The Differences Between Kurepa Trees And Jech–Kunen Trees [1]

Renling Jin

Abstract

By an ω_1–tree we mean a tree of power ω_1 and height ω_1. An ω_1–tree is called a Kurepa tree if all its levels are countable and it has more than ω_1 branches. An ω_1–tree is called a Jech–Kunen tree if it has κ branches for some κ strictly between ω_1 and 2^{ω_1}. In §1, we construct a model of *CH* plus $2^{\omega_1} > \omega_2$, in which there exists a Kurepa tree with no Jech–Kunen subtrees and there exists a Jech–Kunen tree with no Kurepa subtrees. This improves two results in [Ji1] by not only eliminating the large cardinal assumption for [Ji1, Theorem 2] but also handling two consistency proofs of [Ji1, Theorem 2 and Theorem 3] simultaneously. In §2, we first prove a lemma saying that an *Axiom A* forcing of size ω_1 over Silver's model will not produce a Kurepa tree in the extension, and then we apply this lemma to prove that in the model constructed for Theorem 2 in [Ji1], there exists a Jech–Kunen tree and there are no Kurepa trees.

0 Introduction

The first model in which there is a Jech–Kunen tree was probably discovered by T. Jech [Je1] in 1971. In fact, the tree in that model is a Kurepa tree with less than 2^{ω_1} branches. Later, in 1975, K. Kunen [K1] showed that, under *CH* and $2^{\omega_1} > \omega_2$, the existence of a Jech–Kunen tree is equivalent to the existence of a compact Hausdorff space with weight ω_1 and cardinality strictly between ω_1 and 2^{ω_1}. Such a space is interesting because a compact Hausdorff space with weight ω cannot have cardinality strictly between ω and 2^{ω}. Let us also look at the natural correspondence between a tree and a linearly ordered set (see [T] for the details). Assuming *CH* and $2^{\omega_1} > \omega_2$, we can easily see that the existence of a Jech–Kunen tree is equivalent to the existence of a (Dedekind) complete dense linear order with density ω_1 and cardinality strictly between 2^{ω_1} and ω_1. Note that a separable complete dense linear order is order–isomorphic to an interval of the real line and therefore has cardinality 2^{ω}. K. Kunen proved also that the non–existence of Jech–Kunen trees under *CH* and $2^{\omega_1} > \omega_2$ is equiconsistent to the existence of an inaccessible cardinal (consult [Ju, Theorem 4.8] for details). Kunen's model of non–existence of Jech–Kunen trees is a slight

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modification of J. Silver's model (see [K2]) for non–existence of Kurepa trees. It is easy to show that there are no Kurepa trees in Kunen's model by imitating Silver's argument. Since in Jech's model both Kurepa trees and Jech–Kunen trees exist and in Kunen's model neither of them exist, it is natural to ask whether or not there are any differences between the existences of both trees.

In [Ji1], the following results were proved.

(1) Assuming the consistency of an inaccessible cardinal, it is consistent with *CH* and $2^{\omega_1} > \omega_2$ that there exists a Jech–Kunen tree which has no Kurepa subtrees [Ji1, Theorem 2].

(2) It is consistent with *CH* and $2^{\omega_1} > \omega_2$ that there exists a Kurepa tree which has no Jech–Kunen subtrees [Ji1, Theorem 3].

(3) *CH* and $2^{\omega_1} > \omega_2$ plus Generalized Martin's Axiom (see [B] and [W] for the definition) imply that every Jech–Kunen tree has a Kurepa subtree [Ji1, Theorem 4].

(4) It is consistent with *CH* and $2^{\omega_1} > \omega_2$ plus Generalized Martin's Axiom that there exists a Kurepa tree with 2^{ω_1} branches and every Kurepa tree has a Jech–Kunen subtree [Ji1, Theorem 5].

In [Ji2], I proved that assuming the consistency of two inaccessible cardinals, it is consistent with *CH* and $2^{\omega_1} > \omega_2$ that there exist Kurepa trees and there are no Jech–Kunen trees.

In [SJ1], we proved that assuming the consistency of an inaccessible cardinal, it is consistent with *CH* and $2^{\omega_1} > \omega_2$ that there exist Jech–Kunen trees and there are no Kurepa trees.

In [SJ2], we proved that one inaccessible cardinal is enough to prove the result of [Ji2].

Since we need an inaccessible cardinal to destroy all Kurepa trees or all Jech–Kunen trees, the assumption of one inaccessible cardinal is necessary to prove the results of [SJ1] and [SJ2]. But we may not need large cardinal to destroy all Kurepa subtrees of a particular Jech–Kunen tree and to destroy all Jech–Kunen subtrees of a particular Kurepa tree. For example, the second result of [Ji1] mentioned above does not use large cardinals. It is now natural to ask whether or not the large cardinal assumption for [Ji1, Theorem 2] can be eliminated. In §1 of this paper, we give a positive answer to that question by not only eliminating the large cardinal assumption but also proving Theorem 2 and Theorem 3 of [Ji1] simultaneously.

Let us look at the model used in the proof of Theorem 2 of [Ji1], in which we use an inaccessible cardinal to kill all Kurepa subtrees of a particular Jech–Kunen tree. Although we believed that there are no Kurepa trees at all in that model, we were not able to prove that until now. In searching for a model in which there exist Jech–Kunen trees and there are no Kurepa trees, Shelah and I in [SJ1] used a different approach. The construction of the model in [SJ1] is more complicated than the construction of the model for Theorem 2 of [Ji1]. In §2 of this paper, we are going to prove that there are no Kurepa trees in the model we constructed for Theorem 2 of [Ji1]. In fact, it is an easy corollary of an interesting lemma we are going to prove. The lemma says that if we take Silver's model (collapse all cardinals between ω_1 and an inaccessible cardinal κ by a countable support Lévy collapsing order (see [K2]) as our ground model, then forcing with any partial order of size at most ω_1 which satisfies Baumgartner's *Axiom A* (see [B]) will never create Kurepa trees. This lemma suggests another interesting question, which I will pose at the end of this paper.

Our notation is fairly standard. We refer the reader to [K2] or [Je2] for set theory. It is helpful for the reader to have copies of [Ji1] and [SJ1] in hand. We write $\dot{a}$ as a name for a and $\ddot{a}$ as a name for $\dot{a}$ in forcing arguments. We always assume that M is a countable transitive model of *ZFC*. By a forcing notion $\mathbb{P}$ we mean that $\mathbb{P}$ is a partial order with a largest element $1_{\mathbb{P}}$. We assume, without loss of generality, that all trees considered in this paper are subtrees of $2^{<\omega_1}$ ordered by reverse inclusion (all trees grow downward). For a tree T, let T_α be the α–th level of T and write $ht(t) = \alpha$ if $t \in T_\alpha$, let $T \restriction \alpha$ be the union of all T_β's for $\beta < \alpha$, and let $ht(T)$, the height of T, be the least ordinal α such that $T_\alpha = \emptyset$. For any tree T, $\emptyset$ is the unique root of T. We call a totally ordered subset B of T a branch of T if B intersects every nonempty level of T. Let $\mathcal{B}(T)$ be the set of all branches of T.

1 A Kurepa (Jech–Kunen) tree without Jech–Kunen (Kurepa) subtrees

In this section we are going to construct, without using large cardinals, a model of *CH* and $2^{\omega_1} > \omega_2$, in which there exists a Kurepa tree with no Jech–Kunen subtrees and there exists a Jech–Kunen tree with no Kurepa subtrees.

We call T a normal tree if the following two conditions are true.

(1) For every $t \in T$, both $t\hat{}\langle 0\rangle$ and $t\hat{}\langle 1\rangle$ are in T.

(2) For every $t \in T_\alpha$ and every β such that $\alpha < \beta < ht(T)$, there exists a $t' \in T_\beta$ such that $t' <_T t$.

(3) For every strictly decreasing sequence in T, there is at most one maximal lower bound in T.

Let κ be a regular cardinal. We let $\mathbb{K}_\kappa$ be the forcing notion

> $\{p : p = \langle A_p, l_p\rangle$ where A_p is a countable normal tree of height $\alpha_p + 1$ for some countable ordinal α_p and l_p is a one to one function from some countable subset of κ onto the α_p–th level of $A_p\}$

with the order defined as follows: for any $p = \langle A_p, l_p\rangle$ and $q = \langle A_q, l_q\rangle$ in $\mathbb{K}_\kappa$, $p \leq q$ if and only if $A_p \restriction \alpha_q + 1 = A_q$, $dom(l_q) \subseteq dom(l_p)$ and for every $\xi \in dom(l_q)$, $l_q(\xi) \subseteq l_p(\xi)$. $\mathbb{K}_\kappa$ is the forcing notion used in [Je1] to force a Kurepa tree. Assuming *CH*, it is easy to show that $\mathbb{K}_\kappa$ is ω_1–closed and has ω_2–*c.c.*.

Lemma 1 (K. Kunen) *Let M be a model of GCH, and let $\kappa > \omega_2$ be an uncountable regular cardinal in M. Suppose G is a $\mathbb{K}_\kappa$–generic filter over M and $T_G = \bigcup_{p\in G} A_p$. Then in $M[G]$, CH holds, $2^{\omega_1} = \kappa > \omega_2$, T_G is a Kurepa tree with κ branches and T_G has no Jech–Kunen subtrees.*

Proof: See [Ji1, Theorem 3]. $\square$

Let T be a tree, let I be an index set. For a function p from I to T we use $supt(p)$, the support of p, for the set $\{i \in I : p(i) \neq \emptyset\}$. Let $\mathbb{P}(T, I)$ be the forcing notion

$$\{p : p \text{ is a function from } I \text{ to } T \text{ such that } |supt(p)| < \omega_1\}$$

with the order defined by the following: for any p and q in $\mathbb{P}(T, I)$, $p \leq q$ if and only if for every $i \in I$, $p(i) \leq_T q(i)$. Note that $\mathbb{P}(T, I)$ is just a countable support product of $|I|$ copies of T. Therefore $\mathbb{P}(T, I)$ has ω_2–*c.c.* if *CH* holds.

An ω_1–tree T is called properly pruned in countable products if for any countable limit ordinal δ, and for any $\{p_s : s \in 2^{<\omega}\} \subseteq \mathbb{P}(T \restriction \delta, \omega)$ such that

(1) $s \subseteq t$ if and only if $p_t \leq p_s$,

(2) for every $n \in \omega$, for every $f \in 2^\omega$, $\bigcup_{m\in\omega} ht(p_{f\restriction m}(n)) = \delta$,

(3) for any two different f and g in 2^ω

$$\{\bigcup_{m\in\omega} p_{f^\frown m}(n) : n \in \omega\} \bigcap \{\bigcup_{m\in\omega} p_{g^\frown m}(n) : n \in \omega\} = \emptyset,$$

then there exist f and g in 2^ω such that

$$\forall n \in \omega, \ \bigcup_{m\in\omega} p_{f^\frown m}(n) \in T_\delta \text{ and}$$

$$\forall n \in \omega, \ \bigcup_{m\in\omega} p_{g^\frown m}(n) \notin T_\delta.$$

We define properly pruned trees in countable products because these trees have some nice properties. (1) If we force with a properly pruned tree, we will not create any new countable sequences. (2) Forcing with a properly pruned tree will not create any new branches of Kurepa trees in the ground model. (3) An ω_1–closed forcing will not create any new branches of a properly pruned tree. The purpose for considering properly pruned trees in countable products is that we want to do a countable support product forcing with a properly pruned tree. The next four lemmas about the trees which are properly pruned in countable products as well as the detailed proofs can be found in [SJ1]. We will give only brief sketches of the proofs here to make the discussion self-contained.

Lemma 2 *(CH) There exists an ω_1–tree T which is properly pruned in countable products.*

Sketch of the proof: Construct the tree recursively on all countable ordinals. For every limit stage, because of CH, a diagonal argument can be applied for picking right points to form a new level of the tree. □

We use PT for a tree which is properly pruned in countable products. In the next two lemmas, we let M be a model of CH, let PT be a fixed properly pruned tree in countable products in M, and let I be an index set in M.

Lemma 3 *Let G be a $\mathbb{P}(PT, I)$–generic filter over M. Then $M^\omega \cap M[G] \subseteq M$.*

Sketch of the proof: It is easy to show that $\mathbb{P}(PT, I)$ is ω_1–Baire [K2] (or ω–distributive in [Je2]). □

Lemma 4 *Let K be an ω_1–tree such that every level of K is countable. Suppose that G is a $\mathbb{P}(PT, I)$–generic filter over M. Then $\mathcal{B}(K) \cap M[G] \subseteq M$.*

Sketch of the proof: Similar to the argument used by J. Silver to prove that any ω_1–closed forcing will not add a new branches to a Kurepa tree. Note that if we embed the Cantor tree $2^{<\omega}$ cofinally into $\mathbb{P}(PT \restriction \delta, I)$ for some limit ordinal δ, we can find uncountably many branches of $2^{<\omega}$ such that the images of those branches have lower bounds in $\mathbb{P}(PT, I)$. $\Box$

Lemma 5 *Let $\mathbb{P}$ be an ω_1–closed forcing notion in M and let G be a $\mathbb{P}$–generic filter over M. Then $\mathcal{B}(PT) \cap M[G] \subseteq M$.*

Sketch of the proof: If there is a new branch of PT in $M[G]$, then by Silver's argument the new branch will create an embedding from $2^{<\omega}$ cofinally into $PT \restriction \delta$ for some countable limit ordinal δ such that every infinite increasing sequence in the image of the embedding has an upper bound in PT. This contradicts that PT is properly pruned in countable products. $\Box$

Lemma 6 *Let M be a model of GCH. Let $\mathbb{P} = \mathbb{K}_{\omega_4}$ (see Lemma 1), let $PT \in M$ and let $\mathbb{Q} = \mathbb{P}(PT, \omega_3)$. Suppose that G is a $\mathbb{P}$–generic filter over M and H is a $\mathbb{Q}$–generic filter over $M[G]$. Then in $M[G][H]$ the following are true.*

(1) CH.

(2) $2^{\omega_1} = \omega_4$.

(3) $T_G = \bigcup_{p \in G} A_p$ is a Kurepa tree with no Jech–Kunen subtrees.

(4) PT is a Jech–Kunen tree with ω_3 branches.

(5) There are no Kurepa subtrees of PT with exactly ω_3 branches.

Proof: (1) is true because this two–step forcing extension does not add any new countable sequences of M. (2) is true because M satisfies GCH and both $\mathbb{P}$ and $\mathbb{Q}$ have ω_2–*c.c.* and both $\mathbb{P}$ and $\mathbb{Q}$ have size less than or equal to ω_4.

We prove (3). Suppose that T' is a Jech–Kunen subtree of T_G in $M[G][H]$. Note that T' is also a Kurepa tree. Since $|T'| = \omega_1$ and $\mathbb{Q}$ has ω_2–*c.c.*, there exists an $I \subseteq \omega_3$ such that $|I| = \omega_1$ and $T' \in M[G][H_I]$, where $H_I = H \cap \mathbb{P}(PT, I)$. Without loss of generality, we can assume that $I \in M$. By Lemma 4, T' is still a Jech–Kunen tree in $M[G][H_I]$. Since

$$M[G][H_I] = M[H_I][G]$$

and the definition of $\mathbb{P}$ is absolute with respect to M and $M[H_I]$, T_G is a Kurepa tree which has no Jech–Kunen subtrees in $M[H_I][G]$ by applying Lemma 1 to $M[H_I]$. This contradicts that T' is a Jech–Kunen subtree of T_G in $M[G][H_I]$.

We prove (4). Notice that

$$M[G][H] = M[H][G].$$

Since $\mathbb{P}$ is ω_1–closed in $M[H]$ (no new countable sequences of M are added), every branch of PT in $M[H][G]$ is already in $M[H]$ by Lemma 5. Besides, PT has only ω_3 branches in $M[H]$, so that PT has ω_3 branches in $M[G][H]$.

We now prove (5). Suppose that K is a Kurepa subtree of PT with exactly ω_3 branches in $M[G][H]$. Then there exists an $I \subseteq \omega_3$ such that $|I| = \omega_1$ and $K \in M[G][H_I]$. By the proof of (3), K has still ω_3 branches in $M[G][H_I]$. But in $M[G][H_I]$ PT has at most ω_2 branches by the same reason as in the proof of (4). It is impossible for PT to have a subtree with ω_3 branches. $\square$

Let λ be a regular cardinal and let I and J be two sets. We use $Fn(I, J, \lambda)$ for the forcing notion

$$\{p : p \subseteq I \times J,\ p \text{ is a function and } |p| < \lambda\}$$

ordered by reverse inclusion. We may omit λ when $\lambda = \omega$.

Theorem 7 *It is consistent with CH and $2^{\omega_1} > \omega_2$ that there exists a Kurepa tree with no Jech–Kunen subtrees and there exists a Jech–Kunen tree with no Kurepa subtrees.*

Proof: Let $M[G][H]$ be the model in Lemma 6, and let $\mathbb{R} = Fn(\omega_1, \omega_2, \omega_1)$ ($\mathbb{R}$ is just a standard collapsing order which collapses ω_2). Let F be an $\mathbb{R}$–generic filter over $M[G][H]$. We want to show that

$$\bar{M} = M[G][H][F]$$

is the model we are looking for.

It is obvious that CH holds. In $\bar{M}$ $2^{\omega_1} = \omega_3$ because ω_2 in $M[G][H]$ has been collapsed. It is also easy to see that T_G is a Kurepa tree with ω_3 branches in $\bar{M}$. PT has ω_2 branches in $\bar{M}$ because $\mathbb{R}$ is ω_1–closed so that forcing with $\mathbb{R}$ will not create any new branches of PT.

Claim 7.1 In $\bar{M}$, T_G has no Jech–Kunen subtrees.

Proof of Claim 7.1: Suppose that T' is a Jech–Kunen subtree of T_G in $\bar{M}$. Then

$$\bar{M} \models (|\mathcal{B}(T')| = \omega_2).$$

Without loss of generality, let $1_{\mathbb{R}}$ force that $\dot{T}'$ is a Jech–Kunen subtree of T_G. Since $\mathbb{R}$ is ω_1–closed, then

$$\mathcal{B}(T_G) \cap \bar{M} \subseteq M[G][H].$$

In $\bar{M}$, since $|\mathbb{R}| = \omega_1$ and T' has ω_2 branches, then there exists an $r_0 \in \mathbb{R}$ such that the set

$$S = \{B \in \mathcal{B}(T_G) : (r_0 \Vdash B \in \mathcal{B}(\dot{T}'))^{M[G][H]}\}$$

has cardinality ω_2. Note that S is actually in $M[G][H]$ and $\omega_2^{\bar{M}} = \omega_3^{M[G][H]}$. So $T'' = \bigcup S$ is a subtree of T_G with ω_3 branches in $M[G][H]$. This contradicts Lemma 6.

Claim 7.2 In $\bar{M}$, PT has no Kurepa subtrees.

Proof of Claim 7.2: Suppose that T' is a Kurepa subtree of PT. Then in $\bar{M}$, T' has ω_2 branches. By the same reason, there exists an $r_0 \in \mathbb{R}$ such that the set

$$S' = \{B \in \mathcal{B}(PT) : r_0 \Vdash B \in \mathcal{B}(\dot{T}')\}$$

has cardinality ω_3 in $M[G][H]$. Let $T'' = \bigcup S'$. Then T'' is a subtree of PT. T'' has now ω_3 branches in $M[G][H]$. T'' is also a Kurepa tree in $M[G][H]$ because T'' is a subtree of the Kurepa tree T' in $\bar{M}$. This contradicts Lemma 6. □

Remark: In [Ji1, Theorem 2], I proved the consistency of a Jech–Kunen tree with no Kurepa subtrees. In the proof, one inaccessible cardinal is used. Here we not only eliminated the large cardinal assumption but also put the results of [Ji1, Theorem 2] and [Ji1, Theorem 3] together in one model.

2 A new consequence of an old model

In this section let κ be always an inaccessible cardinal. We use $Lv(\kappa, \lambda)$ for *Lévy collapsing order* [K2], which collapses all cardinals between κ and λ in the ground model. Let $\mathbb{P} = Lv(\kappa, \omega_1)$ in M and let G be a $\mathbb{P}$–generic filter over M. J. Silver showed that there are no Kurepa trees in $M[G]$. Let's call the model $M[G]$ a Silver's model.

In [Ji1, Theorem 2], I constructed, by assuming an inaccessible cardinal, a model of *CH* and $2^{\omega_1} > \omega_2$, in which there exists a Jech–Kunen tree with no Kurepa subtrees. It is a three–step iterated forcing extension. Let $M[G]$ be the Silver's model mentioned above. In $M[G]$ let H be a $Fn(\omega_1, 2)$–generic filter over $M[G]$. Note that $Fn(\omega_1, 2)$ is absolute with respect to M and $M[G]$. Let $CT = (\langle 2^{<\omega_1}, \subseteq \rangle)^{M[G]}$, *i.e.* a complete binary tree with height ω_1 in $M[G]$. Since forcing with $Fn(\omega_1, 2)$ adds a lot of new reals to $M[G]$, *CT* becomes very incomplete in $M[G][H]$. This incompleteness enable us to increase the size of 2^{ω_1} by an ω_1–closed forcing without adding any new branches to *CT*. As a consequence, *CT* becomes a Jech–Kunen tree in $M[G][H][F]$, where F is an $(Fn(\omega_3, 2, \omega_1))^{M[G][H]}$–generic filter over $M[G][H]$. If *CT* has a Kurepa subtree T' in $M[G][H][F]$, then it is easy to show that *CT* has a Kurepa subtree in $M[G][H]$ because we need only ω_1 conditions to determine T' while T' has more than ω_1 branches, which are in $M[G][H]$. Now by an argument due to K. Devlin (see [D] or [Ji1]), there are no Kurepa trees in $M[G][H]$. That leads a contradiction. We can easily see that the last step forcing above did not create Kurepa subtrees in *CT*, but it is harder to show that in $M[G][H][F]$, there are no Kurepa trees at all. In order to construct a model of *CH* and $2^{\omega_1} > \omega_2$, in which there are Jech–Kunen trees but there are no Kurepa tree, S. Shelah and I tried a different approach [SJ1]. Recently I discovered that we can directly prove that there are no Kurepa trees in $M[G][H][F]$. This result is a simple corollary of a lemma, which says that if $\mathbb{P}$ is a forcing notion of size ω_1 and satisfies Baumgartner's *Axiom A* (see [B]) in a Silver's model, then forcing with $\mathbb{P}$ will not create any Kurepa trees in the forcing extension.

Let $\mathbb{P}$ be a forcing notion. $\mathbb{P}$ is said to satisfy *Axiom A* iff there exists a collection $\{\leq_n\}_{n\in\omega}$ of partial orderings on $\mathbb{P}$ such that

(1) $\leq_0$ is the original order of $\mathbb{P}$;

(2) for any $p, q \in \mathbb{P}$, $p \leq_{n+1} q$ implies $p \leq_n q$;

(3) if $\{p_n\}_{n\in\omega}$ is a sequence in $\mathbb{P}$ such that for every $n \in \omega$, $p_{n+1} \leq_n p_n$, then there is a $q \in \mathbb{P}$ such that $q \leq_n p_n$ for all $n \in \omega$;

(4) for every $p \in \mathbb{P}$ and for every $n \in \omega$, if $A \subseteq \mathbb{P}$ is predense below p, then there exist a countable subset B of A and a $q \leq_n p$ such that B is predense below q.

Remarks: (1) Forcing with $\mathbb{P}$ which satisfies *Axiom A* will not collapse ω_1. (2) If

$\mathbb{P}$ has size ω_1 in addition, then $\mathbb{P}$ has obviously ω_2–*c.c.*, and hence forcing with $\mathbb{P}$ will preserve all cardinals. (3) If $\mathbb{P}$ satisfies *Axiom A* in a ground model M and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$–name such that $1_{\mathbb{P}} \Vdash$"$\dot{\mathbb{Q}}$ satisfies *Axiom A*", then the forcing notion of the two–step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ satisfies *Axiom A* in M. The proofs for all these facts can be found in [B].

Lemma 8 *In a model M, let $\mathbb{P}$ be a forcing notion of size ω_1, which satisfies Axiom A, and let $\mathbb{Q}$ be a forcing notion which is ω_1–closed. Let G be a $\mathbb{P}$–generic filter over M and let H be a $\mathbb{Q}$–generic filter over $M[G]$. Suppose in $M[G]$, T is an ω_1–tree such that for every $\alpha < \omega_1$, T_α is countable. Then*

$$\mathcal{B}(T) \cap M[G][H] \subseteq M[G].$$

Proof: Suppose that there exists a branch

$$B \in \mathcal{B}(T) \cap (M[G][H] \smallsetminus M[G]).$$

Without loss of generality we can assume that

$$1_{\mathbb{P}} \Vdash 1_{\mathbb{Q}} \Vdash (\ddot{B} \in \mathcal{B}(\dot{T}) \cap (M[\dot{G}][\ddot{H}] \smallsetminus M[\dot{G}])).$$

In order to carry on the induction let's first enumerate $2^{<\omega}$ in the order type ω. Let

$$I(s) = 2^n + \sum_{i<n} s(i) 2^{n-1-i}$$

for each $n \in \omega$ and $s \in 2^n$. It is easy to see that $I(s)$ is obtained by attaching a 1 to the beginning of s and reading the result as the binary notation for an integer. Next we construct inductively three sequences $\{p_n : n \in \omega\} \subseteq \mathbb{P}$, $\{q_s : s \in 2^{<\omega}\} \subseteq \mathbb{Q}$ and $\{X_s : s \in 2^{<\omega}\} \subseteq [\omega_1]^{\leq\omega}$ (where $[\omega_1]^{\leq\omega}$ means the set of all finite or countable subset of ω_1) such that the following are true:

(1) $p_0 = 1_{\mathbb{P}}$, $q_{\langle\rangle} = 1_{\mathbb{Q}}$ and $X_{\langle\rangle} = \{0\}$;

(2) for every $n \in \omega$, $p_{n+1} \leq_n p_n$;

(3) for any $s, t \in 2^{<\omega}$, $s \subseteq t$ implies $q_t \leq q_s$;

(4) for every $n \in \omega$, there exists a $\delta_n < \omega_1$ such that $\bigcup\{\bigcup X_s : s \in 2^{<n}\} < \delta_n \leq \min\{\min X_s : s \in 2^n\}$;

(5) for $i = 0, 1$, for every $s \in 2^{<\omega}$,

$$p_{I(s)} \Vdash (\exists \beta \in X_s \, \exists x_i \in \dot{T}_\beta \, (x_0 \neq x_1) \, (q_{s\hat{}\langle i\rangle} \Vdash (x_i \in \ddot{B}))).$$

Claim 8.1 The lemma follows from the construction.

Proof of Claim 8.1: Let $p \in \mathbb{P}$ such that $p \leq p_n$ for every $n \in \omega$. For every $f \in 2^\omega$ let q_f be the lower bound of $\{q_{f \restriction n} : n \in \omega\}$. Let $\delta = \bigcup_{n\in\omega} \delta_n$. Since we have that

$$p \Vdash q_f \Vdash (\exists x \in \ddot{B} \cap \dot{T}_\delta),$$

then we have

$$p \Vdash (\exists x \in \dot{T}_\delta \text{ and } \exists q_f' \leq q_f \text{ such that } q_f' \Vdash x \in \ddot{B}).$$

Let G be a $\mathbb{P}$–generic filter over M containing p. Working within $M[G]$, let $x_f \in T_\delta$ and $q_f' \leq q_f$ for every $f \in 2^\omega \cap M$ such that

$$q_f' \Vdash x_f \in \dot{B}.$$

If $f \neq g$, then $x_f \neq x_g$. Since $(2^\omega)^M$ is still uncountable in $M[G]$, we have now $|T_\delta| \geq \omega_1$. This contradicts that every level of T is countable in $M[G]$.

We now do the construction.

Assume that we have constructed $\{p_{I(s)} : s \in 2^{<n}\}$, $\{q_{s\hat{}\langle i\rangle} : s \in 2^{<n}, i = 0, 1\}$ and $\{X_s : s \in 2^{<n}\}$. We want now to construct $p_{I(s)}$, $q_{s\hat{}\langle i\rangle}$ and X_s inductively for all $s \in 2^n$ and $i = 0, 1$. First, we fix a countable ordinal δ_n such that

$$\delta_n > \bigcup\{\bigcup X_s : s \in 2^{<n}\}.$$

Assume that we have found $p_{I(s)}$, $q_{s\hat{}\langle i\rangle}$ and X_s for all s such that $I(s) \leq m-1$, where $2^n \leq m < 2^{n+1}$ and $i = 0, 1$. Let $s \in 2^n$ such that $I(s) = m$.

Since $p_{I(s)-1} \leq p_{I(s \restriction n-1)}$, then

$$p_{I(s)-1} \Vdash (\exists \beta \in X_{s \restriction n-1}\, \exists x \in \dot{T}_\beta\ (q_s \Vdash x \in \ddot{B}))$$

by (5).

Claim 8.2 There exists a $p' \leq_{I(s)-1} p_{I(s)-1}$, there exists a countable $X_s \subseteq \omega_1 \smallsetminus \delta_n$ and there exist $q_i' \leq q_s$ for $i = 0, 1$ such that

$$p' \Vdash (\exists \beta \in X_s,\ \exists x_i \in \dot{T}_\beta\ (x_0 \neq x_1)\ (q_i' \Vdash x_i \in \ddot{B})).$$

Proof of Claim 8.2: Let $\mathbb{P} = \{\bar{p}_\alpha : \alpha \in \omega_1\}$ be an enumeration of $\mathbb{P}$. Let $i = 0, 1$. We construct inductively $\{p_\alpha, q_\alpha^i, \beta_\alpha : \alpha < \mu\}$ for some $\mu \in \omega_1$ or $\mu = \omega_1$ such that

(1) $\{p_\alpha : \alpha < \mu\}$ is a maximal antichain below $p_{I(s)-1}$,
(2) $q^i_0 \leq q_s$ and $q^i_\alpha \leq q^i_{\alpha'}$ if $\alpha' \leq \alpha$,
(3) $\beta_\alpha \geq \delta_n$ for every $\alpha < \mu$ and
(4) $p_\alpha \Vdash (\exists x_i \in \dot{T}_{\beta_\alpha}(x_0 \neq x_1)\ (q^i_\alpha \Vdash x_i \in \ddot{B}))$.

$\alpha = 0$. Let $p'_0 \leq p_{I(s)-1}$. In addition, we require $p'_0 \leq \bar{p}_0$ if $\bar{p}_0$ and $p_{I(s)-1}$ are compatible. Since

$$p'_0 \Vdash (q_s \Vdash (\ddot{B} \notin M[\ddot{G}][\dot{H}])),$$

then

$$p'_0 \Vdash (\exists \beta > \delta_n\ \exists x_i \in \dot{T}_{\beta_0}(x_0 \neq x_1)\ \exists q^i_0 \leq q_s\ (q^i_0 \Vdash x_i \in \ddot{B})),$$

and hence in M there exists a $p_0 \leq p'_0$, there exists a $\beta_0 > \delta_n$ and there exist $q^i_0 \leq q_s$ such that

$$p_0 \Vdash (\exists x_i \in \dot{T}_{\beta_0}(x_0 \neq x_1)\ (q^i_0 \Vdash x_i \in \ddot{B})).$$

Assume that we have found p_α, q^i_α and β_α for all $\alpha < \eta$ for some $\eta < \omega_1$.

We are done if $\{p_\alpha : \alpha < \eta\}$ is already maximal under $p_{I(s)-1}$. Note that the set $\{p_\alpha : \alpha < \eta\}$ is countable. Let $p_{I(s)} = p_{I(s)-1}$, let $q_{s\hat{}\langle i\rangle}$ be a lower bound of the decreasing sequence $\{q^i_\alpha : \alpha < \eta\}$ ($\mathbb{Q}$ is ω_1–closed in M) and let $X_s = \{\beta_\alpha : \alpha < \eta\}$. Otherwise, let $p'_\eta \leq p_{I(s)-1}$ such that p'_η is incompatible with every p_α for all $\alpha < \eta$. We also require that $p'_\eta < \bar{p}_\eta$ if $\bar{p}_\eta$ is compatible with $p_{I(s)-1}$ and incompatible with every p_α for all $\alpha < \eta$. Let q_i be a lower bound of $\{q^i_\alpha : \alpha < \eta\}$. We have now

$$p'_\eta \Vdash (q_i \Vdash (\ddot{B} \notin M[\dot{G}][\ddot{H}])).$$

Then

$$p'_\eta \Vdash (\exists \beta > \delta_n\ \exists x^j_i \in \dot{T}_\beta\ (x^0_i \neq x^1_i)\ \exists q^j_i \leq q_i\ (q^j_i \Vdash x^j_i \in \ddot{B})),$$

where $j = 0, 1$. Hence in M there exists a $p_\eta \leq p'_\eta$, there exist $q^j_i \leq q_i$ and there exists a $\beta_\eta > \delta_n$ such that

$$p_\eta \Vdash (\exists x^j_i \in \dot{T}_{\beta_\eta}\ (x^0_i \neq x^1_i)\ (q^j_i \Vdash x^j_i \in \ddot{B})).$$

We can now pick j_0 and j_1 such that

$$p_\eta \Vdash (\exists x^{j_i}_i \in \dot{T}_{\beta_\eta}\ (x^{j_0}_0 \neq x^{j_1}_1)\ (q^{j_i}_i \Vdash x^{j_i}_i \in \ddot{B})).$$

Choose $q^i_\eta = q^{j_i}_i$.

Note that if $\{p_\alpha : \alpha < \eta\}$ has never been a maximal antichain below $p_{I(s)-1}$ for every countable ordinal η, then $\{p_\alpha : \alpha < \omega_1\}$ must be a maximal antichain below $p_{I(s)-1}$ because for every $\bar{p}_\alpha \in \mathbb{P}$ such that $\bar{p}_\alpha \leq p_{I(s)-1}$, either $\bar{p}_\alpha$ is compatible with some p_β for some $\beta < \alpha$ or $\bar{p}_\alpha$ is above p_α. Because $\mathbb{P}$ satisfies *Axiom A*, we can find a $p' \leq_{I(s)-1} p_{I(s)-1}$ and $\eta < \omega_1$ such that $\{p_\alpha : \alpha < \eta\}$ is predense under p'. Now let $q'_i = q^i_\eta$ and $X_s = \{\beta_\alpha : \alpha < \eta\}$. Then we have

$$p' \Vdash (\exists \beta \in X_s\ \exists x_i \in \dot{T}_\beta\ (x_0 \neq x_1)\ (q'_i \Vdash x_i \in \ddot{B})).$$

This ends the construction of the claim. We choose $p_{I(s)} = p'$ and $q_{s\hat{}\langle i\rangle} = q'_i$. Note that we have X_s already.

We have shown that the lemma follows from Claim 8.2. $\square$

Recall that $M[G]$ is said to be a Silver's model if G is a $(Lv(\kappa, \omega_1))^M$–generic filter over M, where κ is an inaccessible cardinal in M.

Lemma 9 *Let $M[G]$ be a Silver's model. In $M[G]$ let $\mathbb{P}$ be a forcing notion of size ω_1, which satisfies Axiom A. Let H be a $\mathbb{P}$–generic filter over $M[G]$. Then there are no Kurepa trees in $M[G][H]$.*

Proof: Without loss of generality we can assume that $\mathbb{P} \in \mathbb{M}$ because $\mathbb{P}$ has size ω_1. Suppose that T is a Kurepa tree in $M[G][H]$. We are going to prove the lemma by deriving a contradiction.

For any subset I of κ, we use $\mathbb{L}_I = Lv(I, \omega_1)$ for the set of all $p \in \mathbb{L} = Lv(\kappa, \omega_1)$ such that $dom(p) \subseteq I$. If G is a subset of $\mathbb{L}$, we use G_I for the set $G \cap \mathbb{L}_I$.

Since T has size ω_1 and $M[G][H] = M[H][G]$, there exists an ordinal $\delta < \kappa$ such that $T \in M[G_\delta][H]$. Note that

$$M[G][H] = M[G_\delta][H][G_{\kappa \smallsetminus \delta}]$$

and

$$M[G_\delta][H] \models (2^{\omega_1} < \kappa),$$

and so there exists a branch B of T such that

$$B \in M[G][H] \smallsetminus M[G_\delta][H].$$

But in $M[G_\delta]$ $\mathbb{P}$ satisfies *Axiom A* and $\mathbb{L}_{\kappa \smallsetminus \delta}$ is ω_1–closed. By Lemma 8, T has no new branches in $M[G][H] \smallsetminus M[G_\delta][H]$. A contradiction. $\square$

Theorem 10 *In M let κ be an inaccessible cardinal, let $\mathbb{P} = Lv(\kappa, \omega_1)$, and let G be a $\mathbb{P}$–generic filter over M. In $M[G]$ let $\mathbb{Q} = Fn(\omega_1, 2)$, and let H be a $\mathbb{Q}$–generic filter over $M[G]$. In $M[G][H]$ let $\mathbb{R} = Fn(\omega_3, 2, \omega_1)$, and let F be an $\mathbb{R}$–generic filter over $M[G][H]$. Then $M[G][H][F]$ is a model of CH and $2^{\omega_1} = \omega_3$, in which there exists a Jech–Kunen tree and there are no Kurepa trees.*

Proof: Besides the proof of [Ji1, Theorem 2], we need only prove that there are no Kurepa trees in $M[G][H][F]$. Suppose that T is a Kurepa tree in $M[G][H][F]$. Since the cardinality of T is ω_1, then there exists a subset $I \subseteq \omega_3$ of size ω_1 such that $T \in M[G][H][F_I]$ where $F_I = F \cap (Fn(I, 2, \omega_1))^{M[G][H]}$. Since $Fn(\omega_3 \smallsetminus I, 2, \omega_1)$ is ω_1–closed in $M[G][H][F_I]$, then T is still a Kurepa tree in $M[G][H][F_I]$ because any ω_1–closed forcing will not create new branches for an ω_1–tree with countable levels. Without loss of generality we can assume that $I = \omega_1$. Let $\mathbb{R}_{\omega_1} = (Fn(\omega_1, 2, \omega_1))^{M[G][H]}$. In $M[G]$ both $\mathbb{Q}$ and $\dot{\mathbb{R}}_{\omega_1}$ have size ω_1, $\mathbb{Q}$ has *c.c.c.* and

$$1_{\mathbb{Q}} \Vdash (\dot{\mathbb{R}}_{\omega_1} \text{ is } \omega_1\text{–closed}).$$

Then $\mathbb{Q} * \dot{\mathbb{R}}_{\omega_1}$ has size ω_1 and satisfies *Axiom A* in $M[G]$. Hence by Lemma 9, there are no Kurepa trees in $M[G][H][F_{\omega_1}]$. This contradicts that T is a Kurepa tree in $M[G][H][F_{\omega_1}]$. $\square$

3 A question

We would like to end this paper by asking a question.

Let $M[G]$ be a Silver's model. Can we find a forcing notion $\mathbb{P}$ of size ω_1 in $M[G]$ such that forcing with $\mathbb{P}$ will preserve ω_1 and produce a Kurepa tree?

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Department of Mathematics
University of California
Berkeley, CA 94720
e-mail: jin@math.berkeley.edu