A MODEL IN WHICH EVERY KUREPA TREE IS THICK [1]

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Abstract

In this paper we show that, assuming the existence of two strongly inaccessible cardinals, it is consistent with *CH* (or $\neg CH$) plus $2^{\omega_1} > \omega_2$ that there exists a Kurepa tree with 2^{ω_1}–many branches and no ω_1–trees have λ–many branches for some λ strictly between ω_1 and 2^{ω_1}.

A tree is a partially ordered set $(T, <_T)$ such that for every $t \in T$, the set $\{s \in T : s <_T t\}$ is well–ordered. $(T', <_{T'})$ is a subtree of $(T, <_T)$ if $T' \subseteq T$ and $<_{T'} = <_T \bigcap T' \times T'$. We shall not distinguish a tree $(T, <_T)$ from its domain T. Let $ht_T(t)$, the height of t in T, be the order type of $\{s \in T : s <_T t\}$, let T_α, the α–th level of T, be the set $\{t \in T : ht_T(t) = \alpha\}$ and let $ht(T)$, the height of T, be the smallest ordinal α such that $T_\alpha = \emptyset$. By a branch of T we mean a linearly ordered subset of T which intersects every non–empty level of T. Let $\mathcal{B}(T)$ be the set of all branches of T.

T is called an κ–tree for some regular cardinal κ if $|T| = \kappa$ and $ht(T) = \kappa$. An ω_1–tree is called a Kurepa tree if $|T_\alpha| < \omega_1$ for every $\alpha < \omega_1$ and $|\mathcal{B}(T)| > \omega_1$. A Kurepa tree T is called thick if $|\mathcal{B}(T)| = 2^{\omega_1}$. An ω_1–tree is called a Jech–Kunen tree if $\omega_1 < |\mathcal{B}(T)| < 2^{\omega_1}$.

It is obvious that under *CH* plus $2^{\omega_1} > \omega_2$, (1) a Jech–Kunen tree T is a Kurepa tree if $|T_\alpha| < \omega_1$ for every $\alpha < \omega_1$; (2) a Kurepa tree T is a Jech–Kunen tree if it is not thick.

The independence of the existence of a Kurepa tree was proved by J. Silver (see [K2]). In [Je], T. Jech constructed by forcing a model of *CH* plus $2^{\omega_1} > \omega_2$, in which there is a Jech–Kunen tree. In fact, it is a Kurepa tree with less than 2^{ω_1}–many branches. The independence of the existence of a Jech–Kunen tree (in terms of a compact Hausdorff space) under *CH* plus $2^{\omega_1} > \omega_2$ was given by K. Kunen [K1]. The detailed proof can be found in [Ju, Theorem 4.8]. In Kunen's model all Kurepa trees, including those with 2^{ω_1}–many branches, are also killed. Is it necessary to kill all Kurepa trees when we kill all Jech–Kunen trees? In [Ji], Kunen proved that it is consistent with *CH* plus $2^{\omega_1} > \omega_2$ that there is a thick Kurepa tree which has no Jech–Kunen subtrees. So it is natural to ask whether it is consistent with *CH* plus $2^{\omega_1} > \omega_2$ that there exists a thick Kurepa tree and there are no Jech–Kunen trees. Next we will give a positive answer by assuming the existence of two strongly inaccessible cardinals. (Note that the assumption of one strongly inaccessible cardinal is necessary for killing all Jech–Kunen trees.)

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Theorem 1 *Assuming the existence of two strongly inaccessible cardinals, it is consistent with CH plus $2^{\omega_1} > \omega_2$ that there exists a thick Kurepa tree and there are no Jech–Kunen trees.*

In order to prove the theorem we need some notations and a lemma from [D] which plays a key role in our proofs. By a poset we mean a partially ordered set with a largest element. We always let $1_{\mathbb{P}}$ be the largest element of a poset $\mathbb{P}$. Let I, J be two sets and λ be a cardinal.

$$Fn(I, J, \lambda) = \{f : f \text{ is a function, } f \subseteq I \times J \text{ and } |f| < \lambda\}$$

is a poset ordered by reverse inclusion. We omit λ if $\lambda = \omega$. Let I be a subset of an ordinal κ and λ be a cardinal.

$$Lv(I, \lambda) = \{f : f \text{ is a function, } f \subseteq (I \times \lambda) \times \kappa, |f| < \lambda \text{ and } \forall \langle \alpha, \beta \rangle \in dom(f)(f(\alpha, \beta) \in \alpha)\}$$

is a poset ordered by reverse inclusion.

Let 2^{α} be the set of all functions from α to 2 and $2^{<\kappa} = \bigcup_{\alpha<\kappa} 2^{\alpha}$. Then $2^{<\kappa}$ is a tree ordered by inclusion.

In forcing arguments we let $\dot{a}$ be a name for a and $\ddot{a}$ be a name for $\dot{a}$. We always assume the consistency of *ZFC* and let M denote a countable transitive model of *ZFC*. The author refers to [K2] for background in forcing and refers to [T2] for background in trees.

Lemma 1 *Let $\mathbb{P}$, $\mathbb{P}'$ be two posets in M such that $\mathbb{P}$ has κ–c.c. and $\mathbb{P}'$ is κ–closed in M, where κ is a regular cardinal in M. Let $G_{\mathbb{P}}$ be a $\mathbb{P}$–generic filter over M and $G_{\mathbb{P}'}$ be a $\mathbb{P}'$–generic filter over $M[G_{\mathbb{P}}]$. Let T be an κ–tree in $M[G_{\mathbb{P}}]$. If T has a new branch B in $M[G_{\mathbb{P}}][G_{\mathbb{P}'}] \smallsetminus M[G_{\mathbb{P}}]$, then T has a subtree T' in $M[G_{\mathbb{P}}]$, which is isomorphic to the tree $\langle 2^{<\kappa} \bigcap M, \subseteq \rangle$.*

Proof: First we work within M. In the proof we always let $i = 0, 1$. Without loss of generality we can assume that $|T_0| = 1$ and

$$1_{\mathbb{P}} \Vdash_{\mathbb{P}} (1_{\mathbb{P}'} \Vdash_{\mathbb{P}'} (\ddot{B} \text{ is a branch of } \dot{T})).$$

Claim 1: Let $\alpha < \kappa$ and $q \in \mathbb{P}'$. Then there is a $q' \leq_{\mathbb{P}'} q$ such that

$$1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Phi(\alpha, q', \dot{T}, \ddot{B})),$$

where

$$\Phi(\alpha, q, \dot{T}, \ddot{B}) \stackrel{\text{def}}{=} (\exists y \in \dot{T}_{\alpha})(q \Vdash_{\mathbb{P}'} (y \in \ddot{B})).$$

Proof of Claim 1: Replace ω_1 by κ in the proof of [D, Lemma 3.6].

Claim 2: Let $\alpha < \kappa$, $q \in \mathbb{P}'$ and $1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Phi(\alpha, q, \dot{T}, \ddot{B}))$. Then there is a $\beta < \kappa$, $\beta > \alpha$ and $q^i \leq_{\mathbb{P}'} q$ such that

$$1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Psi(\alpha, \beta, q, q^0, q^1, \dot{T}, \ddot{B})),$$

where

$\Psi(\alpha, \beta, q, q^0, q^1, \dot{T}, \ddot{B}) \stackrel{\text{def}}{=} [$ if $x \in \dot{T}_\alpha$ and $q \Vdash_{\mathbb{P}'} (x \in \ddot{B})$, then there are $x^i \in \dot{T}_\beta, x^0 \neq x^1$ and $x <_T x^i$ such that $q^i \Vdash_{\mathbb{P}'} (x^i \in \ddot{B})$].

Proof of Claim 2: Replace ω_1 by κ in the proof of [D, Lemma 3.6].

Claim 3: Let δ be an ordinal below κ. Let $\langle q_\gamma : \gamma < \delta \rangle$ be a decreasing sequence in $\mathbb{P}'$ and $\langle \alpha_\gamma : \gamma < \delta \rangle$ be an increasing sequence in κ such that

$$1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Phi(\alpha_\gamma, q_\gamma, \dot{T}, \ddot{B}))$$

for all $\gamma < \delta$. Let $\alpha_\delta = \sup\{\alpha_\gamma : \gamma < \delta\}$. Then there is a $q \leq_{\mathbb{P}'} q_\gamma$ for all $\gamma < \delta$ such that

$$1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Phi(\alpha_\delta, q, \dot{T}, \ddot{B})).$$

Proof of Claim 3: Since $\mathbb{P}'$ is κ–closed in M, there is a $q' \in \mathbb{P}'$ such that $q' \leq_{\mathbb{P}'} q_\gamma$ for all $\gamma < \delta$. By Claim 1 there is a $q \leq_{\mathbb{P}'} q'$ such that

$$1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Phi(\alpha_\delta, q, \dot{T}, \ddot{B})).$$

This ends the proof of Claim 3.

We now prove the lemma. We construct a subset $\bar{\mathbb{P}} = \{p_s : s \in 2^{<\kappa}\}$ of $\mathbb{P}'$ and a subset $O = \{\alpha_s : s \in 2^{<\kappa}\}$ of κ in M such that

(1) the map $s \mapsto p_s$ is an isomorphic imbedding from $\langle 2^{<\kappa}, \subseteq \rangle$ to $\mathbb{P}'$ in M.
(2) $\forall s, t \in 2^{<\kappa}$ $(s \subseteq t$ and $s \neq t \to \alpha_s < \alpha_t)$.
(3) $\alpha_{s\hat{}\langle 0\rangle} = \alpha_{s\hat{}\langle 1\rangle}$ for all $s \in 2^{<\kappa}$.
(4) $1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Phi(\alpha_s, p_s, \dot{T}, \ddot{B}))$ for all $s \in 2^{<\kappa}$.
(5) $1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Psi(\alpha_s, \alpha_{s\hat{}\langle 0\rangle}, p_s, p_{s\hat{}\langle 0\rangle}, p_{s\hat{}\langle 1\rangle}, \dot{T}, \ddot{B}))$ for all $s \in 2^{<\kappa}$.

Let $\alpha_{\langle\rangle} = 0$ and $p_{\langle\rangle} = 1_{\mathbb{P}'}$. Assume that we have α_s and p_s for all $s \in 2^{<\kappa}$.

Case 1: $\alpha = \gamma + 1$.
Let $s \in 2^\gamma$. Since

$$1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Phi(\alpha_s, p_s, \dot{T}, \ddot{B})),$$

then there is a $\beta < \kappa$, $\beta > \alpha_s$ and $q^i \leq_{\mathbb{P}'} p_s$ such that

$$1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Psi(\alpha_s, \beta, p_s, q^0, q^1, \dot{T}, \ddot{B}))$$

by Claim 2. Let $\alpha_{s\hat{}\langle i\rangle} = \beta$ and $p_{s\hat{}\langle i\rangle} = q^i$. (Note that q^0, q^1 are incompatible by Claim 2.)

Let G be any $\mathbb{P}$–generic filter over M. Then

$$M[G] \models [\Phi(\alpha_s, p_s, T, \dot{B})].$$

Hence in $M[G]$ there is an $x \in T_{\alpha_s}$ such that $p_s \Vdash_{\mathbb{P}'} (x \in \dot{B})$. Since

$$M[G] \models [\Psi(\alpha_s, \alpha_{s\hat{}\langle 0\rangle}, p_s, p_{s\hat{}\langle 0\rangle}, p_{s\hat{}\langle 1\rangle}, T, \dot{B}) \text{ and } x \in T_{\alpha_s}],$$

then there are $x^i \in T_{\alpha_{s\hat{}\langle i\rangle}}$ such that

$$M[G] \models [p_{s\hat{}\langle i\rangle} \Vdash_{\mathbb{P}'} (x^i \in \dot{B})].$$

This implies that

$$1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Phi(\alpha_{s\hat{}\langle i\rangle}, p_{s\hat{}\langle i\rangle}, \dot{T}, \ddot{B})).$$

Case 2: α is a limit ordinal below κ.

Let $s \in 2^\alpha$. Since $\langle \alpha_{s^{-}\beta} : \beta < \alpha\rangle$ is increasing in κ, $\langle p_{s^{-}\beta} : \beta < \alpha\rangle$ is decreasing in $\mathbb{P}'$ and

$$1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Phi(\alpha_{s^{-}\beta}, p_{s^{-}\beta}, \dot{T}, \ddot{B}))$$

for all $\beta < \alpha$, then there is an

$$\alpha_s = \sup\{\alpha_{s^{-}\beta} : \beta < \alpha\}$$

and a $p_s \leq_{\mathbb{P}'} p_{s^{-}\beta}$ for all $\beta < \alpha$ such that

$$1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\Phi(\alpha_s, p_s, \dot{T}, \ddot{B}))$$

by Claim 3.

We now work within $M[G_{\mathbb{P}}]$ to construct a subtree $T' = \{t_s : s \in 2^{<\kappa} \bigcap M\}$ of T such that

(1) the map $s \mapsto t_s$ is an isomorphic imbedding from $\langle 2^{<\kappa} \bigcap M, \subseteq\rangle$ to T.

(2) $t_s \in T_{\alpha_s}$ and $p_s \Vdash_{\mathbb{P}'} (t_s \in \dot{B})$ for all $s \in 2^{<\kappa} \bigcap M$.

Let $t_{\langle\rangle}$ be the element in T_0. Assume that we have t_s for all $s \in 2^{<\alpha} \bigcap M$.

Case 1: $\alpha = \beta + 1$.

Let $s \in 2^\beta \bigcap M$. Since $p_s \Vdash_{\mathbb{P}'} (t_s \in \dot{B})$ and $\Psi(\alpha_s, \alpha_{s\hat{}\langle 0\rangle}, p_s, p_{s\hat{}\langle 0\rangle}, p_{s\hat{}\langle 1\rangle}, T, \dot{B})$ is true, there are $t^i \in T_{\alpha_{s\hat{}\langle 0\rangle}}$ such that $t <_T t^i, t^0 \neq t^1$ and $p_{s\hat{}\langle i\rangle} \Vdash_{\mathbb{P}'} (t^i \in \dot{B})$.

Let $t_{s\hat{}\langle i\rangle} = t^i$ for $i = 0, 1$.

Case 2: α is a limit ordinal below κ.

Let $s \in 2^\alpha \bigcap M$. Since $\Phi(\alpha_s, p_s, T, \dot{B})$ is true, there is an $x \in T_{\alpha_s}$ such that $p_s \Vdash_{\mathbb{P}'} (x \in \dot{B})$. Since $\forall\beta < \alpha\ (p_s \leq p_{s^{-}\beta})$, then $p_s \Vdash_{\mathbb{P}'} (t_{s^{-}\beta} \in \dot{B})$. Now $t_{s^{-}\beta} <_T x$ because $\alpha_s > \alpha_{s^{-}\beta}$ for all $\beta < \alpha$.

Let $t_s = x$.

We have now finished construction and T' is a desired subtree of T. $\square$

Proof of Theorem 1: Let $\kappa_1 < \kappa_2$ be two inaccessible cardinals in M. Let $\mathbb{P}_1 = Lv(\kappa_2, \kappa_1)$, $\mathbb{P}_2 = Fn(\kappa_2^+, 2, \kappa_1)$ and $\mathbb{P}_3 = Lv(\kappa_1, \omega)$ in M. Let G_1 be a $\mathbb{P}_1$–generic filter over M, $M' = M[G_1]$, G_2 be a $\mathbb{P}_2$–generic filter over M', $M'' = M'[G_2]$, G_3 be a $\mathbb{P}_3$–generic filter over M'' and $M''' = M''[G_3]$. We want to show that $M''' \models [CH,\ 2^{\omega_1} = \omega_3$, there exists a thick Kurepa tree and there exist no Jech–Kunen trees].

We list some simple facts first:

(1) $M' \models [2^{\kappa_1} = \kappa^+ = \kappa_2]$.

(2) $M'' \models [2^{\kappa_1} = \kappa_1^{++} = \kappa_2^+]$.

(3) $M''' \models [CH,\ \kappa_1 = \omega_1,\ 2^{\omega_1} = \omega_3 = \kappa_1^{++}$ and $T = \langle 2^{<\kappa} \bigcap M'', \subseteq\rangle$ is a thick Kurepa tree.].

See [K2, pp. 232] for the proof of this.

We now show that in M''' there are no Jech–Kunen trees.

Suppose that T is a Jech–Kunen tree in M'''. Since the cardinality of T is $\omega_1 = \kappa_1$, there exists a $\theta < \kappa_2$ and a subset $I \subseteq \kappa_2^+$ of power κ_1 such that

$$T \in M[G_1 \bigcap Lv(\theta, \kappa_1)][G_2 \bigcap Fn(I, 2, \kappa_1)][G_3].$$

Let $G_1' = G_1 \bigcap Lv(\theta, \kappa_1)$, $G_1'' = G_1 \bigcap Lv(\kappa_2 \setminus \theta, \kappa_1)$, $G_2' = G_2 \bigcap Fn(I, 2, \kappa_1)$ and $G_2'' = G_2 \bigcap Fn(\kappa_2^+ \setminus I, 2, \kappa_1)$. Then the cardinality of $\mathcal{B}(T)$ in $M[G_1'][G_2'][G_3]$ is less than κ_2. Since the cardinality of $\mathcal{B}(T)$ in M''' is at least $\omega_2 = \kappa_2$, there exists a new branch of T in $M''' \setminus M[G_1'][G_2'][G_3]$.

$\mathbb{P}_3$ has κ_1–*c.c.* and $Lv(\kappa_2 \setminus \theta, \kappa_1) \times Fn(\kappa_2^+ \setminus I, 2, \kappa_1)$ is κ_1–closed. By Lemma 2, there exists a subtree T' of T in $M[G_1'][G_2'][G_3]$, which is isomorphic to the tree $\langle 2^{<\kappa_1} \bigcap M[G_1'][G_2'], \subseteq\rangle$.

Now we have that $M''' \models [|\mathcal{B}(T')| = 2^{\kappa_1} = \kappa_2^+ = 2^{\omega_1}]$. Since

$$M''' \models [|\mathcal{B}(T)| \geq |\mathcal{B}(T')| = 2^{\omega_1}],$$

T can not be a Jech–Kunen tree in M'''. A contradiction. $\square$

Remark: In the proof above $\mathbb{P}_2$ can be $Fn(\lambda, 2, \kappa_1)$ for any regular cardinal $\lambda > \kappa_1$. As a result 2^{ω_1} can be very large in the final model.

Corollary 1 *Assuming the existence of two strongly inaccessible cardinals, it is consistent with CH plus $2^{\omega_1} > \omega_2$ that every Kurepa tree is thick.*

Remark: We call that a Kurepa tree T is thin if $|\mathcal{B}(T)| = \omega_2$. If we start from M, a model of *GCH*, let $\mathbb{P} = Fn(\kappa, 2, \omega_1)$ for some regular cardinal $\kappa > \omega_2$ in M and G be a $\mathbb{P}$–generic filter over M, then $M[G]$ is a model of *CH* plus $2^{\omega_1} > \omega_2$, in which every Kurepa tree is thin. It is interesting to compare this with above corollary.

Under $\neg CH$, an ω_1–tree is called a Canadian tree [B] (or a weak Kurepa tree [T1]) if $|\mathcal{B}(T)| > \omega_1$.

Corollary 2 *Assuming the existence of two strongly inaccessible cardinals, it is consistent with* $\neg CH$ *plus* $2^{\omega_1} > \omega_2$ *that there exists a thick Kurepa tree and every Canadian tree has* 2^{ω_1}*–many branches.*

Proof: Let $M, \mathbb{P}_1, \mathbb{P}_2, G_1, G_2, M'$ and M'' be the same as in the proof of Theorem 1. Let

$$\mathbb{P}_3 = Lv(\kappa_1, \omega) \times Fn(\kappa_2^+, 2),$$

G_3 be a $\mathbb{P}_3$–generic filter over M'' and $M''' = M''[G_3]$. Then

$$M''' \models [2^\omega = 2^{\omega_1} = \omega_3 \text{ and there exists a thick Kurepa tree. }].$$

Let T be a Canadian tree in M'''. Then there exists a subset I of κ_2^+ with $|I| \leq \kappa_1$ such that

$$T \in M''[G_3 \bigcap Lv(\kappa_1, \omega) \times Fn(I, 2)].$$

Let $G_3' = G_3 \bigcap Lv(\kappa_1, \omega) \times Fn(I, 2)$. Since $Fn(\kappa_2^+ \smallsetminus I, 2)$ is σ–centered, every branch of T in M''' is already in $M''[G_3']$. Since $Lv(\kappa_1, \omega) \times Fn(I, 2)$ is also κ_1–*c.c.*, then by the same argument as in the proof of Theorem 1, we can show that T has $2^{\omega_1} = \kappa_2^+ = \omega_3$ many branches in $M''[G_3']$. Hence T has $2^{\omega_1} = \omega_3$ many branches in M'''. $\square$

We would like to end this paper by asking some questions.

(1) Can we find a model of *CH* plus $2^{\omega_1} > \omega_2$, in which there exists a Jech–Kunen tree but there are no Kurepa trees ?

The author [Ji] found, by assuming the existence of one inaccessible cardinal, a model of *CH* plus $2^{\omega_1} > \omega_2$, in which there exists a Jech–Kunen tree which has no Kurepa subtrees.

(2) Can we assume the existence of only one inaccessible cardinal in Theorem 1 ?

(3) Can we add Martin's Axiom to the model in Corollary 4 ?

(4) Can we find a model of *CH* plus $2^{\omega_1} = \omega_4$, in which only Kurepa trees with ω_3–many branches exist ?

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