(0,H)$. So we can assume $\gamma \leqslant \frac{1}{2}$.

Choose an a with $0 \prec a \leqslant H$ such that $\frac{B(0,a)}{a+1} \lessapprox \frac{4}{3}\gamma$ and let $b = \max\left\{x \in B : x \leqslant \frac{1}{2}a\right\}$. If $b \prec a$, then

$$
\begin{aligned}
(2B)(0, \left[\frac{a}{2}\right]) &\succeq B(0,b) + |b + B[0, \left[\frac{a}{2}\right] - b]| \\
&\succeq B(0, \left[\frac{a}{2}\right]) + B(0, \left[\frac{a}{2}\right] - b) \\
&\succ B(0, \left[\frac{a}{2}\right]).
\end{aligned}
$$

This contradicts $(2B)(0, \left[\frac{a}{2}\right]) \sim B(0, \left[\frac{a}{2}\right])$. So we can assume $b \sim \frac{a}{2}$.

By Lemma 1.5, we have $|B[0,b] + B[0,b]| \succeq \min\{3B(0,b), b + B(0,b)\}$. Then

$$
\begin{aligned}
\frac{4}{3}\gamma &\gtrapprox \frac{B(0,a)}{a+1} \\
&\approx \frac{(2B)(0,a)}{a+1} \gtrapprox \frac{|B[0,b] + B[0,b]|}{2b+1} \\
&\gtrapprox \min\left\{\frac{3B(0,b)}{2b+1}, \frac{b + B(0,b)}{2b+1}\right\} \\
&\gtrapprox \min\left\{\frac{3}{2}\gamma, \frac{1+\gamma}{2}\right\} \gtrapprox \frac{3}{2}\gamma,
\end{aligned}
$$

which is absurd. This ends the proof. $\square$(Corollary 3.3)

If $\alpha \in Q$, then the corollary is no longer true. The following is an example for showing that.

Example 3.4 *Let $\alpha \in Q$ and $\alpha = \frac{2}{k}$. Note that $\alpha \leqslant \frac{1}{2}$. For each $0 \leqslant \beta \leqslant \alpha$, we need to construct A such that $\bar{d}(A) = \alpha$, $\underline{d}(A) = \beta$, and $\bar{d}(2A) = \frac{3}{2}\alpha$. Let*

$$B = \{kn : n \in \mathbb{N}\} \cup \{1 + kn : n \in \mathbb{N}\}.$$

Then $\bar{d}(B) = \alpha$. If $\beta = 0$, then let

$$A = B \smallsetminus \bigcup_{n \in \mathbb{N}} [2^{2^{2n}}, 2^{2^{2n+1}}],$$

and if $0 < \beta \leqslant \alpha$, then let

$$A = B \smallsetminus \bigcup_{n \in \mathbb{N}} [\left[\frac{\beta}{\alpha} 2^{2^n}\right], 2^{2^n}].$$

It is easy to check that $0 \in A$, $\gcd(A) = 1$, $\bar{d}(A) = \alpha$, $\bar{d}(2A) = \frac{3}{2}\alpha$, and $\underline{d}(A) = \beta$.

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