Inverse Problem
For Upper Asymptotic Density II

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Abstract

Inverse problems study the structure of a set A when the $A+A$ is "small". In the article, the structure of an infinite set A of natural numbers is described when $A+A$ has the least possible upper asymptotic density and A contains two consecutive numbers. For example, if the upper asymptotic density α of A is between 0 and $\frac{1}{2}$, the upper asymptotic density of $A+A$ is less than or equal to $\frac{3}{2}\alpha$, and A contains two consecutive numbers, then A is either a large subset of the union of two arithmetic sequences with same common difference $k=\frac{2}{\alpha}$, or for any increasing sequence h_n of positive integers such that the relative density of A in $[0,h_n]$ approaches α, the set $A\cap[0,h_n]$ can be partitioned into two parts $A\cap[0,c_n]$ and $A\cap[b_n,h_n]$ such that c_n/h_n approaches 0, *i.e.* the cardinality of $A\cap[0,c_n]$ is relatively very small, and $(h_n-b_n)/h_n$ approaches to α, *i.e.* the cardinality of $A\cap[b_n,h_n]$ is relatively the same as the cardinality of the interval $[b_n,h_n]$.

1 Introduction

Let $\mathbb{N}$ be the set of all natural numbers, including 0. A and B will always denote the sets of natural numbers, and a,b,c,h,i,j,k,m,n,x,y,z will always denote natural numbers. For any m,n and A, we write $[m,n]$ for the set $\{k\in\mathbb{N}: m\leqslant k\leqslant n\}$, $A[m,n]$ for the set $A\cap[m,n]$, and $A(m,n)$ for the number of elements in $A[m,n]$. Sometimes, we writed $A(1,n)$ as $A(n)$. For any A and B we write $A\pm B$ for the set $\{a\pm b: a\in A \text{ and } b\in B\}$, and $2A$ for $A+A$. For a set A and a number b we write $A\pm b$ for $A\pm\{b\}$. In this paper we often use $(2A)(m,n)$ for $|\{x\in 2A: m\leqslant x\leqslant n\}|$ and $2A(m,n)$ for 2 times $A(m,n)$. We write *a.p.* as an abbreviation for arithmetic progression. The upper asymptotic density of a set A is defined by

$$\bar{d}(A)=\limsup_{n\to\infty}\frac{A(n)}{n}.$$

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The main results of this paper describe the structural properties of A when A contains two consecutive numbers and

$$\bar{d}(2A) = \inf\{\bar{d}(2B) : B \text{ contains two consecutive numbers and } \bar{d}(B) \geqslant \bar{d}(A)\}.$$

To motivate of the main result, we quote a few sentences from the preface of the book [9]: "The classical problems in additive number theory are *direct problems*, in which we start with a set A of integers and proceed to describe the h–fold sumset hA, that is, the set of all sums of h elements of A. In an *inverse problems*, we begin with the sumset hA and try to deduce information about the underlying set A. In the last few years, there has been remarkable progress in the study of inverse problems for finite sets in additive number theory. There are important inverse theorems due to Freiman, Kneser, Plünnecke, Vosper, and others. In particular, Ruzsa recently discovered a new method to prove a generalization of Freiman's theorem." Although the results in this paper are not directly related to the Freiman's Theorem mentioned above, they share the same pattern, which says that if $2A$ is small, then A must have some structure.

In fact, the idea of inverse problem occurs also in some of the theorems involving densities. The theorems about Shnirel'man pairs and Mann's pairs in [4] deduce information about the Shnirel'man density of A and Shnirel'man density of B when the Shnirel'man density of $A+B$ is small. Kneser's Theorem (cf. [3] and [1]) deduces information about $A+B$, which gives information about A and B when the lower asymptotic density of $A+B$ is small. In [7], the inverse problems for upper asymptotic density are considered. We describe the structural properties of A when the upper asymptotic density of $2A+\{0,1\}$ is small. However, adding $\{0,1\}$ to $2A$ seems to be a non–traditional condition. The result will be more interesting if the condition can be replaced by some condition more natural to number theorists.

Why do we need to add $\{0,1\}$ to $2A$ in the first place in [7]? Let $\alpha = \bar{d}(A)$ and $a_0 = \min A$. By Lemma 1.5, one can prove the following. If $\alpha \leqslant \frac{1}{2}$ and $\gcd(A-a_0)$, the greatest common divisor of all numbers in $A - a_0$, is one, then $\bar{d}(2A) \geqslant \frac{3}{2}\alpha$. If $\alpha > \frac{1}{2}$, then $\bar{d}(2A) \geqslant \frac{\alpha+1}{2}$. Note that the two inequalities above are optimal. There exists two kinds of sets, which witness that the equalities hold.

Example 1.1 *For any real number* $0 \leqslant \alpha \leqslant 1$*, let*

$$A = \bigcup_{n=1}^{\infty} [\lceil (1-\alpha)2^{2^n} \rceil, 2^{2^n}].$$

Then $\bar{d}(A) = \alpha$, $\bar{d}(2A) = \bar{d}(2A+\{0,1\}) = \frac{1+\alpha}{2}$ *if* $\alpha \geqslant \frac{1}{2}$, *and* $\bar{d}(2A) = \bar{d}(2A+\{0,1\}) = \frac{3}{2}\alpha$ *if* $\alpha \leqslant \frac{1}{2}$.

Example 1.2 *Let* $k, m, n \in \mathbb{N}$ *be such that* $k \geqslant 4$ *and* $2m, 2n, m+n$ *are not pairwise equivalent modulo* k. *Let*

$$A = \{m + ik : i \in \mathbb{N}\} \cup \{n + ik : i \in \mathbb{N}\}.$$

Then $\bar{d}(A) = \frac{2}{k} = \alpha \leqslant \frac{1}{2}$ *and* $\bar{d}(2A) = \frac{3}{k} = \frac{3}{2}\alpha$. *One can choose* k, m, n *such that* $\gcd(A - a_0) = 1$. *Note that* $\bar{d}(2A + \{0,1\}) \geqslant 2\alpha$.

We believe that if A is a set with positive upper asymptotic density such that $\gcd(A-a_0) = 1$ and the upper asymptotic density of $2A$ reaches its smallest possible value, then A should be a set similar to the one in Example 1.1 or to the one in Example 1.2. If one requires that $\bar{d}(2A + \{0,1\}) = \frac{3}{2}\alpha$ when $\bar{d}(A) = \alpha \leqslant \frac{1}{2}$, then A cannot be the set similar to the one in Example 1.2. Hence one needs only to show that A is a set similar to the one in Example 1.1 as done in [7]. This greatly simplifies the proof. Besides, adding $\{0,1\}$ to the set $2A$ makes it possible to apply Besicovitch's Theorem [3, page 6] to the proof of [7, Lemma 2.1]. Without adding $\{0,1\}$, one needs not only consider that A can be a set similar to the one in Example 1.2, but also find a new way of proofs by-passing Besicovitch's Theorem. Of course, a new condition must be added. The ideal condition should be $\gcd(A - a_0) = 1$. However, so far we are unable to derive the same conclusion as in Part II of Theorem 1.3 with this condition. Instead, a condition slightly stronger than $\gcd(A - a_0) = 1$ is added: *the set* A *contains two consecutive numbers*, and leave the case with $\gcd(A - a_0) = 1$ as an unsolved question near the end of this paper.

Next, we state the main theorem of this paper.

Theorem 1.3 *Let* A *be a set of natural numbers and* $\bar{d}(A) = \alpha > 0$.

Part I: Assume $\alpha > \frac{1}{2}$. *Then* $\bar{d}(2A) = \frac{1+\alpha}{2}$ *implies that for any increasing sequence* $\langle h_n : n \in \mathbb{N}\rangle$ *with* $\lim_{n\to\infty} \frac{A(0,h_n)}{h_n+1} = \alpha$, *one has*

$$\lim_{n\to\infty} \frac{(2A)(0,h_n)}{h_n+1} = \alpha.$$

Part II: Assume $\alpha < \frac{1}{2}$ *and* A *contains two consecutive numbers. Then* $\bar{d}(2A) = \frac{3}{2}\alpha$ *implies that either (a) there exist* k *and* c *such that* $\alpha = \frac{2}{k}$ *and*

$$A \subseteq \{c + ik : i \in \mathbb{N}\} \cup \{c + 1 + ik : i \in \mathbb{N}\}$$

or (b) for any increasing sequence $\langle h_n : n \in \mathbb{N}\rangle$ *with* $\lim_{n\to\infty} \frac{A(0,h_n)}{h_n+1} = \alpha$*, there exist two sequences* $0 \leqslant c_n \leqslant b_n \leqslant h_n$ *such that*

$$\lim_{n\to\infty} \frac{A(b_n, h_n)}{h_n - b_n + 1} = 1,$$

$$\lim_{n\to\infty} \frac{c_n}{h_n} = 0,$$

and $[c_n, b_n] \cap A = \emptyset$ *for every* $n \in \mathbb{N}$.

Part III: Assume $\alpha = \frac{1}{2}$ *and* A *contains two consecutive numbers. Then* $\bar{d}(2A) = \frac{3}{2}\alpha$ *implies that either (a) there exist* $c \in \{0, 1, 2, 3\}$ *such that*

$$A \subseteq \{c + 4i : i \in \mathbb{N}\} \cup \{c + 1 + 4i : i \in \mathbb{N}\}$$

or (b) for any increasing sequence $\langle h_n : n \in \mathbb{N}\rangle$ *with* $\lim_{n\to\infty} \frac{A(0,h_n)}{h_n+1} = \alpha$*, one has*

$$\lim_{n\to\infty} \frac{(2A)(0, h_n)}{h_n + 1} = \alpha.$$

Remark 1.4 *(1) If* $\bar{d}(A) > \frac{1}{2}$*, then* A *automatically contains two consecutive numbers.*

(2) The proof of Part I of Theorem 1.3 is easy and will be omitted. See [7, (2) of the remarks in Section 4] for an explanation.

(3) Given a set $A \subseteq [0, n]$*, the set* $2A$ *is a subset of* $[0, 2n]$*. The structural property of* A *in Part I of Theorem 1.3 says that the number* $(2A)(0, h_n)$ *is close to* $A(0, h_n)$*. Hence the growth of the size of* $(2A)(0, 2h_n)$ *occurs mostly in* $[h_n, 2h_n]$*.*

(4) In Part I and Part III (b) of Theorem 1.3, one cannot expect the set A *has a structural property similar to the one in (b) of Part II. See [7, (1) of the remarks in Section 4] for an example.*

In the next section, we will prove several lemmas necessary for the proof of Theorem 1.3. Then in the third section, the proof of Theorem 1.3 is presented and a corollary is given. In both sections, the techniques from nonstandard analysis are used. Although these nonstandard techniques might not be unavoidable, we strongly believe that they significantly shorten the length of the proofs. Besides, nonstandard analysis is one of my favorite subjects, which gives me a handy tool and offers me a better insight.

Since two of Freiman's theorems (cf.[9, Theorem 1.15 and Theorem 1.16, page 28] or cf.[2, Proposition 1.1]) will frequently be cited, we state them as lemmas in this section. The proofs are entirely standard.

Lemma 1.5 (G. Freiman) *Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be such that $0 = a_0 < a_1 < \cdots < a_{k-1} = n$ and $\gcd(A) = 1$. If $k \leqslant \frac{n+3}{2}$, then $(2A)(0, 2n) \geqslant 3k - 3$. If $k \geqslant \frac{n+3}{2}$, then $(2A)(0, 2n) \geqslant k + n$.*

Lemma 1.6 (G. Freiman) *Let $A \subseteq \mathbb{N}$ be such that $|A| = k > 2$. If $|2A| = 2k - 1 + b \leqslant 3k - 4$, then A is a subset of an* a.p. *of the length $k + b \leqslant 2k - 3$.*

2 Lemmas

As mentioned in §1, techniques from nonstandard analysis in are needed in §2 and §3. One of the advantages of nonstandard methods is that an asymptotic argument such as upper asymptotic density in the standard world can be translated into a *finite argument in a nonstandard world, so that instead of dealing with a sequence of intervals in an upper asymptotic density argument, we can deal with only one interval of *finite length in the nonstandard world. For basic knowledge of nonstandard analysis, the reader is referred to [8], [5], or [6].

We work within a fixed $\aleph_1$–saturated nonstandard universe $^*V$ in this paper. For each set A, write $^*A$ for the nonstandard version of A in $^*V$. For example, $^*\mathbb{N}$ is the set of all natural numbers in $^*V$, and if A is the set of all even numbers in $\mathbb{N}$, then $^*A$ is the set of all even numbers in $^*\mathbb{N}$. If we do not specify that A, B are sets of standard natural numbers, A, B are always assumed to be *internal* sets of (standard and nonstandard) natural numbers. Now $a, b, c, h, i, j, k, m, n, x, y, z$ can take values in $^*\mathbb{N}$. The integers in $^*\mathbb{N} \smallsetminus \mathbb{N}$ are called hyperfinite integers. The letters H, K and N are exclusively used for hyperfinite integers. The Greek letters α, β, γ, δ, and ϵ are reserved exclusively for standard real numbers.

We introduce here some useful notation for comparisons. For any real numbers r, s in $^*V$, by $r \approx s$ we mean that $r - s$ is an infinitesimal, by $r \ll s$ ($r \gg s$) we mean that $r < s$ ($r > s$) and $r \not\approx s$, and by $r \lessapprox s$ ($r \gtrapprox s$) we mean $r < s$ ($r > s$) or $r \approx s$. Given a hyperfinite integer H and two real numbers r, s, by $r \sim_H s$ we mean that $\frac{s-r}{H} \approx 0$, by $r \prec_H s$ ($r \succ_H s$) we mean that $r < s$ ($r > s$) and $r \not\sim_H s$, and by $r \preceq_H s$ ($r \succeq_H s$) we mean that $r \prec_H s$ ($r \succ_H s$) or $r \sim_H s$. Say that a is <u>insignificant</u> with respect to H if $a \sim_H 0$. In the most cases the subscript H is clearly given, so it will be dropped as a subscript for convenience. Note that the comparison relations $\ll$, $\gg$, $\approx$, etc. can be interpreted in terms of $\prec$, $\succ$, $\sim$, etc., or vice versa, when H is given. For example, $\frac{a}{H} \lessapprox \frac{b}{H}$ iff $a \preceq b$. We use $\preceq$ more often than $\lessapprox$ because fractions can be avoided. When using $\sim$, $\prec$, $\preceq$, etc., insignificant quantities can often be neglected. For example, instead of using $A(0,H) \sim \alpha(H+1)$ we can use its equivalent form $A(0,H) \sim \alpha H$. For another example, when $a \leqslant c \leqslant b$, we often write $A(a,c) \sim A(a,b) + A(b,c)$ instead of $A(a,c) = A(a,b) + A(b+1,c)$. For a real number $r \in {}^*\mathbb{R}$ bounded by a standard real number, let $\mathrm{st}(r)$, the standard part of r, be the unique standard real number α such that $r \approx \alpha$. A set T of three integers is called a crowded triple if $T = \{a, a+1, a+2\}$ or $T = \{a, a+1, a+3\}$ or $T = \{a, a+2, a+3\}$ for some integer a.

The next lemma shows how upper asymptotic density can be translated into a nonstandard version.

Lemma 2.1 *Let $A \subseteq \mathbb{N}$ and let $0 \leqslant \alpha \leqslant 1$. Then $\bar{d}(A) \geqslant \alpha$ if and only if there is a hyperfinite integer H such that ${}^*A(0,H) \succeq \alpha H$.*

Proof: Left to the reader. The lemma is an easy consequence of the transfer property in nonstandard analysis. □(Lemma 2.1)

Lemma 2.2 *Suppose $A \subseteq [0,H]$ with $A(0,H) \succ 0$. Then there exists an $a \in [0,H]$ such that $A(0,a) \sim 0$ and for any $a \prec b \leqslant H$, $A(a,b) \succ 0$.*

Proof: Let $S = \{\mathrm{st}(\frac{x}{H}) : x \in [0,H] \text{ and } A(0,x) \sim 0\}$. Then S is a subset containing 0 of the standard unit interval. Let β be the least upper bound of S and let $a \in [0,H]$ be such that $\mathrm{st}(\frac{a}{H}) = \beta$. Note that $x \sim y$ iff $\mathrm{st}(\frac{x}{H}) = \mathrm{st}(\frac{y}{H})$.

Claim 2.2.1 $A(0,a) \sim 0$.

Proof of Claim 2.2.1: Suppose $A(0,a) \succ 0$. Then there is a standard real number $\gamma > 0$ such that $\frac{A(0,a)}{H} \approx \gamma$. Choose a $c \prec a$ such that $\frac{a-c}{H} < \frac{\gamma}{2}$. Then for any $c \leqslant x \leqslant a$

$$\frac{A(0,x)}{H} \geqslant \frac{A(0,a)}{H} - \frac{a-x}{H} \gtrapprox \frac{\gamma}{2}.$$

So $\mathrm{st}(\frac{x}{H})$ is not in S, which contradicts that β is the least upper bound of S. □(Claim 2.2.1)

Claim 2.2.2 For any $a \prec b \leqslant H$, $A(a,b) \succ 0$.

Proof of Claim 2.2.2: If there is a $b \succ a$ with $A(a,b) \sim 0$, then $A(0,b) \sim A(0,a) + A(a,b) \sim 0$. So $\mathrm{st}(\frac{b}{H})$ is in S and is greater than β. This contradicts that β is the least upper bound of S. □(Lemma 2.2)

Lemma 2.3 *Let* $0 \leqslant \alpha \leqslant 1$ *and let* $A \subseteq [0,H]$ *be such that* $A(0,H) \preceq \alpha H$. *Let* $0 \prec a \leqslant H$ *be such that for every* $0 \prec b \prec a$, $A(0,b) \succ \alpha b$. *Then there is a* $c \geqslant a$ *such that* $A(0,c) \sim \alpha c$ *and for every* $0 \prec b \prec c$, $A(0,b) \succ \alpha b$.

Proof: Let $S = \{\mathrm{st}(\frac{x}{H}) : x \in [0,H] \text{ and } A(0,x) \preceq \alpha x\}$ and let β be the greatest lower bound of S. Let $c \in [0,H]$ be such that $\mathrm{st}(\frac{c}{H}) = \beta$. Now using a similar argument as in the proof of Lemma 2.2, one can verify that c is the desired number. □(Lemma 2.3)

Lemma 2.4 *Suppose* $A \subseteq [0,H]$ *does not contain any crowded triple. Then*

(1) $A(0,H) \preceq \frac{1}{2}H$,

(2) $|A + \{a, a+1\}| \succeq \frac{3}{2}|A|$,

(3) if T *is a crowded triple, then* $|A+T| \succeq 2|A|$.

Proof: Let $A_1 = \{x \in A : x+1 \notin A \text{ and } x-1 \notin A\}$ and $A_2 = A \smallsetminus A_1$. Note that A_1 contains all "isolated" singleton numbers in A. Since A does not contain any crowded triple, then A_2 contains all paired numbers in A and any two different pairs in A are at least two units apart.

(1) If $x \in A_1$, then let $f(x) = x+1$. If $\{x, x+1\} \subseteq A_2$, then let $f(x) = x+2$ and $f(x+1) = x+3$. It is easy to see that $f : A \mapsto [0, H+2] \smallsetminus A$ is a one-one internal function. Hence $A(0,H) \preceq \frac{1}{2}H$.

(2) It suffices to show $|A + \{0,1\}| \succeq \frac{3}{2}|A|$. Clearly $A_1 + \{0,1\} = 2|A_1|$ and $|A_2 + \{0,1\}| = \frac{3}{2}|A_2|$. Since A does not contain any crowded triple, then

$$(A_1 + \{0,1\}) \cap (A_2 + \{0,1\}) = \emptyset.$$

Hence

$$|A + \{0,1\}| = |A_1 + \{0,1\}| + |A_2 + \{0,1\}| = 2|A_1| + \frac{3}{2}|A_2| \geqslant \frac{3}{2}|A|.$$

(3) Without loss of generality, we assume that $T = \{0,1,2\}$ or $T = \{0,1,3\}$. Let f be the function from A to $[0, H+2] \smallsetminus A$ defined in (1). Then $A_1 \cup f[A_1] \subseteq A_1 + T$ and $A_2 \cup f[A_2] \subseteq A_2 + T$. Hence

$$|A+T| \geqslant |A_1 \cup f[A_1]| + |A_2 \cup f[A_2]| = 2|A_1| + 2|A_2| = 2|A|.$$

□(Lemma 2.4)

Lemma 2.5 *Let $A \subseteq [0,H]$ and $0 \leqslant x_1 \prec x_2 \leqslant H$ be such that*

(1) $(2A)(2x_1, 2x_2) \succ 3A(x_1, x_2)$,

(2) if $0 \prec x_1$, then $A[0,x]$ is not a subset of an a.p. of difference $d \geqslant 2$ and $A(0,x) \preceq \frac{1}{2}(x+1)$ for some $x \sim x_1$,

(3) if $x_2 \prec H$, then $A[x,H]$ is not a subset of an a.p. of difference $d \geqslant 2$ and $A(x,H) \preceq \frac{1}{2}(H-x+1)$ for some $x \sim x_2$,

Then $(2A)(0,2H) \succ 3A(0,H)$.

Proof: By Lemma 1.6, one has $(2A)(0,2x_1) \succeq 3A(0,x_1)$ and $(2A)(2x_2,2H) \succeq 3A(x_2,H)$. Hence

$$\begin{aligned}&(2A)(0,2H)\\&\quad\sim (2A)(0,2x_1) + (2A)(2x_1,2x_2) + (2A)(2x_2,2H)\\&\quad\succ 3A(0,x_1) + 3A(x_1,x_2) + 3A(x_2,H)\\&\quad\sim 3A(0,H).\end{aligned}$$

□(Lemma 2.5)

The next three lemmas are the main ingredients of the proof of Theorem 1.3.

Lemma 2.6 *Let $A \subseteq [0,H]$ be such that*

(1) $A \cap \mathbb{N}$ contains two consecutive numbers,

(2) $A(0,H) \prec \frac{1}{2}H$ and $A(0,x) \preceq \frac{1}{2}x$ for any $0 \prec x \leqslant H$,

(3) $(2A)(0,2H) \sim 3A(0,H)$,

(4) there exists a $k_0 \prec H$ such that for every $k_0 \prec k \prec H$, $A(k,H) \succ \frac{1}{2}(H-k)$.

Then there exist $0 \leqslant c \prec b \prec H$ such that $c \sim 0$, $[c,b] \cap A = \emptyset$, and $A(b,H) \sim H-b$.

Proof: By (1) we can, without loss of generality, assume $0, 1 \in A$. By Lemma 2.3, one can choose k_0 such that $A(k_0,H) \sim \frac{1}{2}(H-k_0)$. By (2), one has $k_0 \succ 0$.

Claim 2.6.1 $(2A)(k_0+H, 2H) \sim H - k_0 \sim 2A(k_0,H)$.

Proof of Claim 2.6.1: Given any $k_0 + H \prec x \prec 2H$, since

$$A(x-H,H) \succ \frac{1}{2}(2H-x),$$

$$A[x-H,H] \subseteq [x-H,H],$$

and

$$x - A[x-H, H] \subseteq [x-H, H],$$

then $A[x-H,H] \cap (x - A[x-H,H]) \neq \emptyset$. This shows

$$x \in A[x-H,H] + A[x-H,H] \subseteq 2A.$$

Hence $2A$ contains all x with $k_0 + H \prec x \prec 2H$. This shows

$$(2A)(k_0+H, 2H) \sim H - k_0 \sim 2A(k_0, H).$$

□(Claim 2.6.1)

The lemma is now divided into two cases.

Case 2.6.1 $A(0, k_0) \sim 0$.

Claim 2.6.2 $(2A)(H, k_0+H) \sim 0$.

Proof of Claim 2.6.2: If $(2A)(H, k_0+H) \succ 0$, then

$$\begin{aligned}
&(2A)(0, 2H) \\
&\quad \sim (2A)(0,H) + (2A)(H, k_0+H) + (2A)(k_0+H, 2H) \\
&\quad \succeq A(0,H) + (2A)(H,k_0+H) + 2A(k_0,H) \\
&\quad \sim 3A(0,H) + (2A)(H,k_0+H) \\
&\quad \succ 3A(0,H),
\end{aligned}$$

which contradicts (3). □(Claim 2.6.2)

Claim 2.6.3 Let $c = \max A[0, k_0]$ and $b = \min A[k_0, H]$. Then $c \sim 0$ and $b \sim \frac{H+k_0}{2}$. Hence the conclusion of Lemma 2.6 holds.

Proof of Claim 2.6.3: The proof is similar to the proof of [7, Claim 1.3.5]. It is clear that $b \preceq \frac{H+k_0}{2}$ because otherwise

$$A(k_0, H) = A(b, H) \preceq H - b \prec \frac{1}{2}(H - k_0),$$

which contradicts the choice of k_0.

For notational convenience, we deal with $B = H - A \subseteq [0, H]$, the reverse of A, instead, in the rest of this claim. Let $m_0 = H - k_0$, let $d = H - b$, and let $a = H - c$. By Claim 2.6.2, $(2B)(m_0, H) \sim 0$.

Choose two standard natural numbers $p < q$ such that $\gcd(p, q) = 1$ and

$$m_0 + \frac{2m_0}{p} + 2 < H.$$

For each $i = 0, 1, \ldots, 2p$, let $x_i = [\frac{m_0}{2p} i]$. Then

$$B[x_{p-i}, x_{p-i+1}] + B[x_{p+i}, x_{p+i+1}] \subseteq (2B)[m_0 - 2, H].$$

Hence for each of $i = 0, 1, \ldots, p-1$, $B(x_{p-i}, x_{p-i+1}) \succ 0$ implies $B(x_{p+i}, x_{p+i+1}) = \emptyset$ and $B(x_{p+i}, x_{p+i+1}) \succ 0$ implies $B(x_{p-i}, x_{p-i+1}) = \emptyset$. Note that when $i = 0$, one has $B(x_p, x_{p+1}) \sim 0$. Since $B(0, m_0) \sim \frac{1}{2} m_0$, then there are exactly half of the i's in $\{1, 2, \ldots, p-1\} \cup \{p+1, p+2, \ldots, 2p-1\}$ such that $B(x_i, x_{i+1}) \sim x_{i+1} - x_i$ and for the rest of the i's, $B[x_i, x_{i+1}] = \emptyset$. Since $B(x_p, x_{p+1}) \sim 0$, then $B(x_0, x_1) \sim x_1 - x_0$.

By the same procedure, one can define $y_j = [\frac{m_0}{2q} j]$ for $j = 0, 1, \ldots, 2q$ so that there are exactly half of the j's in $\{1, 2, \ldots, q-1\} \cup \{q+1, q+2, \ldots, 2q-1\}$ such that $B(y_j, y_{j+1}) \sim y_{j+1} - y_j$ and for the rest of the j's, $B[y_j, y_{j+1}] = \emptyset$. Since $B(y_q, y_{q+1}) \sim 0$, then $B(y_0, y_1) \sim y_1 - y_0$.

Since p and q are relatively prime, for any $i \in \{1, 2, \ldots, p-1\}$ and $j \in \{1, 2, \ldots, q-1\}$, one has $x_i \not\sim y_j$. Hence there is no $i \in \{1, 2, \ldots, p-1\}$ such that $B(x_{i-1}, x_i) \not\sim B(x_i, x_{i+1})$ because otherwise, one can take a $j \in \{1, 2, \ldots, q-1\}$ such that $x_i \in [y_j, y_{j+1}]$, which would make $B(y_j, y_{j+1}) \not\sim 0$ and $B(y_j, y_{j+1}) \not\sim y_{j+1} - y_j$ at the same time.

Since $B(x_0, x_1) \sim x_1 - x_0$, then $B(0, x_p) \sim x_p$ and $B[x_{p+1}, m_0] = \emptyset$. So $d < x_{p+1}$. By the fact that p and q can be chosen arbitrarily large in $\mathbb{N}$, we have $d \sim \frac{m_0}{2}$.

If $m_0 \leqslant a \prec H$, then

$$\begin{aligned}
&(2B)(0, 2H) \\
&\quad \succeq (2B)(0, m_0) + (2B)(a, H) + (2B)(H, 2H) \\
&\quad \succeq 2B(0, m_0) + |B[0, H-a] + a| + B(0, H) \\
&\quad \succeq 3B(0, H) + B(0, H-a) \succ 3B(0, H),
\end{aligned}$$

which contradicts (3). So $a \sim H$.

It is easy to see that $c = H - a$ and $b = H - d$ are the numbers satisfying the conclusion of the lemma. □(Claim 2.6.3)

Case 2.6.2 $A(0, k_0) \succ 0$.

By Lemma 2.2, there is a $0 \prec k \preceq k_0$ such that $A(k, k_0) \sim 0$ and for any $0 \leqslant k' \prec k$, $A(k', k) \succ 0$. Without loss of generality, we assume that $k \in A$. Choose $k' \in A$ such that $k_0 + k - H \prec k' \prec k$. Then

$$\begin{aligned}
&(2A)(0, 2H) \\
&\quad \succeq (2A)(0, 2k) + (2A)(k_0 + k, H + k') + (2A)(H + k', H + k) + (2A)(H + k_0, 2H) \\
&\quad \succeq 3A(0, k) + |A[k_0 + k - k', H] + k'| + |A[H + k' - k, H] + k| + 2A(k_0, H) \\
&\quad \succ 3A(0, k) + \frac{1}{2}(H - k_0 - k + k') + \frac{1}{2}(k - k') + 2A(k_0, H) \\
&\quad \sim 3A(0, k_0) + \frac{1}{2}(H - k_0) + 2A(k_0, H) \\
&\quad \sim 3A(0, H),
\end{aligned}$$

which contradicts (3). $\square$(Lemma 2.6)

Lemma 2.7 *Let $A \subseteq [0, H]$ and $\alpha \leqslant \frac{1}{2}$ be such that*

(1) $0, 1 \in A$,

(2) $A(0, H) \sim \alpha H$,

(3) for any $0 \prec x \leqslant H$, $A(0, x) \preceq \alpha x$,

(4) for any $0 \prec x \leqslant 2H$, $(2A)(0, x) \preceq \frac{3}{2}\alpha x$,

(5) there is no $d \in \mathbb{N}$ *such that* $\alpha = \frac{2}{d}$ *and* $A \subseteq \{nd : 0 \leqslant n \leqslant \frac{H}{d}\} \cup \{1 + nd : 0 \leqslant n \leqslant \frac{H}{d}\}$.

Then A contains a crowded triple.

Proof: Suppose that A satisfies (1)–(5) and A does not contain a crowded triple. We derive a contradiction.

Let $P = \{x \in A : x + 1 \in A \text{ or } x - 1 \in A\}$ be the set of all paired numbers in A.

Claim 2.7.1 $|A \smallsetminus P| \sim 0$.

Proof of Claim 2.7.1: Let $Q = A \smallsetminus P$. Since A contains no three consecutive numbers, then

$$|Q + \{0, 1\}| = 2|Q|,$$

$$|P + \{0, 1\}| = \frac{3}{2}|P|,$$

$$\text{and } (Q + \{0, 1\}) \cap (P + \{0, 1\}) = \emptyset.$$

Hence

$$\begin{aligned}&(2A)(0,H)\\&\succeq |Q+\{0,1\}|+|P+\{0,1\}|\\&=2|Q|+\frac{3}{2}|P|\\&=\frac{3}{2}A(0,H)+\frac{1}{2}|Q|.\end{aligned}$$

By (2) and (4), one has $|Q|\sim 0$. □(Claim 2.7.1)

By Claim 2.7.1, one can assume that A contains no isolated points, *i.e.* $A=P$. Since A contains no crowded triple, then for any $x,y\in A$, $|x-y|\neq 2$, *i.e.* every gap of A has the length at least 2. Note also that for any $0\leqslant x\prec y\leqslant H$, we have $A(x,y)\preceq \frac{1}{2}(y-x)$ because every subinterval of length 4 can contain at most two elements from A. Let A_0 contain all even numbers in A and A_1 contain all odd numbers in A. Since A contains only pairs of two consecutive numbers, then $|A_0|=|A_1|$. Let $A_0^+=\{x\in A_0 : x+1\in A_1\}$ and let $A_0^-=A_0\smallsetminus A_0^+$. Note that $0\in A_0^+$.

Claim 2.7.2 For any $0\preceq x\prec y\preceq H$, if $A(x,y)\succ 0$, then $A[x,y]\cap A_0^+\neq\emptyset$.

Proof of Claim 2.7.2: Suppose the claim is not true and let $a=\max A_0^+[0,x]$. Without loss of generality, one can assume $x=\min A_0^-[a,H]$ and $y\in A_0^-$. Let

$$B=A_0^-[x,y]\cup(A_0^-[x,y]+y).$$

Then

$$B+\{-2,-1,0\}\subseteq 2A \text{ and } |B+\{-2,-1,0\}|=3|B|.$$

For each $z\in A_0^-[x,y]$, if $a+z\in B$, then let $f(z)=a+1+z$, and if $a+z\notin B$, then let $f(z)=a+z-1$. Since $a+1\in A$ and $z-1\in A$, then

$$f[A_0^-[x,y]]\subseteq(2A)\smallsetminus(B+\{-2,-1,0\}).$$

Hence

$$\begin{aligned}&(2A)(2a,2y)\\&\succeq |B+\{-2,-1,0\}|+|f[A_0^-[x,y]]|\\&=3|B|+A_0^-(x,y)\\&\sim 7A_0^-(x,y)\sim 3.5A(x,y)\sim 3.5A(a,y)\\&\succ 3A(a,y).\end{aligned}$$

By Lemma 2.5, $(2A)(0, 2H) \succ 3A(0, H)$, which contradicts (4). $\square$(Claim 2.7.2)

By Claim 2.7.2, $\max A_0^+ \sim H$. Note that the existence of $a = \max A_0^+[0, x]$ depends on the fact $A_0^+[0, x] \neq \emptyset$ (because $0 \in A_0^+$). However, a parallel proof can be given if one has $A_0^+[y, H] \neq \emptyset$ instead of $A_0^+[0, x] \neq \emptyset$ and let $a = \min A_0^+[y, H]$. So the proof needs only the condition $A_0^+ \neq \emptyset$ instead of $0 \in A_0^+$. By the same argument, one can show that if $A_0^- \neq \emptyset$, then $A(x, y) \succ 0$ implies $A_0^-[x, y] \neq \emptyset$. So $\max A_0^- \sim H$ and if $b = \min A_0^-$, then $A(0, b-1) \sim 0$.

Claim 2.7.3 If $A_0^- \neq \emptyset$, then $A_0^-(0, H) \sim A_0^+(0, H)$.

Proof of Claim 2.7.3: Let $c = \min A_0^-$ and $b = \max A_0^+$. Let

$$B^+ = A_0^+ \cup (A_0^+ + b) \text{ and } B^- = A_0^- \cup (A_0^- + b).$$

Let $B = B^+ \cup B^-$. Then

$$B^+ + \{0, 1, 2\} \subseteq 2A,$$

$$B^- + \{-1, 0, 1\} \subseteq 2A,$$

$$|B^+ + \{0, 1, 2\}| = 3|B^+|,$$

$$\text{and } |B^- + \{-1, 0, 1\}| = 3|B^-|.$$

Let

$$C = (B^+ + \{0, 1, 2\}) \cup (B^- + \{-1, 0, 1\}).$$

Then $|C| \sim 3|A|$.

Suppose $z \in A_0^+$ and $c + z \notin B^-$. If $c + z \notin C$, let $f(z) = z + c$. If $c + z \in B^+$, then let $f(z) = z + c - 1$. If $z + c \in B^+ + 2$, then let $f(z) = z + 1 + c$. It is easy to see that

$$f[\{z \in A_0^+ : z + c \notin B^-\}] \cap C = \emptyset.$$

So by (4),

$$|\{z \in A_0^+ : z + c \notin B^-\}| \sim 0.$$

Hence

$$|A_0^+| \preceq B^-(c, b) + B^-(b, b + c) \sim |A_0^-|.$$

By a similar argument, one can show

$$|\{z \in A_0^- : z + c \notin B^+ + 2\}| \sim 0.$$

Hence one has $|A_0^-| \preceq |A_0^+|$. $\square$(Claim 2.7.3)

Claim 2.7.4 If $A_0^- \neq \emptyset$, then A_0^+ is a subset of an *a.p.* of length $\sim |A_0^+|$ with difference d and A_0^- is a subset of an *a.p.* of length $\sim |A_0^-|$ with difference d.

Proof of Claim 2.7.4: We use the notation from the last claim. If

$$z \in (A_0^- + A_0^-) \setminus (B^+ + 2),$$

then either

$$z \notin C, \text{ or } z - 1 \notin C, \text{ or } z - 2 \notin C.$$

Since $|C| \sim 3|A|$, then

$$|(A_0^- + A_0^-) \setminus (B^+ + 2)| \sim 0.$$

So

$$|(A_0^- + A_0^-)| \preceq |B^+| \preceq 2|A_0^+| \sim 2|A_0^-|.$$

By Lemma 1.6, A_0^- is a subset of an *a.p.* of length $\sim |A_0^-|$ with difference d.

By a similar argument, one can show

$$|(A_0^+ + A_0^+) \setminus B^+| \sim 0.$$

Hence

$$|(A_0^+ + A_0^+)| \preceq 2|A_0^+|$$

and this implies that A_0^+ is a subset of an *a.p.* of length $\sim |A_0^+|$ with a difference $d' \in \mathbb{N}$.

Clearly, $d'A_0^+(0, H) \sim H$ since $0 \in A_0^+$ and $\max A_0^+ \sim H$. And since $A_0^+(0, x) \succ 0$ for any $x \succ 0$ by the fact that A_0^+ is almost an *a.p.* (A_0^+ is a subset of an *a.p.* of the length $\sim A_0^+$), then $\min A_0^- \sim 0$ by the comments after Claim 2.7.2. Hence $dA_0^-(0, H) \sim H$. This shows $d = d'$. $\square$(Claim 2.7.4)

Claim 2.7.5 $A_0^- = \emptyset$.

Proof of Claim 2.7.5: Assume the claim is not true. By Claim 2.7.4,

$$A_0^+ \subseteq \{dn : 0 \leqslant n \leqslant H/d\}$$

and

$$A_0^- \subseteq \{b + dn : 0 \leqslant n \leqslant H/d\}$$

for some $0 < b < d$. It is easy to see that $\frac{1}{d} = \frac{\alpha}{4}$ because

$$|A_0^+| \sim |A_0^-| \sim \frac{1}{2}|A_0| \sim \frac{1}{4}|A|.$$

Since there are

$$z \in (A_0^- + A_0^-) \cap (B^+ + 2),$$

then $2b \equiv 2 \pmod{d}$. Note that A_0 is a set of even numbers, then d is even. Hence $b \equiv 1 \pmod{\frac{d}{2}}$. This shows

$$A_0^- - 1 \subseteq \{\frac{d}{2}n : 0 \leqslant n \leqslant \frac{2H}{d}\}.$$

Hence

$$(A_0^- - 1) \cup A_0^+ \subseteq \{\frac{d}{2}n : 0 \leqslant n \leqslant \frac{2H}{d}\}$$

and

$$A_0^- \cup (A_0^+ + 1) \subseteq \{1 + \frac{d}{2}n : 0 \leqslant n \leqslant \frac{2H}{d}\}.$$

Now we have

$$\begin{aligned} A &= (A_0^- + \{-1, 0\}) \cup (A_0^+ + \{0, 1\}) \\ &\subseteq \{\frac{d}{2}n : 0 \leqslant n \leqslant \frac{2H}{d}\} \cup \{1 + \frac{d}{2}n : 0 \leqslant n \leqslant \frac{2H}{d}\} \end{aligned}$$

and $\alpha = \frac{2}{d/2}$, which contradicts (5) with d replaced by $d/2$. □(Claim 2.7.5)

Now we can assume that $A_0 = A_0^+$ and derive the contradiction. Let $b = \max A_0$ and let $B = A_0 \cup (A_0 + b)$. Then

$$B + \{0, 1, 2\} \subseteq 2A$$

and

$$|B + \{0, 1, 2\}| = 3|B| \sim 6|A_0| = 3A(0, H).$$

For each $x \in A_0 + A_0$, if $x \notin B$, then

$$x + 1 \notin B + \{0, 1, 2\}.$$

Hence

$$|A_0 + A_0| \preceq |B| = 2|A_0|.$$

By Lemma 1.6, A_0 is an *a.p.* of length $\sim |A_0|$ with difference d. Clearly,

$$A \subseteq \{dn : 0 \leqslant n \leqslant \frac{H}{d}\} \cup \{1 + dn : 0 \leqslant n \leqslant \frac{H}{d}\}$$

and $\alpha = \frac{2}{d}$, which contradict (5). $\Box$(Lemma 2.7)

The next lemma is a weak version of Kneser's theorem in nonstandard analysis. In order to state the lemma, some notation needs to be introduced.

An infinite initial segment U of $^*\mathbb{N}$ is a <u>cut</u> if $\mathbb{N} \subseteq U$ and $U + U \subseteq U$. A cut is usually an external set. For example, $\mathbb{N}$ is a cut. The set

$$U_H = \bigcap_{n \in \mathbb{N}} [0, [H/n]]$$

is a cut. In the next lemma and in the next section write $U = U_H$ H is clearly given.

Suppose U is a cut such that $U \subseteq D \subseteq {}^*\mathbb{N}$. Given a function $f : D \mapsto {}^*\mathbb{R}$ (not necessarily internal) bounded by a standard real number, the lower U–density of f is defined as the following:

$$\underline{d}_U(f) = \sup\{\inf\{\mathrm{st}(f(n)) : n \in U \smallsetminus [0, m]\} : m \in U\}.$$

A set $C \subseteq {}^*\mathbb{N}$ is called U–<u>internal</u> if for any $m \in U$, the set $C[0, m]$ is internal. Note that if $A \subseteq [0, H]$ is internal, then A is U–internal. Given a set $A \subseteq [0, H]$, let $f_A(x) = \frac{A(0,x)}{x+1}$ for any $x \in [0, H]$. The lower U–density of A is defined as

$$\underline{d}_U(A) = \underline{d}_U(f_A).$$

For any $x \in {}^*\mathbb{N} \smallsetminus U$, define also

$$\underline{d}_{x+U}(A) = \underline{d}_U((A - x) \cap {}^*\mathbb{N})$$

and

$$\underline{d}_{x-U}(A) = \underline{d}_U((x - A) \cap {}^*\mathbb{N}).$$

Remark 2.8 *(1) For any* $A \subseteq \mathbb{N}$, $\underline{d}(A) = \underline{d}_U(A)$ *with* $U = \mathbb{N}$, *where* $\underline{d}(A) = \liminf_{n\to\infty} \frac{A(n)}{n}$.

(2) $\underline{d}_{x+U}(A)$ *is a forward lower* U*–density of* A *from* x *and* $\underline{d}_{x-U}(A)$ *is a backward lower* U*–density of* A *from* x.

(3) It is easy to check that for any $a \in U$,

$$\underline{d}_U(A + a) = \underline{d}_U(A)$$

and

$$\underline{d}_U(A \smallsetminus [0, a]) = \underline{d}_U(A).$$

(4) Let $U \subseteq [0, H]$ be a cut and $A \subseteq [0, H]$. If $\underline{d}_U(A) > \gamma$, then by the overspill principle, there are $x \in U$ and $y \in [0, H] \smallsetminus U$ such that for any $x \leqslant z \leqslant y$, $\frac{A(0,z)}{z+1} > \gamma$.

Another definition needed is called e–transform (cf.[9, page 42]). It is also called τ–transformation (cf.[3, page 58]). Let $A, B \subseteq {}^*\mathbb{N}$ and $a \in A$. An e_a–transform of (A, B) is a pair $(A', B') = e_a(A, B)$ such that

$$A' = A \cup (B + a) \text{ and } B' = B \cap (A - a).$$

The important facts of the e_a–transform are the following:

(1) $A' \supseteq A$ and $B' \subseteq B$.

(2) $A' + B' \subseteq A + B$. Hence $\underline{d}_U(A' + B') \leqslant \underline{d}_U(A + B)$ when $a \in U$.

(3) $\underline{d}_U(f_{A'} + f_{B'}) = \underline{d}_U(f_A + f_B)$ when $a \in U$.

The essential idea of the proof of the next lemma can be found in [3, page 61]. The detailed proof is included here because a nonstandard setting is involved.

Lemma 2.9 *Let U be a cut and let $A \subseteq {}^*\mathbb{N}$ be such that $0 < \underline{d}_U(A) = \alpha \leqslant \frac{1}{2}$ and $A \cap U$ contains a crowded triple. Then $\underline{d}_U(2A) \geqslant \frac{8}{5}\alpha$.*

Proof: Let $T \subseteq A \cap U$ be the crowded triple. Without loss of generality, one can assume $0 = \min T$. By performing a sequence of e_a–transform with $a \in U$ to (A, A) one can produce a pair (A', B') such that A' contains four consecutive numbers $[a, a + 3]$ and

$$[a, a + 3] + B' \subseteq A'.$$

(Because A contains a crowded triple T, after one e–transform A' contains four consecutive numbers $[a, a+3]$. After four additional e–transforms A' contains $B' + [a, a+3]$.) It suffices now to show

$$\underline{d}_U(A' + B') \geqslant \frac{8}{5}\alpha.$$

For simplicity we assume $[0, 3] \subseteq A'$ and $[0, 3] + B' \subseteq A'$. Note that $B' \subseteq A'$.

Choose an arbitrary $0 < \gamma < 2\alpha \leqslant 1$. We want to show

$$\underline{d}_U(2A) \geqslant \underline{d}_U(A' + B') \geqslant \frac{4}{5}\gamma.$$

Since

$$\underline{d}_U(f_{A'} + f_{B'}) = \underline{d}_U(f_A + f_A) = 2\underline{d}_U(A) = 2\alpha > \gamma,$$

then there exists an $x_0 \in U$ such that for any $x \in U \setminus [0, x_0 - 1]$,

$$\frac{A'(0,x) + B'(0,x)}{x+1} > \gamma.$$

Let x_0 be the least such number. Then

$$\frac{A'(0,x_0-1) + B'(0,x_0-1)}{x_0} \leqslant \gamma$$

if $x_0 > 0$. This shows

$$\frac{A'(x_0,x_0+x) + B'(x_0,x_0+x)}{x+1} > \gamma$$

for any $x \geqslant 0$. It is easy to see that $x_0 \in A' \cup B' \subseteq A'$. One can assume $\underline{d}_U(B') > 0$ because otherwise

$$\underline{d}_U(A' + B') \geqslant \underline{d}_U(A') = \underline{d}_U(f_{A'} + f_{B'}) > \gamma > \frac{4}{5}\gamma.$$

Let

$$x_1 = \min(B' \cap (U \setminus [0, x_0 - 1]),$$

$$\bar{A} = (A' - x_0) \cap [0, H],$$

$$\text{and } \bar{B} = (B' - x_1) \cap [0, H].$$

It is easy to check that $0 \in \bar{A} \cap \bar{B}$ and

$$\underline{d}_U(\bar{A} + \bar{B}) = \underline{d}(A' + B').$$

Claim 2.9.1 For any $x \in U$, $1 + \bar{A}(x) + \bar{B}(x) \geqslant \frac{4}{5}\gamma(x+1)$.

Proof of Claim 2.9.1: The proof is divided into four cases.

Case 2.9.1.1 $x \geqslant \frac{5}{\gamma} - 1$.

Then $\gamma(x+1) \geqslant 5$ and hence

$$5\gamma(x+1) - 5 \geqslant 4\gamma(x+1).$$

So

$$\gamma(x+1) - 1 \geqslant \frac{4}{5}\gamma(x+1).$$

Since

$$\bar{A}(x) = A'(x_0, x_0 + x) - 1$$

and

$$\bar{B}(x) = B'(x_1 + 1, x_1 + x) \geqslant B'(x_0, x_1 + x) - 1 \geqslant B'(x_0, x_0 + x) - 1,$$

then

$$\begin{aligned}
&1 + \bar{A}(x) + \bar{B}(x) \\
&\quad \geqslant A'(x_0, x_0 + x) + B'(x_0, x_0 + x) - 1 \\
&\quad > \gamma(x+1) - 1 \geqslant \frac{4}{5}\gamma(x+1).
\end{aligned}$$

Case 2.9.1.2 $x < x_1 - x_0$.

Since $B'(x_0, x_0 + x) = 0$, then

$$\begin{aligned}
&1 + \bar{A}(x) + \bar{B}(x) \\
&\quad \geqslant A'(x_0, x_0 + x) = A'(x_0, x_0 + x) + B'(x_0, x_0 + x) \\
&\quad > \gamma(x+1) > \frac{4}{5}\gamma(x+1).
\end{aligned}$$

Case 2.9.1.3 $x_1 - x_0 \leqslant x \leqslant x_1 - x_0 + 3$.

Since $x_1 + [0, 3] = [x_1, x_1 + 3] \subseteq A'$, then

$$\begin{aligned}
&1 + \bar{A}(x) + \bar{B}(x) \\
&\quad \geqslant A'(x_0, x_1 - 1) + x - (x_1 - x_0) + B'(x_0, x_1 - 1) + 1 \\
&\quad > \gamma(x_1 - x_0) + x - (x_1 - x_0) + 1 \\
&\quad > \gamma(x_1 - x_0) + \gamma(x - (x_1 - x_0) + 1) \\
&\quad = \gamma(x+1) > \frac{4}{5}\gamma(x+1).
\end{aligned}$$

Case 2.9.1.4 $x_1 - x_0 + 3 \leqslant x < \frac{5}{\gamma} - 1$.

Then one has $\gamma(x+1) < 5$,which implies $\frac{4}{5}\gamma(x+1) < 4$. Hence

$$\begin{aligned}
&1 + \bar{A}(x) + \bar{B}(x) \\
&\quad \geqslant A'(x_1, x_1 + 3) \\
&\quad = 4 > \frac{4}{5}\gamma(x+1).
\end{aligned}$$

By the four cases above, one has that for any x

$$1+\bar{A}(x)+\bar{B}(x)>\frac{4}{5}\gamma(x+1).$$

By van der Corput's Theorem (cf.[3, Theorem 9, page 22]), one has

$$1+(\bar{A}+\bar{B})(x)>\frac{4}{5}\gamma(x+1).$$

This shows

$$\underline{d}_U(2A)\geqslant\underline{d}_U(\bar{A}+\bar{B})\geqslant\frac{4}{5}\gamma.$$

Since $\gamma<2\alpha$ is arbitrary, then one has $\underline{d}_U(2A)\geqslant\frac{8}{5}\alpha$. □(Lemma 2.9)

3 Proofs of the main theorem

In order to use nonstandard techniques, we first translate Theorem 1.3 into a non-standard equivalent. We translate the part II of Theorem 1.3 into Theorem 3.1 and part III of Theorem 1.3 into Theorem 3.2. We then prove Theorem 3.1 and Theorem 3.2.

Theorem 3.1 *Let $A\subseteq[0,H]$ and $0<\alpha<\frac{1}{2}$ be such that*

(1) $A\cap\mathbb{N}$ contains two consecutive numbers,

(2) $A(0,H)\sim\alpha H$,

(3) for any $0\prec x\leqslant H$, $A(0,x)\preceq\alpha x$,

(4) for any $0\prec x\leqslant 2H$, $(2A)(0,x)\preceq\frac{3}{2}\alpha x$.

Then

(a) either there are $a,d\in\mathbb{N}$ such that $\alpha=\frac{2}{d}$ and

$$A\subseteq\{a+dn:0\leqslant n\leqslant\frac{H}{d}\}\cup\{a+1+dn:0\leqslant n\leqslant\frac{H}{d}\}$$

(b) or there are $0\leqslant c\leqslant b\leqslant H$ such that $c\sim 0$, $[c,b]\cap A=\emptyset$, and $A(b,H)\sim H-b$.

Proof of Part II of Theorem 1.3 from Theorem 3.1 Let $A\subseteq\mathbb{N}$ contain two consecutive numbers and $0<\alpha<\frac{1}{2}$ such that $\bar{d}(A)=\alpha$ and $\bar{d}(2A)=\frac{3}{2}\alpha$. Take any increasing sequence $h_n\in\mathbb{N}$ such that $\lim_{n\to\infty}\frac{A(h_n)}{h_n}=\alpha$. Let K be a hyperfinite integer, let $H=h_K$, and let $B={}^*\!A\cap[0,H]$. We now check that (1)–(4) of Theorem 3.1 are true for B in the place of A.

(1) is trivially true. (2) is true because $\frac{A(h_n)}{h_n} \to \alpha$ implies $\frac{B(0,h_K)}{h_K} \approx \alpha$. (3) is true because otherwise $\bar{d}(A) > \alpha$ by Lemma 2.1. (4) is true because otherwise $\bar{d}(2A) > \frac{3}{2}\alpha$ by Lemma 2.1.

By Theorem 3.1, either (a) or (b) of Theorem 3.1 is true for B. If (a) of Theorem 3.1 is true, then there are $a, d \in \mathbb{N}$ such that $\alpha = \frac{2}{d}$ and

$$^*\!A \cap [0,H] \subseteq \{a + dn : 0 \leqslant n \leqslant \frac{H}{d}\} \cup \{a + 1 + dn : 0 \leqslant n \leqslant \frac{H}{d}\}.$$

So clearly (a) of Part II of Theorem 1.3 is true for A. If (b) of Theorem 3.1 is true, then by the underspill principle, there are $0 \leqslant c_n \leqslant b_n \leqslant h_n$ such that $\lim_{n\to\infty} \frac{c_n}{h_n} = 0$, $[c_n, b_n] \cap A = \emptyset$, and $\lim_{n\to\infty} \frac{A(b_n,h_n)}{h_n - b_n + 1} = 1$. □(Part II of Theorem 1.3)

Theorem 3.2 *Let $A \subseteq [0,H]$ be such that*

(1) $A \cap \mathbb{N}$ contains two consecutive numbers,

(2) $A(0,H) \sim \frac{1}{2}H$,

(3) for any $0 \prec x \leqslant H$, $A(0,x) \preceq \frac{1}{2}x$,

(4) for any $0 \prec x \leqslant 2H$, $(2A)(0,x) \preceq \frac{3}{4}x$.

Then either

(a) for some $a < 4$,

$$A \subseteq \{a + 4n : 0 \leqslant n \leqslant \frac{H}{4}\} \cup \{a + 1 + 4n : 0 \leqslant n \leqslant \frac{H}{4}\}, \text{ or}$$

(b) $(2A)(0,H) \sim A(0,H)$.

Proof of Part III of Theorem 1.3 from Theorem 3.2 Let $A \subseteq \mathbb{N}$ contain two consecutive numbers such that $\bar{d}(A) = \frac{1}{2}$ and $\bar{d}(2A) = \frac{3}{4}$. Take any increasing sequence $h_n \in \mathbb{N}$ such that $\lim_{n\to\infty} \frac{A(h_n)}{h_n} = \frac{1}{2}$. Let K be a hyperfinite integer, let $H = h_K$, and let $B = {}^*\!A \cap [0,H]$. By the same idea as in the proof above, we can check that (1)–(4) of Theorem 3.2 are true for B in the place of A.

Now it is easy to see that (a) of Theorem 3.2 for B implies (a) of Part III of Theorem 1.3 for A and (b) of Theorem 3.2 for B implies (b) of part III of Theorem 1.3 for A. □(Part III of Theorem 1.3)

Now we are ready to prove Theorem 3.1 and Theorem 3.2.

Proof of Theorem 3.1 Suppose $0 < \alpha < \frac{1}{2}$ and $A \subseteq [0,H]$ such that A satisfies (1)–(4) of Theorem 3.1. Suppose that A does not satisfy (a) of Theorem 3.1. We want to show that A satisfies (b) of Theorem 3.1.

In the proof, we show that $(2A)(0,2H) \succ 3A(0,H)$ unless A satisfies (b) of Theorem 3.1.

Without loss of generality, we assume that $0, 1 \in A$. By Lemma 2.7, A contains a crowded triple. Let $a = \max\{x : x \text{ is the least element of a crowded triple in } A\}$.

Claim 3.1.1 $a \succ 0$.

Proof of Claim 3.1.1: Let $T \subseteq A$ be the crowded triple with $a = \min T$. Suppose $a \sim 0$, then $A[a+3, H]$ does not contain any crowded triple. By (3) of Lemma 2.4,

$$|A[a+3, H] + T| \succeq 2A(a+3, H).$$

Hence

$$\begin{aligned}
&(2A)(0,H) \\
&\quad \succeq (2A)(2a+3, H+a+3) \\
&\quad \succeq |A[a+3, H] + T| \\
&\quad \succeq 2A(a+3, H) \sim 2A(0,H) \sim 2\alpha H.
\end{aligned}$$

This contradicts (4) of this theorem. □(Claim 3.1.1)

Claim 3.1.2 $A(0,a) \succ 0$.

Proof of Claim 3.1.2: Suppose $A(0,a) \sim 0$. Hence $A(0,H) \sim A(a,H)$.

If $2a < H$, then

$$\begin{aligned}
&(2A)(0, H+a) \\
&\quad \sim (2A)(0,H) + (2A)(H, H+a) \\
&\quad \succeq |A[0,H] + \{0,1\}| + |A[H-a, H] + T| \\
&\quad \succeq \frac{3}{2}A(0,H) + 2A(H-a, H) \\
&\quad \succeq \frac{3}{2}\alpha H + 2\alpha a \succ \frac{3}{2}\alpha(H+a),
\end{aligned}$$

which contradicts (4).

If $2a \geqslant H$, then

$$\begin{aligned}
&(2A)(0, H+a) \\
&\quad \sim (2A)(0,H) + (2A)(H, H+a) \\
&\quad \succeq |A[a+3, H] + \{0,1\}| + |A[H-a, H] + T|
\end{aligned}$$

$$\begin{aligned}
&\succeq \frac{3}{2}A(a+3,H) + |A[a+3,H]+T| \\
&\succeq \frac{3}{2}A(0,H) + 2A(a,H) \\
&\sim \frac{3}{2}A(0,H) + 2A(0,H) \\
&\succeq \frac{3}{2}\alpha H + 2\alpha H \succ \frac{3}{2}\alpha(H+a),
\end{aligned}$$

which again contradicts (4). □(Claim 3.1.2)

Claim 3.1.3 If $a \sim H$, then A satisfies (b) of Theorem 3.1.

Proof of Claim 3.1.3: There are two cases. The first case is when $\underline{d}_{H-U}(A) > \frac{1}{2}$ and the second case is when $\underline{d}_{H-U}(A) \leqslant \frac{1}{2}$. It is shown that only the first case can be true and if the first case is true, then A satisfies (b) of this theorem.

Case 3.1.3.1 $\underline{d}_{H-U}(A) > \frac{1}{2}$.

Then there exists a $K \prec H$ such that for any $K \leqslant x \prec H$, $A(x,H) \succ \frac{1}{2}(H-x)$. Hence by Lemma 2.6, A satisfies (b).

Case 3.1.3.2 $\underline{d}_{H-U}(A) \leqslant \frac{1}{2}$.

By (2) and (3), $\underline{d}_{H-U}(A) = \beta > 0$. Note that $A[x,H]$ contains a crowded triple for some $x \sim H$. By Lemma 2.9, one has $\underline{d}_{2H-U}(2A) \geqslant \frac{8}{5}\beta$. Let $1 < \delta < \sqrt{16/15}$. Then there is a $K \prec H$ in A such that

$$A(K,H) \leqslant \beta\delta(H-K)$$

and

$$(2A)(2K,2H) \geqslant \frac{8\beta}{5\delta}2(H-K).$$

Hence

$$(2A)(2K,2H) \geqslant \frac{8}{5\delta^2}2A(K,H) \succ 3A(K,H).$$

Note that $A(0,K) \preceq \frac{1}{2}K$ by (3) of this theorem. Now Lemma 2.5 implies $(2A)(0,2H) \succ 3A(0,H)$, which contradicts (4). □(Claim 3.1.3)

Now the only case left is $a \prec H$ with $A(0,a) \succ 0$. The next claim shows that this case does not occur.

Claim 3.1.4 If $a \prec H$ and $A(0,a) \succ 0$, then $(2A)(0,2H) \succ 3A(0,H)$, which contradicts (4).

The proof is divided into four cases. Assume $a \prec H$ and $A(0,a) \succ 0$. Since there is no crowded triple in $A[a+1,H]$, one has $A(a,H) \preceq \frac{1}{2}(H-a)$ by (1) of Lemma 2.4.

Case 3.1.4.1 $0 < \underline{d}_{a-U}(A) \leqslant \frac{1}{2}$.

By the same argument in Case 3.1.3.2, one can find an $x \prec a$ in A such that

$$(2A)(2x, 2(a+3)) \succ 3A(x, a+3).$$

Hence by Lemma 2.5, $(2A)(0,2H) \succ 3A(0,H)$.

Case 3.1.4.2 $\underline{d}_{a-U}(A) > \frac{1}{2}$.

If $(2A)(0,2(a+3)) \succ 3A(0,a+3)$, then by Lemma 2.5 one has $(2A)(0,2H) \succ 3A(0,H)$. So we can assume that

$$(2A)(0, 2(a+3)) \sim 3A(0, a+3).$$

It is now easy to check that $A[0,a+3]$ satisfies (1)–(4) of Lemma 2.6 in place of A. (1) is trivially true. (2) is true because of (3) of this theorem. (3) is true by the assumption $(2A)(0,2(a+3)) \sim 3A(0,a+3)$. (4) is true because of the assumption of this case. Hence by Lemma 2.6, there are $0 \leqslant c \leqslant b \leqslant a+3$ such that $c \sim 0$, $[c,b] \cap A = \emptyset$, and

$$A(b, a+3) \sim a+3-b.$$

Note that $2b \succ a$ because

$$\frac{1}{2}a \succ \alpha a \succeq A(0,a+3) \sim A(b,a+3) \sim a+3-b.$$

By Lemma 2.2, there is a $k \in [a+3,H]$ such that $A(a+3,k) \sim 0$ and for any $k \prec x \leqslant H$, $A(k,x) \succ 0$. One can choose $k \in A$.

Subcase 3.1.4.2.1 $k \sim a$.

Note that $a, a+1 \in A[a,H]$. Then one has

$$\begin{aligned}
&(2A)(0,2H) \\
&\quad \sim (2A)(0,b) + (2A)(b,a) + (2A)(a,2b) + (2A)(2b,2a) + (2A)(2a,2H) \\
&\quad \succeq A(0,b) + A(b,a) + A(a,\min\{2b,H\}) + 2A(b,a) + 3A(a,H) \\
&\quad \sim 3A(0,H) + A(a,\min\{2b,H\}) \succ 3A(0,H).
\end{aligned}$$

Subcase 3.1.4.2.2 $k \succ a$.

Let $T \subseteq A$ be the crowded triple with $a = \min T$. Note that

$$|(T \cup A[k,H]) + (T \cup A[k,H])| \succeq 3|(T \cup A[k,H])|$$

and

$$((T \cup A[k,H]) + (T \cup A[k,H])) \cap [2a+7, a+k-1] = \emptyset.$$

Then

$$\begin{aligned}
&(2A)(a+k, 2H) \\
&\quad \succeq |(T \cup A[k,H]) + (T \cup A[k,H])| - |T+T| \\
&\quad \succeq 3|(T \cup A[k,H])| \sim 3A(k,H).
\end{aligned}$$

Let $b \prec x \prec a$ be such that $a - x \leqslant k - a - 7$. Then

$$\begin{aligned}
&(2A)(0, 2H) \\
&\quad \succeq (2A)(b,a) + (2A)(2b,2a) + (2A)(k+x, k+a) + (2A)(k+a, 2H) \\
&\quad \succeq A(b,a) + 2A(b,a) + A(x,a) + 3A(k,H) \\
&\quad \sim 3A(0,H) + A(x,a) \succ 3A(0,H).
\end{aligned}$$

So from now on we assume $\underline{d}_{a-U}(A) = 0$ because of Case 3.1.4.1 and Case 3.1.4.2.

Case 3.1.4.3 $\underline{d}_{a+U}(A) = \beta > 0$.

Since $A[a+3, H]$ does not contain any crowded triple, by (1) of Lemma 2.4, one has $\beta \leqslant \frac{1}{2}$.

Subcase 3.1.4.3.1 for any $x \sim a$, $A[x,H]$ is not a subset of an *a.p.* of difference $d > 1$.

By the overspill principle, there exists a $a \prec K \prec H$ such that $A[K,H]$ is not a subset of an *a.p.* of difference $d > 1$. By Lemma 2.9, $\underline{d}_{2a+U}(2A) \geqslant \frac{8}{5}\beta$. Hence similar to Case 3.1.4.2, there exists a k with $a \prec k \leqslant K$ such that $(2A)(2a, 2k) \succ 3A(a,k)$. This shows $(2A)(0, 2H) \succ 3A(0,H)$ by Lemma 2.5.

Subcase 3.1.4.3.2 There is an $x \sim a$ such that $A[x,H]$ is a subset of an *a.p.* of difference $d > 1$.

Then one has that either

$$(A[x,H] + A[x,H]) \cap (A[x,H] + a') = \emptyset$$

$$\text{or } (A[x,H]+A[x,H])\cap(A[x,H]+a'+1)=\emptyset,$$

for any two consecutive numbers a' and $a'+1$. Since $\underline{d}_{a+U}>0$, one can assume $x\in A$. If $(2A)(2x,2H)\succ 2A(x,H)$, then

$$\begin{aligned}&(2A)(2a,2H)\\ &\quad\succeq A(x,H)+(2A)(2x,2H)\\ &\quad\succ 3A(x,H)\sim 3A(a,H)\end{aligned}$$

because T contains two consecutive numbers. Hence $(2A)(0,2H)\succ 3A(0,H)$.

So one can now assume that $(2A)(2x,2H)\preceq 2A(x,H)$. By Lemma 1.6, $A[x,H]$ is a subset of an *a.p.* of difference d of length $\sim A(x,H)$. This implies $A(x,H)\sim\frac{1}{d}(H-x)$.

If there is a $y\prec a$ with $y\in A$ such that $A(y,a)\sim 0$, then

$$\begin{aligned}&(2A)(0,2H)\\ &\quad\sim(2A)(0,2y)+(2A)(2y,2a)+(2A)(2a,2H)\\ &\quad\succeq 3A(0,y)+|A[y,\min\{2a-y,H\}]+y|+3A(a,H)\\ &\quad\succ 3A(0,H)\end{aligned}$$

because $2a-y\succ a$ and

$$A(a,2a-y)\sim\frac{1}{d}(a-y)\succ 0$$

when $2a-y\leqslant H$. Otherwise, choose a $y\prec a$ such that $y\in A$, $a-y<H-a$, and $A(y,a)\prec\frac{1}{4d}(a-y)$. Such a y exists because of $\underline{d}_{a-U}(A)=0$. Then

$$\begin{aligned}&(2A)(0,2H)\\ &\quad\sim(2A)(0,2y)+(2A)(2y,2a)+(2A)(2a,2H)\\ &\quad\succeq 3A(0,y)+|A[y,2a-y]+y|+3A(a,H)\\ &\quad\succeq 3A(0,y)+A(a,2a-y)+3A(a,H)\\ &\quad\sim 3A(0,y)+\frac{1}{d}(a-y)+3A(a,H)\\ &\quad\succ 3A(0,y)+4A(y,a)+3A(a,H)\succ 3A(0,H).\end{aligned}$$

Now the only case left is the following.

Case 3.1.4.4 $\underline{d}_{a+U}(A)=0$ and $\underline{d}_{a-U}(A)=0$.

This case will be divided into four subcases.

Subcase 3.1.4.4.1 There is a $c \succ a$ such that $A(a,c) \sim 0$.

By Lemma 2.2, c can be chosen so that for any $c \prec x \leqslant H$, $A(c,x) \succ 0$. Without loss of generality, we can choose $c \in A$. By Lemma 2.2, there is a $b \preceq a$ in A such that $A(b,a) \sim 0$ and for any $0 \leqslant x \prec b$, $A(x,b) \succ 0$. Now one has

$$\begin{aligned}
&(2A)(0,2H) \\
&\quad \sim (2A)(0,2b) + (2A)(2b,b+c) + (2A)(a+c,2H) \\
&\quad \succeq 3A(0,b) + |A[2b-c,b]+c| + |(T \cup A[c,H]) + (T \cup A[c,H])| \\
&\quad \succeq 3A(0,b) + A(2b-c,b) + 3A(c,H) \succ 3A(0,H).
\end{aligned}$$

Subcase 3.1.4.4.2 There is a $b \prec a$ with $A(b,a) \sim 0$.

Again b can be chosen so that for any $0 \leqslant x \prec b$, $A(x,b) \succ 0$. Now the proof is similar to the proof above. Assume $b \in A$. Then

$$\begin{aligned}
&(2A)(0,2H) \\
&\quad \sim (2A)(0,2b) + (2A)(2b,2a) + (2A)(2a,2H) \\
&\quad \succeq 3A(0,b) + |A[\max\{2b-a,0\},a]+a| + 3A(a,H) \\
&\quad \succeq 3A(0,H) + A(\max\{2b-a,0\},b) \succ 3A(0,H).
\end{aligned}$$

So we can now assume that for any $x \prec a \prec y$, $A(x,a) \succ 0$ and $A(a,y) \succ 0$.

Subcase 3.1.4.4.3 For any $c \sim a$, the set $A[c,H]$ is not a subset of an *a.p.* of difference $d > 1$.

By the overspill principle, one can choose a $c \succ a$ small enough such that $2c - a < H$ and $A[c,H]$ is not a subset of an *a.p.* of difference $d > 1$. Furthermore, one can require that $A(a,c) \preceq A(c,2c-a)$. This can be done by the following steps. Suppose the original c does not work. First, let $0 < \epsilon < \frac{A(a,c)}{c-a}$. Let $a \prec x \leqslant c$ be the smallest number such that $\frac{A(a,x)}{x-a} > \epsilon$ (x exists because of $\underline{d}_{a+U}(A) = 0$). Then re-define $c = [\frac{x+a}{2}]$. It is easy to see the new c is the number we want. Hence

$$\begin{aligned}
&(2A)(0,2H) \\
&\quad \sim (2A)(0,2a) + (2A)(2a,2c) + (2A)(2c,2H) \\
&\quad \succeq 3A(0,a) + |A[a,2c-a]+T| + 3A(c,H) \\
&\quad \succeq 3A(0,a) + 2A(a,2c-a) + 3A(c,H) \\
&\quad \succeq 3A(0,a) + 4A(a,c) + 3A(c,H) \succ 3A(0,H).
\end{aligned}$$

Subcase 3.1.4.4.4 There is a $c \sim a$ such that $A[c, H]$ is a subset of an *a.p.* of difference $d > 1$.

One can assume $c \in A$ by the fact that $A(a, y) \succ 0$ for any $a \prec y \leqslant H$. If $(2A)(2c, 2H) \succ 2A(c, H)$, then

$$(2A)(2a, 2H) \succ 3A(c, H) \sim 3A(a, H)$$

because T contains two consecutive numbers. Hence $(2A)(0, 2H) \succ 3A(0, H)$. Therefore, one can assume

$$(2A)(2c, 2H) \sim 2A(c, H).$$

By Lemma 1.6, $A[c, H]$ is a subset of an *a.p.* of length $\sim A(c, H)$ with difference

$$d \approx \frac{H - c}{A(c, H)}.$$

This implies that

$$A(c, x) \sim \frac{1}{d}(x - c)$$

for any $c \prec x \leqslant H$. Hence one has $\underline{d}_{a+U}(A) = \frac{1}{d}$, which contradicts $\underline{d}_{a+U}(A) = 0$ as assumed in this case. $\square$(Claim 3.1.4)

The theorem now follows from above four claims. $\square$(Theorem 3.1)

Proof of Theorem 3.2:

If $\alpha < \frac{1}{2}$ in the proof of Theorem 3.1 is replaced by $\alpha = \frac{1}{2}$, everything still holds except the proof of Lemma 2.6, which is used in Case 3.1.3.1 and Case 3.1.4.2.

In Lemma 2.6, the inequality $\alpha < \frac{1}{2}$ is used is to guarantee that $k_0 \succ 0$. When $\alpha = \frac{1}{2}$, Case 2.6.1 cannot occur. Under Case 2.6.2, one can still derive a contradiction using the same proof. Hence $k_0 \sim 0$ must be true. If $k_0 \sim 0$, then it is easy to see that $(2A)(H, 2H) \sim H$. Hence $(2A)(0, 2H) \sim \frac{3}{2}H$ implies $(2A)(0, H) \sim \frac{1}{2}H$. So $(2A)(0, H) \sim A(0, H)$ is true. This shows that in Claim 3.1.3, if $\alpha = \frac{1}{2}$, then the conclusion needs to be changed to $(2A)(0, H) \sim A(0, H)$, which is (b) of Theorem 3.2.

In Case 3.1.4.2, one needs to derive a contradiction when $\alpha = \frac{1}{2}$. In fact if $\alpha = \frac{1}{2}$, then the condition of Claim 3.1.4 cannot occur. Let $a = \max\{x \in A :$ x is the least element of a crowded triple in $A\}$ and let $T \subseteq A$ be the crowded triple with $a = \min T$. If $0 \prec a \prec H$, then $A[a + 3, H]$ contains no crowded triple. Hence

$$A(a + 3, H) \preceq \frac{1}{2}(H - a - 3),$$

which implies $A(0,a) \sim \frac{1}{2}a$, and

$$|A[a+3,H]+T| \succeq 2A(a+3,H).$$

So

$$\begin{aligned}
&(2A)(0,H+a)\\
&\quad\succeq (2A)(0,2a)+(2A)(2a,a+H)\\
&\quad\succeq 3A(0,a)+2A(a,H)\\
&\quad\sim 2A(0,H)+A(0,a)\\
&\quad\succeq H+\frac{1}{2}a\\
&\quad=\frac{3}{4}(H+a)+\frac{1}{4}(H-a)\\
&\quad\succ \frac{3}{4}(H+a).
\end{aligned}$$

This contradicts (4) of Theorem 3.2. $\square$(Theorem 3.2)

Next we present a corollary to Theorem 1.3. Let

$$Q=\{\frac{2}{k}:k\geqslant 4\}.$$

Corollary 3.3 *Suppose A contains two consecutive numbers and $\bar{d}(A)=\alpha<1$. If $\alpha\notin Q$ and*

$$\bar{d}(2A)=\min\{\bar{d}(2B): B \textit{ contains two consecutive numbers and } \bar{d}(B)\geqslant\alpha\},$$

then $\underline{d}(A)=0$.

Proof: Since $\alpha\notin Q$, then A cannot have the structure described in the conclusion of Part II (a) or Part III (a) of Theorem 1.3. If A has the structure described in the conclusion of Part II (b) of Theorem 1.3, then it is easy to see that $\lim_{n\to\infty}\frac{A(b_n)}{b_n}\leqslant \lim_{n\to\infty}\frac{c_n}{b_n}=0$. Hence $\underline{d}(A)=0$. Suppose that A has the structure described in the conclusion of Part I or Part III (b). Suppose $\underline{d}(A)=\beta>0$. Then $\underline{d}_U({}^*\!A)=\gamma\geqslant\beta$, where $U=U_H$ and H is the hyperfinite integer chosen in Theorem 3.1.

If $\gamma>\frac{1}{2}$, then there is $0\prec a\prec H$ such that for any x with $0\prec x\leqslant a$, one has ${}^*\!A(0,x)\succ\frac{1}{2}x$. Hence by Pigeonhole principle $(2\,{}^*\!A)(0,a)\sim a$. This implies $\bar{d}(2A)=1$, which contradicts the fact that $\bar{d}(2A)=\frac{1+\alpha}{2}<1$.

If $0 < \beta \leqslant \gamma \leqslant \frac{1}{2}$, then by Lemma 2.9 one has $\underline{d}_U(2\,{}^*\!A) \geqslant \frac{8}{5}\gamma$. Hence there is an a with $0 \prec a \leqslant H$ such that ${}^*\!A(0,a) \leqslant \frac{11}{10}\gamma a$ and $(2\,{}^*\!A)(0,a) \geqslant \frac{15}{10}\gamma a$. This contradicts the condition $(2\,{}^*\!A)(0,H) \sim {}^*\!A(0,H)$. $\square$(Corollary 3.3)

If $\alpha \in Q$, then the corollary is no longer true. The following is an example.

Example 3.4 *Let $\alpha \in Q$. Note that $\alpha \leqslant \frac{1}{2}$. Fix any $0 \leqslant \beta \leqslant \alpha$. Let*

$$B = \{kn : n \in \mathbb{N}\} \cup \{1 + kn : n \in \mathbb{N}\}.$$

Then $\bar{d}(B) = \alpha$. If $\beta = 0$, then let

$$A = B \setminus \bigcup_{n \in \mathbb{N}} [2^{2^{2n}}, 2^{2^{2n+1}}],$$

and if $0 < \beta \leqslant \alpha$ then let

$$A = B \setminus \bigcup_{n \in \mathbb{N}} [[\frac{\beta}{\alpha} 2^{2^n}], 2^{2^n}].$$

It is easy to check that A contains two consecutive numbers, $\bar{d}(A) = \alpha$, $\bar{d}(2A) = \frac{3}{2}\alpha$, and $\underline{d}(A) = \beta$.

4 An Unsolved Case

We end this paper by asking the following question. A less vigorous version of the question is also asked in [7].

Question 4.1 *Is it true that if $\bar{d}(A) = \alpha < \frac{1}{2}$ and $\gcd(A - \min A) = 1$, then $\bar{d}(2A) = \frac{3}{2}\alpha$ implies that either (1) there exist k, c, c' such that $\alpha = \frac{2}{k}$ and*

$$A \subseteq \{c + ik : i \in \mathbb{N}\} \cup \{c' + ik : i \in \mathbb{N}\}$$

or (2) for any increasing sequence $\langle h_n : n \in \mathbb{N}\rangle$ with $\lim_{n\to\infty} \frac{A(0,h_n)}{h_n+1} = \alpha$, there exist two sequences $0 \leqslant c_n \leqslant b_n \leqslant h_n$ such that

$$\lim_{n\to\infty} \frac{A(b_n, h_n)}{h_n - b_n + 1} = 1,$$

$$\lim_{n\to\infty} \frac{c_n}{h_n} = 0,$$

and $[c_n, b_n] \cap A = \emptyset$ for every $n \in \mathbb{N}$?

We conjecture that the answer is "yes".

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