A similar but better result has been proven by I. Bardaji and D. J. Grynkiewicz in their paper *Long Arithmetic Progressions in Sets with Small Sumset*, Integers 10 (2010), 335–350, (electronic) A28

Detailed structure on the sum of two distinct sets of integers

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Abstract

In [2] Freiman reveals some detailed structural information on a set A when the size of the tow-fold sum of A is less than $3|A|-3$. In this paper we slightly generalize the main theorem in [2] and apply the idea to obtain some detailed structural information and improve the some results in [4, 6] for the sum of two distinct sets.

1 Introduction

Let A be a finite set of integers and $|A|$ be the cardinality of A. Freiman's inverse problem for small doubling constant searches the structural information of A when the size of $2A = \{a + a' : a, a' \in A\}$ is small, say for example, less than 4 times the size of A.

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In Theorem A.1[1] it is shown that if $|2A| = 2|A| - 1 + b < 3|A| - 3$, then A is a subset of an arithmetic progression of length at most $|A| + b$. Recently, Freiman obtained in [2] more detailed structural information of A in addition to that of A being a large subset of an arithmetic progression when $|2A| < 3|A| - 3$. In the next section we slightly generalize the main theorem in [2] using Freiman's idea. Instead of defining e and c according to the stability type of holes in [2], we define l and r according to the density of elements in A on the left and the density of elements in A on the right. This slight change leads to a convenient approach to the case when the sum of two distinct sets is considered.

Freiman's inverse problem for the sum of two distinct sets turns out to be much more difficult. Theorem A.2 gives a lower bound for the size of $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Theorem A.3 improves the appearance of Theorem A.2 under the condition that $|A| + |B| - 2 - \delta \geqslant n$ where δ is defined in (5). These theorems yield Corollary A.4 that if $|A + B| = |A| + |B| - 1 + b < |A| + |B| + \min\{|A|, |B|\} - 2 - \delta$, then A and B both are subsets of two arithmetic progressions I_A and I_B with the same difference d such that $|I_A| \leqslant |A| + b$ and $|I_B| \leqslant |B| + b$. In [3, Chapter 2], some structural properties of A and B are discussed with some restrictive conditions such as $|A| = |B|$ and $\max A - \min A = \max B - \min B$. In the third section we apply Freiman's idea to gain more detailed structural information of sets A and B when $|A + B| < |A| + |B| + \min\{|A|, |B|\} - 2 - \delta$ along the same line as in [2] as well as to improve the lower bound of $|A + B|$ in [4] and in [6] without requiring $|A| = |B|$ or $\max A - \min A = \max B - \min B$.

Let a, b be integers. Throughout this paper we will write $[a, b]$ for the interval of *integers* between a and b including a and b. Notice that $[a, b] = \emptyset$ if $a > b$. For any set of integers A we will use the following notation:

$$A[a, b] := A \cap [a, b] \text{ and } A(a, b) := |A[a, b]|.$$

Hence $A[a, b]$ is a set and $A(a, b)$ is an integer.

[1] We will frequently refer to Theorem A.1, Theorem A.2, and Theorem A.3. They are included in the appendix for the reader's convenience.

2 The sum of two copies of the same set A

Throughout this section the letter A always represents a set of integers such that $0 = \min A$ and $n = \max A > 1$. Let $H = [0, n] \smallsetminus A$ and $h = |H|$. The set H contains all "holes" of A in $[0, n]$.

Theorem 2.1 *Suppose that* $|A| > 2$, $\gcd(A) = 1$, *and*

$$|2A| = 2|A| - 1 + b < 3|A| - 3. \tag{1}$$

Let $l, r \in [-1, n]$ *be such that* $A(0, l) \leqslant \frac{1}{2}(l + 1)$ *and* $A(n - r, n) \leqslant \frac{1}{2}(r + 1)$. *Then* $l < n - r$.

Proof: The idea of the proof is due to Freiman [2]. We reproduce the proof here because a slight generalization is made as well as that the reader can benefit from a warm-up before reading the proofs in the next section.

Notice that $b < |A| - 2$. By Theorem A.1 we have that $n + 1 \leqslant |A| + b < 2|A| - 2$, which implies that $|A| > \frac{1}{2}(n + 1) + 1$. For each $x \in [0, n]$ we have that $A(0, x) > \frac{1}{2}(x + 1)$ or $A(x, n) > \frac{1}{2}(n - x + 1)$ because otherwise

$$\begin{aligned} 2|A| &\leqslant 2A(0, x) + 2A(x, n) \\ &\leqslant (x + 1) + (n - x + 1) = n + 2 \leqslant |A| + b + 1, \end{aligned}$$

which implies that $b \geqslant |A| - 1$. This implies that for each $x \in [0, n]$ either $x \in 2A$ or $x + n \in 2A$ by the pigeonhole principle.

Assume in contrary that $l \geqslant n - r$. Notice that $l = n - r$ is impossible by the definition of l, r and the argument above. Hence we can assume that $l > n - r$.

Let

- $h_1 = |\{x \in H : x \notin 2A \text{ and } x + n \in 2A\}|$,
- $h_2 = |\{x \in H : x \in 2A \text{ and } x + n \notin 2A\}|$, and
- $h_3 = |\{x \in H : x \in 2A \text{ and } x + n \in 2A\}|$.

The letter h_1 represents the number of left stable holes, h_2 represents the number of right stable holes, and h_3 represents the number of unstable holes of A in [2].

Since $b = |(2A) \setminus (A \cup (n + A))|$ by (1), we have that $h = h_1 + h_2 + h_3$ and $b = h_1 + h_2 + 2h_3$. Hence $b - h = h_3$. Let r' be such that

$$n - r' = \max\left(\left\{x \in [n - r, l\,] : A(x, n) \leqslant \frac{1}{2}(n - x + 1)\right\} \cup \{n - r\}\right).$$

Since $A(0, l) \leqslant \frac{1}{2}(l + 1)$, we have that $A(l, n) > \frac{1}{2}(n - l + 1)$, which implies that $n - r' < l$. Let

$$l' = \min\left(\left\{x \in [n - r', l] : A(0, x) \leqslant \frac{1}{2}(x + 1)\right\} \cup \{l\}\right).$$

Then $n - r' < l'$ by the same reason above. Notice that $n - r', l'$ may or may not be the elements of A. The important point is that $A(0, l') \leqslant \frac{1}{2}(l' + 1)$ and $A(n - r', n) \leqslant \frac{1}{2}(r' + 1)$ are true, and for any $x \in H[n - r' + 1, l' - 1]$ we have that $A(0, x) > \frac{1}{2}(x + 1)$ and $A(x, n) > \frac{1}{2}(n - x + 1)$ by the definition of $n - r'$ and l', which imply that $x \in 2A$ and $x + n \in 2A$ by the pigeonhole principle. Hence $H(n - r' + 1, l' - 1) \leqslant h_3$. Notice that $H(n - r', l') \leqslant l' - (n - r') + 1$. Now we have that

$$\begin{aligned} h &\geqslant H(0, l') + H(n - r', n) - H(n - r', l') \\ &\geqslant \frac{1}{2}(l' + 1) + \frac{1}{2}(r' + 1) - H(n - r', l') \\ &\geqslant \frac{1}{2}(n + 1) + \frac{1}{2}(l' - (n - r') + 1) - H(n - r', l') \\ &\geqslant \frac{1}{2}(|A| + h) - \frac{1}{2}H(n - r', l'). \end{aligned}$$

Hence $h \geqslant |A| - H(n - r', l')$. Now we have that

$$\begin{aligned} 0 &\geqslant |A| - b + (b - h) - H(n - r', l') \\ &\geqslant |A| - b + h_3 - H(n - r' + 1, l' - 1) - 2 \\ &\geqslant |A| - (|A| - 3) - 2 = 1, \end{aligned}$$

which is absurd. Therefore, we conclude that $l < n - r$.

Theorem 2.2 *Suppose that* $|A| > 2$, $\gcd(A) = 1$, *and* A *satisfies (1). Let*

$$L = \max\left(\left\{x \in [0,n] : A(0,x) \leqslant \frac{1}{2}(x+1)\right\} \cup \{-1\}\right) \quad \textit{and} \tag{2}$$

$$R = \max\left(\left\{x \in [0,n] : A(n-x,n) \leqslant \frac{1}{2}(x+1)\right\} \cup \{-1\}\right). \tag{3}$$

Then

1. $n - R - L \geqslant n + 2 - 2h > 3$,
2. $J = [L+1, 2n-R-1] \subseteq (2A)$,
3. $b \geqslant h + H(L+1, n-R-1)$,
4. $L+1, n-R-1 \in A$, *and*
5. $A(0,L) = \frac{1}{2}(L+1)$ *and* $A(n-R,n) = \frac{1}{2}(R+1)$, *hence* $L+1$ *and* $R+1$ *are even numbers.*
6. $|J| = 2n - R - L - 1 = 2|A| - 1 + 2H(L+1, n-R-1)$,

   Furthermore, if $n+1 = |A| + b$, *or equivalently* $h = b$, *then*

7. $(2A)(0,L) = A(0,L)$,
8. $(2A)(2n-R, 2n) = A(n-R, n)$, *and*
9. $[L+1, n-R-1] \subseteq A$.

Proof: By Theorem 2.1 we have that $n-R-L > 0$. Since $n+1 = |A|+h > \frac{1}{2}(n+1) + 1 + h$, we have that $h < \frac{1}{2}(n+1) - 1$. Since

$$\frac{1}{2}(L+1) + \frac{1}{2}(R+1) \leqslant H(0,L) + H(n-R,n) \leqslant h$$

and

$$\begin{aligned} n+1-h = |A| &= A(0,L) + A(L+1, n-R-1) + A(n-R,n) \\ &\leqslant \frac{1}{2}(L+1) + n - R - L - 1 + \frac{1}{2}(R+1) \leqslant h + n - R - L - 1, \end{aligned}$$

we have that $n-R-L \geqslant n+2-2h > n+2-(n+1)+2 = 3$. Hence item 1 is true. Notice that when $|A|$ is larger, h becomes smaller. Hence $n+2-2h$ becomes larger, which drives L and R farther apart.

Item 2 follows from the definition of L, R and the pigeonhole principle.

Item 3 follows from the fact that $b-h = h_3 \geqslant H(L+1, n-R-1)$.

Item 4 is true because if $L+1 \notin A$, then

$$A(0, L+1) = A(0, L) \leqslant \frac{1}{2}(L+1) < \frac{1}{2}(L+2),$$

which contradicts the maximality of L, and if $n-R-1 \notin A$, then

$$A(n-R-1, n) < \frac{1}{2}(R+2),$$

which contradicts the maximality of R.

Item 5 is true by the following argument: If $A(0, L) < \frac{1}{2}(L+1)$, then $2A(0, L) \leqslant L$. Hence $2A(0, L+1) \leqslant L+2$ and $A(0, L+1) \leqslant \frac{1}{2}(L+2)$, which contradicts the maximality of L. By the same reason $A(n-R, n) = \frac{1}{2}(R+1)$.

Item 6 is a consequence of item 5:

$$\begin{aligned} |J| &= 2n - R - L - 1 \\ &= R + 1 + 2(n-R-L-1) + L + 1 - 1 \\ &= 2A(n-R, n) + 2A(L+1, n-R-1) \\ &\quad +2H(L+1, n-R-1) + 2A(0, L) - 1 \\ &= 2|A| - 1 + 2H(L+1, n-R-1). \end{aligned}$$

Assume that $n+1 = |A| + b$. Since

$$\begin{aligned} |2A| &= 2|A| - 1 + b \\ &\geqslant (2A)(0, L) + (2A)(L+1, 2n-R-1) + (2A)(2n-R, 2n) \\ &\geqslant A(0, L) + 2n - R - L - 1 + A(n-R, n) \\ &= 2|A| + b - 1 + n - R - L - 1 - A(L+1, n-R-1) \\ &\geqslant 2|A| - 1 + b + H(L+1, n-R-1), \end{aligned}$$

we have that $H[L+1, n-R-1] = \emptyset$, $A(0, L) = (2A)(0, L)$, and $A(n-R, n) = (2A)(2n-R, 2n)$. Now Item 7, 8, 9 follow.

Remark 2.3 *Suppose that A satisfies all conditions in Theorem 2.2. Let*

$$e = \max(\{x \in [0, n] : x \notin 2A\} \cup \{-1\}) \text{ and}$$

$$c = \min(\{x \in [0, n] : x + n \notin 2A\} \cup \{n + 1\}).$$

It is shown in [2] that $e < c$. Since $x \in [L + 1, n]$ implies $x \in 2A$ and $x \in [0, n - R - 1]$ implies $x + n \in 2A$, we have that $e \leqslant L$ and $c \geqslant n - R$ where L, R are defined in (2) and (3). Hence $L < n - R$ implies that $e < c$.

Example 2.4 *Let*

$$A = \{0\} \cup [10, 90] \cup \{100\}.$$

Then

$$2A = \{0\} \cup [10, 190] \cup \{200\},$$

$|A| = 83$, and

$$|2A| = 183 = 2|A| - 1 + 18 < 3|A| - 3.$$

Notice that $e = 9 < L = 17$ and $c = 91 > n - R = 83$. Hence Theorem 2.1 is a proper generalization of [2, Lemma 6].

3 The sum of two distinct sets A and B

Throughout this section the letters A, B always represent two sets of integers with

$$0 = \min A = \min B \text{ and } 1 < m = \max A \leqslant \max B = n. \tag{4}$$

Let $H_A = [0, m] \setminus A$, $h_A = |H_A|$, $H_B = [0, n] \setminus B$, and $h_B = |H_B|$. The set H_A contains all "A-holes" in $[0, m]$ and H_B contains all "B-holes" in $[0, n]$.

$$\text{Let } \delta = 1 \text{ if } n = m \text{ and } \delta = 0 \text{ otherwise.} \tag{5}$$

Theorem 3.1 *Suppose that $|A|, |B| > 1$, A, B satisfy (4),*

$$|A| + |B| - 2 - \delta \geqslant n, \text{ and} \tag{6}$$

$$|A + B| = |A| + |B| - 1 + b < |B| + 2|A| - 2 - \delta. \tag{7}$$

If $l, r \in [-1, m]$ such that $A(0, l) + B(0, l) \leqslant l + 1$ and $A(m - r, m) + B(n - r, n) \leqslant r + 1$, then $l < m - r$.

Proof Let

- $S_{d,l} = \{x \in [0,n] \setminus (A \cup B) : x \notin A+B\}$ and $h_{d,l} = |S_{d,l}|$,
- $S_{d,r} = \{x \in [0,n] \setminus (A \cup B) : x+n \notin A+B\}$ and $h_{d,r} = |S_{d,r}|$,
- $S_{d,f} = \{x \in [0,n] \setminus (A \cup B) : x, x+n \in A+B\}$ and $h_{d,f} = |S_{d,f}|$,
- $S_{A,f} = \{x \in B \setminus A : x+n \in A+B\}$ and $h_{A,f} = |A_{A,f}|$

We call $S_{d,l}$ the set of all left double holes, $S_{d,r}$ the set of all right double holes, $S_{d,f}$ the set of all filled double holes, and $S_{A,f}$ the set of all filled A-holes. Notice that any $x \in B[m+1,n]$ is an unfilled A-holes and any $x \in H_B[m+1,n]$ is a right double hole by the arguments below.

Let $x \in [0,n]$. Suppose that $x \in [0,m]$. If $x \notin A+B$ and $x+n \notin A+B$, then $A(0,x)+B(0,x) \leqslant x+1$ and $A(x,m)+B(n-m+x,n) \leqslant m-x+1$. Hence

$$\begin{aligned} n &\leqslant |A|+|B|-2-\delta \\ &\leqslant A(0,x)+A(x,m)+B(0,x)+B(n-m+x,n) \\ &\quad +n-m-1+\delta-2-\delta \\ &\leqslant x+1+m-x+1+n-m-3 = n-1, \end{aligned}$$

which is absurd. Thus we can assume that either $x \in A+B$ or $x+n \in A+B$. Suppose that $x \in [m+1,n]$. Then $\delta = 0$. If $x \notin A+B$, then $A \cap (x - B[x-m,x]) = \emptyset$, which implies that $|A| + B(x-m,x) \leqslant m+1$. Hence

$$\begin{aligned} n &\leqslant |A|+|B|-2 \\ &\leqslant m+1+B(0,x-m-1)+B(x+1,n)-2 \\ &\leqslant m+1+x-m+n-x-2 = n-1, \end{aligned}$$

which is absurd. Hence $x \in A+B$. Combining all arguments we can conclude that either $x \in A+B$ or $x+n \in A+B$ for any $x \in [0,n]$. As a corollary we have that

$$H_B = S_{d,l} \cup S_{d,r} \cup S_{d,f} \cup (A \setminus B)$$

or equivalently,

$$h_B = h_{d,l} + h_{d,r} + h_{d,f} + |A \setminus B|.$$

By (7) we have that

$$b = |(A+B) \setminus (B \cup (n+A))|.$$

For each $y \in (A+B) \setminus (B \cup (n+A))$, if $y \in [0,n]$, then y is a B-hole, and if $y \in [n+1, n+m]$, then $x = y - n$ is either a filled double hole when $x \notin B$ or a filled A-hole when $x \in B$. Hence

$$b = h_{d,l} + h_{d,r} + 2h_{d,f} + |A \setminus B| + h_{A,f}.$$

Therefore,

$$b - h_B = h_{d,f} + h_{A,f}.$$

Suppose that the theorem is not true, i.e., $l \geqslant m-r$. Since $l = m-r = x$ is impossible by an argument above, we can assume that $l > m-r$. Let r' be such that

$$\begin{aligned} m - r' = \max\big(\{x \in [m-r, l\,] :\\ A(x,n) + B(n-m+x, n) \leqslant (n-x+1)\} \cup \{r\}\big). \end{aligned}$$

Since $A(0,l)+B(0,l) \leqslant l+1$, we have that $A(l,m)+B(n-m+l,n) > m-l+1$, which implies that $m-r' < l$. Let

$$l' = \min\left(\{x \in [m-r', l] : A(0,x) + B(0,x) \leqslant x+1\} \cup \{l\}\right).$$

Again we have that $m - r' < l'$.

By the definition of r', l' we have that $A(0,x) + B(0,x) > x+1$ and $A(x,m) + B(n-m+x,n) > m-x+1$, which imply $x, x+n \in A+B$ for any $x \in [m-r'+1, l'-1]$. Notice also that $A(0,l') + B(0,l') \leqslant l'+1$ implies $H_A(0,l') + H_B(0,l') \geqslant l'+1$ and $A(m-r',m) + B(n-r',n) \leqslant r'+1$ implies $H_A(m-r',m) + H_B(n-r',n) \geqslant r'+1$.

Clearly, $H_A(m-r', l') \leqslant l' - (m-r') + 1$ and

$$H_A(m-r'+1, l'-1) \leqslant h_{d,f} + h_{A,f} = b - h_B.$$

If $n-r'>l'$, then $n>m$, $\delta=0$, and

$$\begin{aligned}h_A+h_B &\geqslant H_A(0,l')+H_A(m-r',m)-H_A(m-r',l')\\ &\quad +H_B(0,l')+H_B(n-r',n)\\ &\geqslant l'+1+r'+1-H_A(m-r',l')\geqslant m+1.\end{aligned}$$

Hence

$$m+n+2=|A|+|B|+h_A+h_B\geqslant n+2+m+1,$$

which is absurd. Thus we can assume that $n-r'\leqslant l'$.

Since $n-r'\leqslant l'$, we have that $H_B(n-r',l')\leqslant l'-(n-r')+1\leqslant l'-(m-r')+\delta$. By (7) we have that $b<|A|-1-\delta$. Hence

$$\begin{aligned}h_A+h_B &\geqslant H_A(0,l')+H_A(m-r',m)-H_A(m-r',l')\\ &\quad +H_B(0,l')+H_B(n-r',n)-H_B(n-r',l')\\ &\geqslant l'+1+r'+1-H_A(m-r',l')-H_B(n-r',l')\\ &\geqslant m+2-H_A(m-r',l')-\delta\\ &\geqslant |A|+h_A-H_A(m-r'+1,l'-1)-1-\delta.\end{aligned}$$

Thus we have that

$$\begin{aligned}0 &\geqslant |A|-b+(b-h_B)-H_A(m-r'+1,l'-1)\\ &\quad -1-\delta>|A|-|A|+1+\delta-1-\delta\geqslant 0,\end{aligned}$$

which is absurd. This completes the proof and we conclude that $l<m-r$.

Theorem 3.2 *Suppose that $|A|,|B|>1$ and A,B satisfy (4), (6), and (7). Let*

$$L=\max\left(\{x\in[0,m]:A(0,x)+B(0,x)\leqslant x+1\}\cup\{-1\}\right)\quad \textit{and}$$

$$R=\max\left(\{x\in[0,m]:A(m-x,m)+B(n-x,n)\leqslant x+1\}\cup\{-1\}\right).$$

Then

1. $m-R-L\geqslant m+2-h_A-h_B\geqslant 2+\delta$,

2. $J=[L+1,m+n-R-1]\subseteq A+B$,

3. $L+1 \in A$, $L+1 \in B$, $m-R-1 \in A$, *and* $n-R-1 \in B$,

4. $A(0,L)+B(0,L) = L+1$ *and* $A(m-R,m)+B(n-R,n) = R+1$,

5. $|J| = m+n-R-L-1 = |A|+|B|-1+p+q$ *where* $p = H_A(L+1, m-R-1)$ *and* $q = H_B(L+1, n-R-1)$,

6.

$$\begin{aligned}
&|A+B| \\
&\quad = (A+B)(0,L)+m+n-R-L-1+(A+B)(m+n-R,n+m) \\
&\quad \geqslant \max\{A(0,L),B(0,L)\}+m+n-R-L-1 \\
&\quad\quad + \max\{A(m-R,m),B(n-R,n)\} \\
&\quad \geqslant |A|+|B|-1+p+q+\max\{p_L,q_L\}+\max\{p_R,q_R\} \\
&\quad \geqslant |A|+|B|-1+\max\{h_A+q, h_B+p\}
\end{aligned}$$

where

(a) $p_L = H_A(0,L)$,

(b) $p_R = H_A(m-R,m)$,

(c) $q_L = H_B(0,L)$, *and*

(d) $q_R = H_B(n-R,n)$.

Furthermore, if in addition $|A+B| = |A|+|B|-1+p+q+\max\{p_L,q_L\}+\max\{p_R,q_R\}$, *or equivalently* $b = p+q+\max\{p_L,q_L\}+\max\{p_R,q_R\}$, *then*

7. $(A+B)(0,L) = \max\{A(0,L),B(0,L)\}$ *and*

8. $(A+B)(m+n-R,n+m) = \max\{A(m-R,m),B(n-R,n)\}$.

Proof: By Theorem 3.1 we have that $m-R-L>0$. Since

$$m+n+2 = |A|+|B|+h_A+h_B \geqslant n+2+\delta+h_A+h_B,$$

we have that $h_A + h_B \leqslant m - \delta$. Since

$$\begin{aligned} m + n + 2 - h_A - h_B &= |A| + |B| \\ &= A(0, L) + A(L+1, m-R-1) + A(m-R, m) \\ &\quad + B(0, L) + B(L+1, n-R-1) + B(n-R, n) \\ &\leqslant (L+1) + (m-R-L-1) + (n-R-L-1) + (R+1) \\ &\leqslant h_A + h_B + n - m + 2(m-R-L-1), \end{aligned}$$

we have that

$$2(m-R-L-1) \geqslant 2m + 2 - 2h_A - 2h_B \geqslant 2m + 2 - 2(m-\delta) = 2 + 2\delta,$$

which implies that

$$m - R - L \geqslant m + 2 - h_A - h_B \geqslant 2 + \delta.$$

Hence item 1 is true. Notice that when $|A| + |B|$ is larger, $h_A + h_B$ becomes smaller. Hence $m + 2 - h_A - h_B$ becomes larger, which drives L and $m - R$ farther apart.

Item 2 follows from the definition of L, R, and the pigeonhole principle.

If $L + 1 \notin A$ or $L + 1 \notin B$, then

$$A(0, L+1) + B(0, L+1) \leqslant A(0, L) + B(0, L) + 1 \leqslant L + 2,$$

which contradicts the maximality of L. By the same reason we have $m - R - 1 \in A$ and $n - R - 1 \in B$. Hence item 3 is true.

Suppose that $A(0, L) + B(0, L) < L + 1$. Then

$$A(0, L+1) + B(0, L+1) \leqslant L + 2,$$

which contradicts the maximality of L. Therefore,

$$A(0, L) + B(0, L) = L + 1.$$

By the same reason

$$A(m-R, m) + B(n-R, n) = R + 1.$$

Hence item 4 is true. Notice that $A(0,L) = H_B(0,L)$, $B(0,L) = H_A(0,L)$, $A(m-R,m) = H_B(n-R,n)$, and $B(n-R,n) = H_A(m-R,m)$.

Item 5 is a consequence of item 4:

$$\begin{aligned}|J| &= m+n-R-L-1\\ &= L+1+m-R-L-1+n-R-L-1+R+1-1\\ &= A(0,L)+B(0,L)+A(L+1,m-R-1)+p\\ &\quad +B(L+1,n-R-1)+q+A(m-R,m)+B(n-R,n)-1\\ &= |A|+|B|-1+p+q.\end{aligned}$$

For item 6 we have that

$$\begin{aligned}|A+B| &= (A+B)(0,L)+(A+B)(L+1,m+n-R-1)\\ &\quad +(A+B)(m+n-R,m+n)\\ &\geqslant \max\{A(0,L),B(0,L)\}+m+n-R-L-1\\ &\quad +\max\{A(m-R,m),B(n-R,n)\}\\ &= |A|+h_A-1+|B|+h_B-1-R-L-1\\ &\quad +\max\{A(0,L),B(0,L)\}+\max\{A(R,m),B(n-m+R,n)\}\\ &= |A|+|B|-1+h_A+h_B-(L+1)-(R+1)\\ &\quad +\max\{A(0,L),B(0,L)\}+\max\{A(R,m),B(n-m+R,n)\}\\ &\geqslant |A|+|B|-1+h_A+h_B-\min\{p_L,q_L\}-\min\{p_R,q_R\}\\ &\geqslant |A|+|B|-1+p+q+\max\{p_L,q_L\}+\max\{p_R,q_R\}.\end{aligned}$$

If $b = p+q+\max\{p_L,q_L\}+\max\{p_R,q_R\}$, then all inequalities in the proof of item 6 become equalities. Hence item 7 and item 8 follow.

Remark 3.3 *1. Besides the structural information of A and B, item 6 of Theorem 3.2 is an improvement of the lower bound of $|A+B|$ in Theorem A.2 and Theorem A.3 under the condition (7). Example 3.6 below gives some idea why the improvement is non-trivial.*

2. The condition (7) can be relaxed to $|A+B| < |A|+|B|+\max\{|A|,|B|\}-3$ only when $n = m$. Example 3.7 below shows that (7) cannot be replaced by $|A+B| < |A|+2|B|-2$ when $n > m$.

Example 3.4 *Let*

$$A = B = [0, 9] \cup [20, 29].$$

Then $L = 19$ *and* $m - R = 10$*. Hence* $L > n - R$*. However,*

$$|A + B| = 57 = |B| + 2|A| - 2 - \delta.$$

Hence (7) is a necessary condition.

Example 3.5 *Let*

$$A = \{3x + i : x \in [0, 20] \ \ and \ \ i = 0, 1\} \ \ and$$

$$B = \{3x : x \in [0, 21]\}.$$

Then $|A| = 42$*,* $m = 61$*,* $|B| = 22$*,* $n = 63$*,*

$$A + B = \{3x + i : x \in [0, 41] \ \ and \ \ i = 0, 1\},$$

$$|A + B| = 84 < |B| + 2|A| - 2,$$

$L = 63$*,* $R = 63$*. However,*

$$|A| + |B| - 2 = 62 < n = 63.$$

Hence (6) is a necessary condition.

The following example shows that item 6 is a proper improvement of the lower bound of $|A + B|$ obtained in Theorem A.2 and Theorem A.3.

Example 3.6 *Let*

$$A = [0, 8] \cup [10, 19] \cup [21, 30] \cup \{40\} \ \ and$$

$$B = \{0\} \cup [10, 19] \cup [21, 30] \cup [32, 40].$$

Then

1. $m = n = 40$*,*

2. $|A| = 30 = |B|$*,*

3. $|A| + |B| - 3 = 57 > n = 40$, *hence (6) is satisfied,*

4. $A + B = [0, 8] \cup [10, 70] \cup [72, 80]$,

5. $|A + B| = 79 = |A| + |B| - 1 + 19 < |B| + 2|A| - 2 = 88$, *hence (7) is satisfied,*

6. $L = 9$, $R = 9$,

7. $p_L = 1$, $p = 1$, $p_R = 9$, $q_L = 9$, $q = 1$, $q_R = 1$, $h_A = 11$, $h_B = 11$,

8. $p + q + \max\{p_L, q_L\} + \max\{p_R, q_R\} = 20 > \max\{h_A, h_B\} = 11$.

Hence $|A + B| \geqslant |A| + |B| - 1 + p + q + \max\{p_L, q_L\} + \max\{p_R, q_R\} = 79$ *is an improvement of* $|A + B| \geqslant |A| + |B| - 1 + \max\{h_A, h_B\} = n + |A| = 70$.

The next example shows that (7) cannot be replaced by $|A + B| < |A| + 2|B| - 2$.

Example 3.7 *Let*

$$A = [0, 9] \cup [30, 39] \text{ and } B = [0, 19] \cup [30, 49].$$

Then $|A| = 20$, $|B| = 40$, $|A| + |B| - 2 = 58 > n = 49$, $A + B = [0, 88]$, $|A + B| = 89 < |A| + 2|B| - 2 = 98$, $L = 29 > m - R = 10$.

A Appendix

Theorem A.1 (G. A. Freiman, [1]) *Let A be a finite set of integers and $|A| > 2$. If $|2A| = 2|A| - 1 + b < 3|A| - 3$, then A is a subset of an arithmetic progression of length at most $|A| + b$.*

The proof of Theorem A.1 can also be found in [5, p.28]

Theorem A.2 (V. Lev & P. Y. Smeliansky, [4]) *Let A and B be two finite set of non-negative integers such that $0 \in A \cap B$, $|A|, |B| > 1$, $\gcd(B) = 1$, $m = \max A \leqslant n = \max B$. Then $|A + B| \geqslant \min\{n + |A|, |B| + 2|A| - 2 - \delta\}$ where δ is defined in (5).*

The proof of Theorem A.2 can also be found in [5, p.118].

Theorem A.3 (Y. V. Stanchescu, [6]) *Let A and B be two finite set of non-negative integers such that $0 \in A \cap B$, $|A|, |B| > 1$, $m = \max A \leqslant n = \max B$, $h_A = m + 1 - |A|$, and $h_B = n + 1 - |B|$. If $n \leqslant |A| + |B| - 2 - \delta$ where δ is defined in (5), then*

$$|A + B| \geqslant |A| + |B| - 1 + \max(h_A, h_B).$$

The following corollary is in [6].

Corollary A.4 *Let A and B be two finite set of non-negative integers such that $0 \in A \cap B$, $|A|, |B| > 1$, $m = \max A$, $n = \max B$, and $\gcd(A \cup B) = 1$. If $|A + B| = |A| + |B| - 1 + b < |B| + |A| + \min\{|A|, |B|\} - 2 - \delta$ where δ is defined in (5), then $m + 1 \leqslant |A| + b$ and $n + 1 \leqslant |B| + b$.*

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