Distinguishing Three Strong Saturation Properties In Nonstandard Analysis [1]

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Abstract

Three results in [14] and one in [8] are analyzed in §§3–6 in order to supply examples on Loeb probability spaces, which distinguish the different strength among three generalizations of κ-saturation, as well to answer some questions in §7 of [15]. In §3 we show that not every automorphism of a Loeb algebra is induced by an internal permutation, in §4 we show that if the $\aleph_1$-special model axiom is true, then every automorphism of a Loeb algebra is induced by a point-automorphism, in §5 we show that not every measure-preserving homomorphism from a small subalgebra to a Loeb algebra is induced by an internal permutation, without assuming full-saturation, in §6 we show that, under some cardinality assumptions, the $\aleph_1$-isomorphism property does not guarantee the compactness of a Loeb space, and in §7 an application of the $\aleph_1$-special model axiom is given on the existence of ergodic transformations of a Loeb space, which partially answers Problem 2.3 of [5].

1. Introduction

The reader is assumed to be familiar, besides logic, with the basic knowledge of nonstandard analysis and nonstandard universes. For references, see §4.4 of [1], [4] or [11].

Let κ be an uncountable regular cardinal. For generalizing κ-saturation three strong saturation properties in nonstandard analysis, the κ-isomorphism property, the κ-special model axiom, and full-saturation are introduced in [3], [14] and [15]. It is also proven there that full-saturation is the strongest, then comes the κ-special model axiom, and then the κ-isomorphism property, and the κ-isomorphism property implies κ-saturation, for all reasonable κ. For brevity let's write IP_κ for the κ-isomorphism property and write SMA_κ for the κ-special model axiom throughout.

When working with those three strong saturation properties together with κ-saturation in nonstandard analysis, two fundamental concerns naturally arise. The first concern is why one does not always adopt a stronger property to work with. In other words, what is the reason for one to work with a weaker property rather than to

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work with a stronger one in some occasions? The second concern is just opposite, *i.e.* can one justify the necessity of adopting a stronger property rather than adopting a weaker one in some other occasions? A part of our answer to the first concern is rather simple. A nonstandard universe possessing a weaker property is easier to construct, and hence is more understandable and more accessible to a nonstandard analysis practitioner. See [8] for the details. The answer to the second concern could be sometimes technical and is a never-ending task. Every time when a result is proven under a stronger property, one can always ask whether or not the result could be a consequence of a weaker property. Most of the questions posed in §7 of [15] are of this kind. The main purpose of this paper is to give answers to those questions, so that some of the answers could be the witnesses which distinguish the different strength among those strong saturation properties. There are theorems in [6], which supply some consequences of $SMA_{\aleph_1}$ unprovable under $IP_{\aleph_1}$. Unfortunately, those consequences are all pathological and, in our point of view, lack of general mathematical interest. In fact, so far as we know, there have been no examples in some mathematical fields other than logic to show the different strength among IP_κ, SMA_κ and full-saturation. In §§5,6 we supply such examples on Loeb probability spaces by analyzing some results in [14] and one in [8].

The first question in §7 of [15] asks for an example, which is a consequence of SMA_κ unprovable under IP_κ, and suggests that Theorem 4.3 of [14] be such an example. We show in §3 that Theorem 4.3 of [14] is true in any $\aleph_1$-saturated nonstandard universe. In fact, $\aleph_1$-saturation is needed only for Loeb measure construction. The second question in §7 of [15] asks for an example, which is a consequence of full-saturation unprovable under SMA_κ, and suggests that Theorem 4.1 of [14] be such an example. We show in §4 that Theorem 4.1 of [14] is a consequence of $SMA_{\aleph_1}$. But in §5 we do give an example on Loeb probability space, which is a consequence of full-saturation unprovable under SMA_κ. That example is Theorem 4.4 of [14]. In §6 we show that $SMA_{\aleph_1}$ could not be replaced by $IP_{\aleph_1}$ in Theorem 3.9 of [8] (Theorem 3.9 of [8] is a special case of Corollary 7 of [10]). In [5] it is proven that there does not exist any Borel ergodic transformation on a Loeb probability space. It is also asked in Problem 2.3 of [5] whether there ever exists any ergodic transformation on a Loeb probability space. We show in §7 that, under $SMA_{\aleph_1}$, every Loeb probability space has a ergodic (even strong mixing) transformation. The proof is just a combination of Ross's idea in [14] together with Maharam Theorem.

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2. Notation, Definition and Maharam Theorem

Let $\mathbb{N}$ be the set of all non-negative integers. Let $V_0 = \mathbb{N}$, let $V_{n+1} = V_n \cup \mathcal{P}(V_n)$ for each $n \in \mathbb{N}$ and let $V = \bigcup_{n\in\mathbb{N}} V_n$. Then we call $(V, \in)$ the standard superstructure. By a nonstandard universe we mean the truncation at $^*\!\in$-rank ω of a proper elementary extension of the standard superstructure. We always denote by $(^*V, ^*\!\in)$ a nonstandard universe, which is at least $\aleph_1$-saturated. For brevity we often write $\in$ instead of $^*\!\in$ and write $^*V$ instead of $(^*V, ^*\!\in)$. Let $^*V$ and $^*V'$ be two nonstandard universes. We call $^*V'$ a bounded elementary extension of $^*V$ iff $^*V'$ is the truncation at $\in$-rank ω of an elementary extension of $^*V$. We write $\alpha, \beta, \gamma, \ldots$ for ordinals and $\kappa, \lambda, \eta, \ldots$ for cardinals. Given a set S let $card(S)$ denote the external (or set-theoretical) cardinality of S and let $|S|$ denote the internal cardinality of S in some $^*V$ provided S is internal in $^*V$. Let $\mathcal{L}$ always denote a first-order language. Given a structure $\mathfrak{A}$ we write $\mathcal{L}_{\mathfrak{A}}$ for the language corresponding to $\mathfrak{A}$. An $\mathcal{L}$-structure $\mathfrak{A}$ is called *internally presented* in $^*V$ iff the base set of $\mathfrak{A}$ and the interpretation in $\mathfrak{A}$ of every relation symbol and every function symbol from $\mathcal{L}$ are internal in $^*V$.

We say that $^*V$ satisfies IP_κ iff the following is true.

> For any language $\mathcal{L}$ with $card(\mathcal{L}) < \kappa$, for any two internally presented $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ in $^*V$, if $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent, then $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic.

An $\mathcal{L}$-structure $\mathfrak{A}$ is called a special model if there is an increasing sequence $\langle \mathfrak{A}_\alpha : \alpha < card(\mathfrak{A})\rangle$ of elementary submodels of $\mathfrak{A}$ such that

(1) $\mathfrak{A}_\alpha$ is $card(\alpha)^+$-saturated for every $\alpha < card(\mathfrak{A})$,

(2) $\mathfrak{A} = \bigcup_{\alpha<card(\mathfrak{A})} \mathfrak{A}_\alpha$,

where $card(\mathfrak{A})$ means the external cardinality of the base set of $\mathfrak{A}$. The sequence above is usually called a specializing sequence of $\mathfrak{A}$.

We say that ${}^*V$ satisfies SMA_κ iff the following is true.

For any language $\mathcal{L}$ with $card(\mathcal{L}) < \kappa$ every internally presented $\mathcal{L}$-structure in ${}^*V$ is a special model.

A structure $\mathfrak{A}$ is called saturated if it is $card(\mathfrak{A})$-saturated.
We say that ${}^*V$ is fully-saturated if the following is true.

Every internally presented $\mathcal{L}$-structure $\mathfrak{A}$ in ${}^*V$ with $card(\mathcal{L}) < card(\mathfrak{A})$ is a saturated model.

By saying "$IP_\kappa \ldots$" in the paper, we really mean "suppose ${}^*V$ satisfies $IP_\kappa \ldots$". Same for "SMA_κ" and for "full-saturation". Let's list some facts about those three properties. The proofs of those facts could be found in [3], [6], [7], [15] and [18].

Facts: (1) Full-saturation $\Longrightarrow$ SMA_κ $\Longrightarrow$ IP_κ $\Longrightarrow$ κ-saturation.
(2) $IP_\lambda \Longrightarrow IP_\kappa$ and $SMA_\lambda \Longrightarrow SMA_\kappa$ when $\lambda > \kappa$.
(3) $IP_\kappa \Longleftrightarrow IP_{\aleph_0} +$ κ-saturation, and $SMA_\kappa \Longleftrightarrow SMA_{\aleph_0} +$ κ-saturation.
(4) $IP_{\aleph_0}$ implies that any two infinite internal sets have same external cardinality.

For any ${}^*V$ satisfying at least $IP_{\aleph_0}$ let's denote by $\Xi_{{}^*V}$ the common external cardinality of every infinite internal set in ${}^*V$. Sometimes we write just Ξ when it is clear which ${}^*V$ we work with. When we say ${}^*V$ satisfies IP_κ or SMA_κ we assume always that $\kappa \leqslant \Xi_{{}^*V}$. Let $\mathbb{Z}$ and $\mathbb{R}$ be the set of all standard integers, and all standard real numbers, respectively. Let ${}^*\mathbb{N}$, ${}^*\mathbb{Z}$ and ${}^*\mathbb{R}$ be the sets of all non-negative integers, all integers, and all real numbers in ${}^*V$, respectively. For an $r \in {}^*\mathbb{R}$ let $[r]$ be the greatest integer in ${}^*\mathbb{Z}$ less than or equal to r. By a hyperfinite integer we mean an integer in ${}^*\mathbb{N} \smallsetminus \mathbb{N}$. Given a hyperfinite integer H and let $\Omega = \{0, 1, \ldots, H-1\} \subseteq {}^*\mathbb{N}$, one can construct a standard atomless probability space $(\Omega, \mathcal{B}, L_\mu)$ called uniform hyperfinite Loeb probability space or simply *Loeb space*, where L_μ is a complete probability measure on the completion $\mathcal{B}$ of the σ-algebra, generated by the standard part of the normalized counting measure μ on the algebra $\mathcal{A}$ of all internal subsets of Ω. We call $(\Omega, \mathcal{A}, \mu)$ an (internal) normalized counting measure space, which generates $(\Omega, \mathcal{B}, L_\mu)$. See [11] or [17] for the details of Loeb measure construction. For each Loeb space $(\Omega, \mathcal{B}, L_\mu)$ generated by $(\Omega, \mathcal{A}, \mu)$ in ${}^*V$ we associate it with an internally presented structure

$$\mathfrak{A} = (\Omega \cup \mathcal{A} \cup {}^*\mathbb{R}; \Omega, \mathcal{A}, {}^*\mathbb{R}, \in, \mu, \cap, \smallsetminus, +, \cdot, <, 0, 1),$$

where Ω, $\mathcal{A}$ and ${}^*\mathbb{R}$ are considered as unary relations, $\in$ is the membership relation between Ω and $\mathcal{A}$, μ is the normalized counting measure from $\mathcal{A}$ to ${}^*\mathbb{R}$, $\cap$ and $\smallsetminus$ are Boolean operators on $\mathcal{A}$ and $({}^*\mathbb{R}; +, \cdot, <, 0, 1)$ is the real field in ${}^*V$.

Given a Loeb space $(\Omega, \mathcal{B}, L_\mu)$ and $B \in \mathcal{B}$, let $\bar{B} = \{C \in \mathcal{B} : L_\mu(B \Delta C) = 0\}$, and let $\bar{\mathcal{B}} = \{\bar{B} : B \in \mathcal{B}\}$. The Boolean algebra $\bar{\mathcal{B}}$ together with the induced measure on $\bar{\mathcal{B}}$ is called the Loeb algebra of the Loeb space $(\Omega, \mathcal{B}, L_\mu)$.

A bijection $\Phi : \bar{\mathcal{B}} \mapsto \bar{\mathcal{B}}$ is called an automorphism iff it is a Boolean algebra homomorphism and preserves the measure, *i.e.* for any $B, C \in \mathcal{B}$, $\Phi(\bar{B}) = \bar{C}$ implies $L_\mu(B) = L_\mu(C)$.

A bijection $F : \Omega \mapsto \Omega$ is called a point-automorphism iff for any $B \in \mathcal{B}$ one has $F[B], F^{-1}[B] \in \mathcal{B}$ and $L_\mu(B) = L_\mu(F[B]) = L_\mu(F^{-1}[B])$. If a point-automorphism is internal we often call it an internal permutation. The name "point-automorphism" is used in [14]. A point-automorphism is also called a measure-preserving transformation in Ergodic Theory. So we use the name "measure-preserving transformation" when Ergodic Theory is involved.

It is easy to see that every point-automorphism $F : \Omega \mapsto \Omega$ naturally induces an automorphism $\Phi_F : \bar{\mathcal{B}} \mapsto \bar{\mathcal{B}}$, *i.e.* for any $B \in \mathcal{B}$, $\Phi_F(\bar{B}) = \overline{F[B]}$.

Maharam Theorem is needed in next several sections. Let $\bar{\mathcal{B}}$ be a complete Boolean algebra and let $a \in \bar{\mathcal{B}}$ be a non-zero element. We denote by $\bar{\mathcal{B}} \restriction a$ the Boolean algebra with the base set $\{b \in \bar{\mathcal{B}} : b \leqslant a\}$, the largest element $1_{\bar{\mathcal{B}} \restriction a} = a$ and the Boolean operators inherited from $\bar{\mathcal{B}}$. Let

$$\tau(\bar{\mathcal{B}} \restriction a) = \min\{card(X) : \bar{\mathcal{B}} \restriction a \text{ is completely generated by } X \subseteq \bar{\mathcal{B}} \restriction a.\}.$$

We write $\tau(\bar{\mathcal{B}})$ if $a = 1_{\bar{\mathcal{B}}}$ and let $\tau(\bar{\mathcal{B}} \restriction 0) = 0$. $\bar{\mathcal{B}}$ is called homogeneous iff $\tau(\bar{\mathcal{B}} \restriction a) = \tau(\bar{\mathcal{B}} \restriction b)$ for any non-zero $a, b \in \bar{\mathcal{B}}$. Given any non-empty index set I, let ν_I be the usual measure on the complete σ-algebra $\mathcal{B}(\{0,1\}^I)$ generated by those A_i's, where the set $A_i = \{f \in \{0,1\}^I : f(i) = 0\}$ has measure $\frac{1}{2}$ for every $i \in I$. We denote by $\bar{\mathcal{B}}(\{0,1\}^I)$ the Boolean algebra of $\mathcal{B}(\{0,1\}^I)$ modulo the ideal of ν_I-measure zero sets.

The following is a version of Maharam Theorem needed in this paper.

Maharam Theorem (see p.911, Theorem 3.5 of [2]) Suppose a measure algebra $\bar{\mathcal{B}}$ is homogeneous and $\tau(\bar{\mathcal{B}}) = \lambda$. Then there exists a measure-preserving isomorphism Φ from $\bar{\mathcal{B}}$ to $\bar{\mathcal{B}}(\{0,1\}^\lambda)$.

Let (Ω, Σ, ν) be any probability space and let $T : \Omega \mapsto \Omega$ be a measure-preserving transformation (or a point-automorphism). The map T is called a ergodic transformation iff for any $B \in \Sigma$ one has that $\nu(B \Delta T[B]) = 0$ implies $\nu(B) = 0$ or $\nu(B) = 1$, where Δ means symmetric difference. Let $\bar{\Sigma}$ be the measure algebra of (Ω, Σ, ν) and $\bar{\nu}$ be the induced measure on $\bar{\Sigma}$. An automorphism $\Phi : \bar{\Sigma} \mapsto \bar{\Sigma}$ is called ergodic iff for any $a \in \bar{\Sigma}$ one has that $\bar{\nu}((a \wedge -\Phi(a)) \vee (-a \wedge \Phi(a))) = 0$ implies $a = 0$ or $a = 1$. It is easy to see that an ergodic transformation induces an ergodic automorphism, and if an ergodic automorphism is induced by a measure-preserving transformation T, then T is an ergodic transformation.

3. Automorphisms $\neq$ Internal Permutations

It is proven in Theorem 4.3 of [14] that, assuming full-saturation, for every Loeb space $(\Omega, \mathcal{B}, L_\mu)$ there exists an automorphism Φ on $\bar{\mathcal{B}}$, which is not induced by any internal permutation of Ω. It is also implicitly conjectured in §7 of [15] that above theorem is a consequence of SMA_κ unprovable under IP_κ. In this section we refute the conjecture by showing that Theorem 4.3 of [14] is true in *any* $\aleph_1$-saturated nonstandard universe. Let ${}^*V$ be a fixed $\aleph_1$-saturated nonstandard universe throughout this section.

It is proven in [5] that for any Loeb space $(\Omega, \mathcal{B}, L_\mu)$ there is no ergodic transformation T on Ω such that the graph of T is a Borel subset of $\Omega \times \Omega$ (a subset of $\Omega \times \Omega$ is called Borel iff it is a member of the σ-algebra generated by the algebra of all internal subsets of $\Omega \times \Omega$).

The proof of the main result in this section is a combination of Maharam Theorem and the result in [5] mentioned above.

Lemma 3.1. *The Loeb algebra $\bar{\mathcal{B}}$ of any Loeb space $(\Omega, \mathcal{B}, L_\mu)$ is homogeneous.*

Proof: Let $\kappa = \min\{\tau(\bar{\mathcal{B}} \restriction a) : a \in \bar{\mathcal{B}} \setminus \{0\}\}$. It suffices to show that $\tau(\bar{\mathcal{B}}) = \kappa$. Since $(\Omega, \mathcal{B}, L_\mu)$ is non-atomic, then κ is infinite. Let $a \in \bar{\mathcal{B}}$ be such that $\tau(\bar{\mathcal{B}} \restriction a) = \kappa$. Then there is an internal set $A \in a$ such that $L_\mu(A) > 0$. This means $|A|/H$ is not infinitesimal, where $H = |\Omega|$. Let $|A| = K$ and let

$$k = \max\{n \in {}^*\mathbb{N} : n \cdot K < H\}.$$

Then $k \in \mathbb{N}$. For each $i = 0, 1, \ldots, k-1$, let $A_i = \{i \cdot K, i \cdot K + 1, \ldots, (i+1) \cdot K - 1\}$ and let $A_* = \{k \cdot K, k \cdot K + 1, \ldots, H-1\}$. Then $|A_*| \leqslant K$ and $\bar{\mathcal{B}} \restriction \bar{A}$ is isomorphic to

$\bar{\mathcal{B}}\restriction\bar{A}_i$ for each $i < k$. Hence

$$\kappa \leqslant \tau(\bar{\mathcal{B}}) \leqslant (\prod_{i=0}^{k-1}\tau(\bar{\mathcal{B}}\restriction\bar{A}_i))\cdot\tau(\bar{\mathcal{B}}\restriction\bar{A}_*) \leqslant \kappa^{k+1} = \kappa. \quad \square$$

Theorem 3.2. *For any Loeb space $(\Omega, \mathcal{B}, L_\mu)$ there exists an automorphism of $\bar{\mathcal{B}}$, which is not induced by any internal permutation of Ω.*

Proof: By Maharam Theorem and Lemma 3.1 there is an infinite cardinal λ (it can be proven that $\lambda = card(\{0, 1, \ldots, 2^{|\Omega|}\})$) such that the Loeb algebra $\bar{\mathcal{B}}$ is measure-preserving isomorphic to $\bar{\mathcal{B}}(\{0,1\}^\lambda)$. Without loss of generality we can replace $\bar{\mathcal{B}}(\{0,1\}^\lambda)$ by $\bar{\mathcal{B}}(\{0,1\}^{\lambda\times\mathbb{Z}})$ and let $\Phi : \bar{\mathcal{B}} \mapsto \bar{\mathcal{B}}(\{0,1\}^{\lambda\times\mathbb{Z}})$ be the isomorphism. Let's define

$$F : \{0,1\}^{\lambda\times\mathbb{Z}} \mapsto \{0,1\}^{\lambda\times\mathbb{Z}}$$

to be a measure-preserving transformation such that for any $f \in \{0,1\}^{\lambda\times\mathbb{Z}}$ the element $F(f)$ is in $\{0,1\}^{\lambda\times\mathbb{Z}}$ such that for any $(\alpha, n) \in \lambda \times \mathbb{Z}$ one has

$$F(f)(\alpha, n) = f(\alpha, n+1).$$

It is well-known in Ergodic Theory that F is a ergodic transformation (in fact it is even a strong mixing transformation). Let $\hat{F}$ be the ergodic automorphism of $\bar{\mathcal{B}}(\{0,1\}^{\lambda\times\mathbb{Z}})$ naturally induced by F and let

$$\hat{T} = \Phi^{-1} \circ \hat{F} \circ \Phi.$$

It is clear that $\hat{T}$ is an ergodic automorphism of $\bar{\mathcal{B}}$. Now the automorphism $\hat{T}$ is not induced by any internal (in fact, any Borel) permutation of Ω because otherwise there would exist an internal ergodic transformation on Ω, which contradicts the result of [5]. $\square$

4. About Automorphisms = Point-Automorphisms

It is proven in Theorem 4.1 of [14] that, assuming full-saturation, every automorphism of a Loeb algebra is induced by a point-automorphism of the Loeb space. It is also implicitly conjectured in §7 of [15] that above theorem is a consequence of full-saturation unprovable under SMA_κ. In this section we refute the conjecture by showing that Theorem 4.1 of [14] is a consequence of $SMA_{\aleph_1}$. The proof here is a variant of the proof in [14].

Theorem 4.1. *Suppose $^*V$ satisfies $SMA_{\aleph_1}$ and let $(\Omega, \mathcal{B}, L_\mu)$ be a Loeb space in $^*V$. Then every automorphism Φ of $\bar{\mathcal{B}}$ is induced by a point-automorphism F of $(\Omega, \mathcal{B}, L_\mu)$.*

Proof: Let $(\Omega, \mathcal{B}, L_\mu)$ be generated by a normalized counting measure space $(\Omega, \mathcal{A}, \mu)$ in $^*V$. Let

$$\mathfrak{A} = (\Omega \cup \mathcal{A} \cup {}^*\mathbb{R}; \Omega, \mathcal{A}, {}^*\mathbb{R}, \in, \mu, \cap, \smallsetminus, +, \cdot, <, 0, 1)$$

be the internally presented structure associated with $(\Omega, \mathcal{B}, L_\mu)$. By $SMA_{\aleph_1}$ there is a specializing sequence $\langle \mathfrak{A}_\alpha : \alpha < \Xi \rangle$ for $\mathfrak{A}$. Let the base set of $\mathfrak{A}_\alpha$ be $\Omega_\alpha \cup \mathcal{A}_\alpha \cup \mathbb{R}_\alpha$. Fix an enumeration $\mathcal{A} = \{A_\alpha : \alpha < \Xi\}$. We construct two sequences $\langle B_\alpha : \alpha < \Xi \rangle$ and $\langle C_\alpha : \alpha < \Xi \rangle$ such that

(1) $\{B_\alpha : \alpha < \Xi\} = \{C_\alpha : \alpha < \Xi\} = \mathcal{A}$,

(2) $\{B_\beta : \beta < \alpha\} \cup \{C_\beta : \beta < \alpha\} \subseteq \mathcal{A}_\alpha$ for every $\alpha < \Xi$,

(3) $\Phi(\bar{B}_\alpha) = \bar{C}_\alpha$ for every $\alpha < \Xi$,

(4) for any $\alpha_0 < \alpha_1 < \ldots < \alpha_{n-1} < \Xi$ and for any $h \in 2^n$ one has

$$\mu(\bigcap_{i=0}^{n-1} B_{\alpha_i}^{h(i)}) = \mu(\bigcap_{i=0}^{n-1} C_{\alpha_i}^{h(i)}),$$

where $A^0 = A$, $A^1 = \Omega \smallsetminus A$.

Claim 4.1.1 The theorem follows from the construction.

Proof of Claim 4.1.1: Note that $\mu(A) = \mu(B)$ iff $|A| = |B|$ for any internal sets $A, B \in \mathcal{A}$. For any $x \in \Omega$ there is an α such that $\{x\} = B_\alpha$ by (1). It is easy to see by (4) that C_α is also a singleton $\{y\}$ for some $y \in \Omega$. Let $F(x) = y$. Then it is easy to check again by (4) that F is a well-defined bijection. Also it is not hard to check that $F[B_\alpha] = C_\alpha$. So one has $|A| = |F[A]|$ for every $A \in \mathcal{A}$. This implies that F and F^{-1} are measurable and preserve the measure. So F is a point-automorphism. By (3) and (4) one can easily see that Φ is induced by F. $\square$(Claim 4.1.1)

We now construct B_α and C_α by induction. Suppose we have found $\{B_\beta : \beta < \alpha\}$ and $\{C_\beta : \beta < \alpha\}$ such that (2), (3) and (4) are true up to stage α.

Case 1: α is even. We pick B_α first. Let

$$O_\alpha = \{\delta < \Xi : A_\delta \not\in \{\mathcal{B}_\beta : \beta \in \alpha\} \wedge A_\delta \in \mathcal{A}_\alpha \wedge (\exists A \in \mathcal{A}_\alpha)(\Phi(\bar{A}_\delta) = \bar{A})\}.$$

If $O_\alpha = \emptyset$, then let $B_\alpha = C_\alpha = \emptyset$. Otherwise let $\gamma = \min O_\alpha$ and let $B_\alpha = A_\gamma$. By the definition of O_α there is an $A \in \mathcal{A}_\alpha$ such that $\Phi(\bar{A}_\delta) = \bar{A}$. Let's define a set of $\mathcal{L}_{\mathfrak{A}}$-formulas $,_{\alpha}(x)$ with one free variable x and all parameters from $\mathfrak{A}_\alpha$, which expresses that x is a candidate for C_α. The set $,_{\alpha}(x)$ contains exactly the following:

(a) $\mathcal{A}(x)$, *i.e.* x is an internal subset of Ω,

(b) $\mu(x\Delta A) < \frac{1}{m}$ for every $m \in \mathbb{N}$, *i.e.* the symmetric difference of x and A will have Loeb measure zero,

(c)

$$\mu((\bigcap_{i=0}^{n-1} B_{\alpha_i}^{h(i)}) \cap B_{\alpha}^{j}) = \mu((\bigcap_{i=0}^{n-1} C_{\alpha_i}^{h(i)}) \cap x^{j})$$

for any $\alpha_0 < \alpha_1 < \ldots < \alpha_{n-1} < \alpha$, for any $h \in 2^n$ and for any $j = 0, 1$.

Now $card(,_{\alpha}(x)) = card(\alpha)$ and all parameters of $,_{\alpha}(x)$ are from $\mathfrak{A}_\alpha$. It is easy to see that $,_{\alpha}(x)$ is finitely realizable because finitely many B_{α_i}'s and C_{α_i}'s could only cut Ω, respectively, into finitely many pieces. So by (3) and (4) it is easy to find an $x \subseteq \Omega$ to fit the requirements from (b) and (c). Since $\mathfrak{A}_\alpha$ is $card(\alpha)^+$-saturated, the type $,_{\alpha}(x)$ is realized by some $C \in \mathcal{A}_\alpha$. Let $C_\alpha = C$. Clearly, (2), (3) and (4) are true up to stage $\alpha + 1$.

Case 2: α is odd. Just repeat the proof of Case 1 by switching B's and C's.

To finish the proof we need only to show that (1) is true.

Claim 4.1.2 $\{B_\alpha : \alpha < \Xi\} = \{C_\alpha : \alpha < \Xi\} = \mathcal{A}$.

Proof of Claim 4.1.2: Suppose not. Without loss of generality we assume that $B_0 = C_0 = \emptyset$ and

$$\mathcal{A} \smallsetminus \{B_\alpha : \alpha < \Xi\} \neq \emptyset.$$

Let

$$\gamma = \min\{\delta < \Xi : A_\delta \notin \{B_\alpha : \alpha < \Xi\}\}$$

and let

$$\gamma' = \min\{\delta < \Xi : A_\delta \in \Phi(\bar{A}_\gamma)\}.$$

Since $card(\gamma) < \Xi$ one can find a large enough even ordinal $\alpha < \Xi$ such that $A_\gamma, A_{\gamma'} \in \mathcal{A}_\alpha$. Without loss of generality one can choose α such that $B_\alpha \notin \{A_\beta : \beta < \gamma\}$. The reason is that if $\alpha < \alpha' < \Xi$ and α' is an even ordinals, then $B_\alpha \neq B_{\alpha'}$ as long as $O_{\alpha'} \neq \emptyset$. Now one has $\gamma \in O_\alpha$ by the definition of O_α. If $\epsilon = \min O_\alpha < \gamma$, then $B_\alpha = A_\epsilon$, which contradicts

$$B_\alpha \notin \{A_\beta : \beta < \gamma\}.$$

So $\gamma = \epsilon$. But now $B_\alpha = A_\gamma$, which contradicts the assumption that

$$A_\gamma \notin \{B_\alpha : \alpha < \Xi\}. \quad \square$$

Problem 4.2. *Could $SMA_{\aleph_1}$ be replaced by $IP_{\aleph_1}$ in Theorem 4.1?*

5. About $SMA_\kappa \neq$ Full-saturation

Let $(\Omega, \mathcal{B}, L_\mu)$ be a Loeb space in a nonstandard universe $^*V$. It is proven in Theorem 4.4 of [14] that if $^*V$ is fully-saturated, then for every subalgebra $\bar{\mathcal{C}} \subseteq \bar{\mathcal{B}}$ with $card(\bar{\mathcal{C}}) < card(\bar{\mathcal{B}})$ and for every measure-preserving homomorphism $h : \bar{\mathcal{C}} \mapsto \bar{\mathcal{B}}$ there exists an internal permutation $F : \Omega \mapsto \Omega$ such that h is induced by F, *i.e.* for every $\bar{C} \in \bar{\mathcal{C}}$ one has $h(\bar{C}) = \overline{F[C]}$. In this section we show that the condition "$^*V$ is fully-saturated" above could not be replaced by that "$^*V$ satisfies SMA_κ".

Theorem 5.1. *Suppose $^*V$ satisfies SMA_κ for some $\kappa \geqslant \aleph_1$ and $^*V$ is not fully-saturated. Then for every Loeb space $(\Omega, \mathcal{B}, L_\mu)$ in $^*V$ there exists an subalgebra $\bar{\mathcal{C}} \subseteq \bar{\mathcal{B}}$ with $card(\bar{\mathcal{C}}) < \bar{\mathcal{B}}$ and there exists a measure-preserving homomorphism $h : \bar{\mathcal{C}} \mapsto \bar{\mathcal{B}}$ (in fact h is an automorphism of $\bar{\mathcal{C}}$) such that h is not induced by any internal permutation of Ω.*

Proof: Let $(\Omega, \mathcal{B}, L_\mu)$ be generated by a normalized counting measure space $(\Omega, \mathcal{A}, \mu)$ and let $\mathfrak{A}$ be the internally presented structure associated with $(\Omega, \mathcal{A}, \mu)$ defined in §2.

Claim 5.1.1 Ξ is a singular cardinal.

Proof of Claim 5.1.1: Suppose Ξ is regular. We want to show that $^*V$ is fully-saturated. Given any internally presented $\mathcal{L}$-structure $\mathfrak{B}$ in $^*V$ with $card(\mathcal{L}) < \Xi$ and given any $\mathcal{L}$-type , (x) with parameters from $\mathfrak{B}$ and consistent with $Th(\mathfrak{B})$ such that $card(, (x)) < \Xi$, it suffices to show that , (x) is satisfiable in $\mathfrak{B}$. Without loss of generality we assume that $\mathcal{L}$ contains no function symbols and no constant symbols. The proof would be simple if one had $card(\mathcal{L}) < \kappa$, by SMA_κ and regularity of Ξ. For the case $\mathcal{L} \geqslant \kappa$ we need to consider an auxiliary internally presented structure $\mathfrak{B}'$. Let B be the base set of $\mathfrak{B}$. For each $n \in {^*\mathbb{N}}$ let B^n be the set of all internal n-tuples of elements in B. Let $\Sigma_0 = \bigcup_{n \in {^*\mathbb{N}}} B^n$ and let $\Sigma_1 = \bigcup_{n \in {^*\mathbb{N}}} {^*\mathcal{P}}(B^n)$, where ${^*\mathcal{P}}(A)$ means the set of all internal subsets of A for any internal set A. Note that B is just B^1. The base set of $\mathfrak{B}'$ is $\Sigma = \Sigma_0 \cup \Sigma_1$. There are unary relations Σ_0, Σ_1, B^n for each $n \in \mathbb{N}$, and ${^*\mathcal{P}}(B^n)$ for each $n \in \mathbb{N}$. There is a binary membership relation $\in$ between B^n and ${^*\mathcal{P}}(B^n)$ for all $n \in {^*\mathbb{N}}$. There are binary relations $E_{n,i} \subseteq B \times B^n$ for each $n \in \mathbb{N}$ and each $i < n$ such that

$$(a, \langle a_0, \ldots, a_{n-1} \rangle) \in E_{n,i} \text{ iff } a = a_i.$$

The language $\mathcal{L}_{\mathfrak{B}'}$ is countable. For every relation symbol P in $\mathcal{L}$ the interpretation of P in $\mathfrak{B}$ is now an element in $\mathfrak{B}'$. More precisely, an n-ary relation of $\mathfrak{B}$ is an element of ${}^*\mathcal{P}(B^n) \subseteq \Sigma$. For each $\mathcal{L}$-formula $\theta(x_0, \ldots, x_{n-1})$ with parameters from $\mathfrak{B}$ one can naturally translate it into an $\mathcal{L}_{\mathfrak{B}'}$-formula $\theta'(x_0, \ldots, x_{n-1})$ with parameters from $\mathfrak{B}'$ such that for any $b_0, \ldots, b_n \in B$, $\theta(b_0, \ldots, b_{n-1})$ is true in $\mathfrak{B}$ iff $\theta'(b_0, \ldots, b_{n-1})$ is true in $\mathfrak{B}'$. For example, if $\theta(x_0, \ldots, x_{n-1})$ is an atomic formula $P(x_0, \ldots, x_{n-1})$ and let $R = P^{\mathfrak{B}}$, then the formula $\theta'(x_0, \ldots, x_{n-1})$ should be

$$\exists x \; (x \in R \wedge \bigwedge_{i=0}^{n-1} E_{n,i}(x_i, x)).$$

Let

$$,\,'(x) = \{\theta'(x) : \theta \in \,,\,(x)\} \cup \{B^1(x)\}.$$

Then $,\,'(x)$ is an $\mathcal{L}_{\mathfrak{B}'}$-type consistent with $Th(\mathfrak{B}')$ and $card(,\,'(x)) < \Xi$. By SMA_κ the structure $\mathfrak{B}'$ has a specializing sequence $\langle \mathfrak{B}'_\alpha : \alpha < \Xi \rangle$. Since Ξ is regular, one can assume that there is a $\beta < \Xi$ such that $card(,\,'(x)) \leqslant card(\beta)$ and all parameters in $,\,'(x)$ are from $\mathfrak{B}'_\beta$. Since $\mathfrak{B}'_\beta$ is $card(\beta)^+$-saturated, the type $,\,'(x)$ is satisfiable in $\mathfrak{B}'_\beta$. Hence $,\,(x)$ is satisfiable in $\mathfrak{B}$. $\quad\Box$ (Claim 5.1.1)

Claim 5.1.2 Ξ is a strong limit cardinal.

Proof of Claim 5.1.2: Let $\eta = cf(\Xi)$ and let $\eta \leqslant \lambda < \Xi$. We want to show that $2^\lambda < \Xi$. Suppose $2^\lambda \geqslant \Xi$. Let $\mathcal{R}$ be the real field in ${}^*V$. By SMA_κ there is a specializing sequence $\langle \mathcal{R}_\alpha : \alpha < \Xi \rangle$ for $\mathcal{R}$. Since $\lambda < \Xi$, $\mathcal{R}_\lambda$ is in the sequence and is λ^+-saturated. So

$$\Xi \geqslant card(\mathcal{R}_\lambda) \geqslant 2^\lambda \geqslant \Xi.$$

This implies $2^\lambda = \Xi$. But it is impossible because

$$2^\lambda = (2^\lambda)^\eta = \Xi^{cf(\Xi)} > \Xi. \quad \Box\text{(Claim 5.1.2)}$$

Recall that $\mathfrak{A}$ is the internally presented structure associated with Loeb space $(\Omega, \mathcal{B}, L_\mu)$. We want to define a new auxiliary structure $\mathfrak{A}'$. The base set of $\mathfrak{A}'$ is $\Omega \cup \mathcal{A} \cup {}^*\mathbb{R} \cup \mathcal{F}$, where $\mathcal{F}$ is the set of all internal permutations of Ω. The relations, the functions and the constants of $\mathfrak{A}'$ are same as those in $\mathfrak{A}$ except one more ternary relation R such that

$$(a, b, f) \in R \text{ iff } a, b \in \Omega \wedge f \in \mathcal{F} \wedge f(a) = b.$$

By SMA_κ there is a specializing sequence $\langle \mathfrak{A}'_\alpha : \alpha < \Xi \rangle$ for $\mathfrak{A}'$, where $\mathfrak{A}'_\alpha$ has the base set $\Omega_\alpha \cup \mathcal{A}_\alpha \cup \mathbb{R}_\alpha \cup \mathcal{F}_\alpha$. Choosing an increasing sequence of regular cardinals $\langle \kappa_\alpha : \alpha < \eta \rangle$ cofinal in Ξ such that $2^{\kappa_\alpha} < \kappa_{\alpha+1}$ for each $\alpha < \eta$. Without loss of generality we assume that $card(\mathfrak{A}_{\kappa_\alpha}) \leqslant 2^{\kappa_\alpha}$ for each $\alpha < \eta$. We want to construct a sequence $\langle A_\alpha : \alpha < \eta \rangle$ of internal subsets of Ω such that for each $\alpha < \eta$

(1) $L_\mu(A_\alpha) = \frac{1}{2}$,

(2) $A_\alpha \in \mathcal{A}_{\kappa_{\alpha+1}} \smallsetminus \mathcal{A}_{\kappa_\alpha}$,

(3) $\langle A_\alpha : \alpha < \eta \rangle$ is an independent sequence in Loeb space $(\Omega, \mathcal{B}, L_\mu)$.

Suppose we have had $\langle A_\alpha : \alpha < \gamma \rangle$ for some $\gamma < \eta$. Let $\Gamma(x)$ be an $\mathcal{L}_{\mathfrak{A}'}$-type containing exactly the following formulas:

(a) "$\mathcal{A}(x)$",

(b) "$-\frac{1}{n} < \mu(x) - \frac{1}{2} < \frac{1}{n}$" for each $n \in \mathbb{N}$,

(c) "$-\frac{1}{n} < \mu(x \cap y) - \mu(x) \cdot \mu(y) < \frac{1}{n}$" for each $n \in \mathbb{N}$ and each $y \in \mathcal{A}_{\kappa_\gamma}$.

Then it is easy to see that $\Gamma(x)$ is consistent with $\mathfrak{A}'$ and all parameters in $\Gamma(x)$ are from $\mathfrak{A}'_{\kappa_\gamma}$. Since $\mathfrak{A}'_{\kappa_{\gamma+1}}$ is $(card(\mathfrak{A}'_{\kappa_\gamma}))^+$-saturated, then $\Gamma(x)$ is realized by some $A_\gamma \in \mathcal{A}_{\kappa_{\gamma+1}}$. This ends the construction. The sequence $\langle A_\alpha : \alpha < \eta \rangle$ satisfies (1) by (a) and (b), and satisfies (2) and (3) by (c).

Now let $\bar{\mathcal{C}}$ be the complete subalgebra of $\bar{\mathcal{B}}$ generated by $\{\bar{A}_\alpha : \alpha < \eta\}$. So

$$card(\bar{\mathcal{C}}) \leqslant \eta^{\aleph_0} < \Xi = card(\bar{\mathcal{B}}).$$

Let

$$h : \{\bar{A}_\alpha : \alpha < \eta\} \mapsto \{\bar{A}_\alpha : \alpha < \eta\}$$

be such that $h(\bar{A}_\alpha) = \bar{A}_{\alpha+1}$ when α is even, and $h(\bar{A}_\alpha) = \bar{A}_{\alpha-1}$ when α is odd. It is easy to check that h could be extended to an automorphism of $\bar{\mathcal{C}}$. To finish the proof we need only to check that h is not induced by any internal permutation.

Claim 5.1.3 h is not induced by any internal permutation of Ω.

Proof of Claim 5.1.3: Suppose not. Let $F \in \mathcal{F}$ be the internal permutation which induces h. Since $\mathcal{F} = \bigcup_{\alpha<\eta} F_{\kappa_\alpha}$, then there is an even ordinal $\alpha < \eta$ such that $F \in \mathcal{F}_{\kappa_\alpha}$. From the construction above one has that A_α is an element in $\mathfrak{A}'_{\kappa_{\alpha+1}}$. So $F[A_\alpha]$ is also an element in $\mathfrak{A}'_{\kappa_{\alpha+1}}$. But $h(\bar{A}_\alpha) = \bar{A}_{\alpha+1}$ and $A_{\alpha+1}$ is independent of any subset of Ω in $\mathfrak{A}'_{\kappa_{\alpha+1}}$. So $h(\bar{A}_\alpha) = \bar{A}_{\alpha+1} \neq \overline{F[A_\alpha]}$. This contradicts that h is induced by F. $\square$

Remark In the proof of Theorem 5.1 the cardinality of $\bar{\mathcal{C}}$ is same as $cf(\Xi)$, which is less than Ξ. In fact it is impossible to find a required $\bar{\mathcal{C}}$ with cardinality $< cf(\Xi)$ when ${}^*V$ satisfies $SMA_{\aleph_1}$. See the proof of Theorem 4.4 of [14].

6. About $IP_{\aleph_1} \neq SMA_{\aleph_1}$

Given any probability space (Ω, Σ, ν), a family $\mathcal{C} \subseteq \Sigma$ is called compact iff for any $\mathcal{D} \subseteq \mathcal{C}$, $\mathcal{D}$ has the finite intersection property implies $\bigcap \mathcal{D} \neq \emptyset$. A probability space (Ω, Σ, ν) is called compact iff there exists a compact family $\mathcal{C} \subseteq \Sigma$ such that for every $B \in \Sigma$

$$\nu(B) = \sup\{\nu(C) : C \in \mathcal{C} \wedge C \subseteq B\}.$$

In Theorem 3.9 of [8], which is a special case of Corollary 7 of [10], it is proven that if CH (Continuum Hypothesis) holds, ${}^*V$ satisfies $SMA_{\aleph_1}$ and $cf(\Xi_{{}^*V}) = \aleph_1$, then every Loeb space in ${}^*V$ is compact. In the following theorem we show that $SMA_{\aleph_1}$ could not be replaced by $IP_{\aleph_1}$.

Theorem 6.1. *Suppose $\lambda > \beth_{\aleph_0}$ and $\lambda^{\aleph_0} = \lambda$. Then there exists a nonstandard universe ${}^*V$ satisfying $IP_{\aleph_1}$ such that $\Xi_{{}^*V} = \lambda$ and no Loeb space in ${}^*V$ is compact.*

Proof: First we choose any nonstandard universe ${}^*V_0$ such that every infinite internal set in ${}^*V_0$ has external cardinality λ. We want to construct a sequence of nonstandard universes $\langle {}^*V_\alpha : \alpha < (2^{\aleph_0})^+\rangle$ such that for any $\alpha < \beta < (2^{\aleph_0})^+$,

(1) ${}^*V_\beta$ is a bounded elementary extension of ${}^*V_\alpha$,

(2) every infinite internal set in ${}^*V_\alpha$ has external cardinality λ,

(3) for any two elementarily equivalent internally presented $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ in ${}^*V_\alpha$ with $card(\mathcal{L}) < \aleph_1$ there is a sequence $\langle i_\beta : \alpha < \beta < (2^{\aleph_0})^+\rangle$ such that i_β is an isomorphism from $\mathfrak{A}^{{}^*V_\beta}$ to $\mathfrak{B}^{{}^*V_\beta}$ and $i_{\beta'}$ extends i_β for any $\alpha < \beta < \beta' < (2^{\aleph_0})^+$,

(4) for every normalized counting measure space $(\Omega, \mathcal{A}, \mu)$ in ${}^*V_\alpha$ there is an internal set A in ${}^*V_{\alpha+1}$ such that $\Omega^{{}^*V_\alpha} \subseteq A$ and $\mu(A)$ is an infinitesimal.

Notice the differences between an internal set A in ${}^*V_\alpha$ and its version $A^{{}^*V_\beta}$ in ${}^*V_\beta$.

Suppose $\langle {}^*V_\alpha : \alpha < \gamma\rangle$ has been constructed for some $\gamma < (2^{\aleph_0})^+$. If γ is a limit, then let ${}^*V_\gamma$ be the union of the sequence $\langle {}^*V_\alpha : \alpha < \gamma\rangle$. For any two elementarily equivalent internally presented $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ in ${}^*V_\alpha$ with $card(\mathcal{L}) < \aleph_1$ for some $\alpha < \gamma$ let i_γ be the union of the correspondent sequence $\langle i_\beta : \alpha < \beta < \gamma\rangle$. Now let's assume $\gamma = \beta + 1$ for some $\beta < (2^{\aleph_0})^+$. We first find a bounded elementary extension ${}^*V'_\beta$ of ${}^*V_\beta$ such that (4) is satisfied.

For each normalized counting measure space $(\Omega, \mathcal{A}, \mu)$ in ${}^*V_\beta$ let $,_\Omega(x)$ be the consistent type of the following three kinds of formulas with parameters from ${}^*V_\beta$ and only free variable x:

(a) "$x \subseteq \Omega$",

(b) "$a \in x$", for every $a \in \Omega^{{}^*V_\beta}$,

(c) "$\mu(x) < \frac{1}{n}$", for every $n \in \mathbb{N}$.

Since $card(,_\Omega(x)) \leqslant \lambda$ and there are at most λ different Ω's, then, by Löwenheim-Skolem Theorem, there exists a bounded elementary extension ${}^*V'_\beta$ of ${}^*V_\beta$ such that $card({}^*V'_\beta) = \lambda$ and every type $,_\Omega(x)$ for some Ω in ${}^*V_\beta$ is realized in ${}^*V'_\beta$.

Next we construct a bounded elementary extension ${}^*V_\gamma$ of ${}^*V'_\beta$ such that (1), (2) and (3) are satisfied. The construction is almost identical to the construction in the proof of Case 2 in Theorem 1 of [6]. This ends the construction.

Let ${}^*V$ be the union of the constructed sequence $\langle {}^*V_\alpha : \alpha < (2^{\aleph_0})^+ \rangle$. We want to show that ${}^*V$ is the nonstandard universe we are looking for. It is easy to see that $IP_{\aleph_1}$ is true in ${}^*V$ by (3) and by the fact that the cofinality of $(2^{\aleph_0})^+$ is uncountable. Let $(\Omega, \mathcal{B}, L_\mu)$ be a Loeb space in ${}^*V$.

Claim 6.1.1 $(\Omega, \mathcal{B}, L_\mu)$ is not compact.

Proof of Claim 6.1.1: There is a sketch of the proof using Boolean algebra in Theorem 13 and Claim 9.1 of [10]. Here we give a simplified proof using Maharam Theorem.

Suppose $(\Omega, \mathcal{B}, L_\mu)$ is compact and let $\mathcal{C} \subseteq \mathcal{B}$ be a compact family exemplifying this. Without loss of generality we assume that $\Omega \in {}^*V_0$. By (4) for every $\alpha < (2^{\aleph_0})^+$ the set $\Omega^{{}^*V_\alpha}$ has Loeb measure zero. Hence there is a $C_\alpha \in \mathcal{C}$ such that $\mu(C) > \frac{1}{2}$ and $C \cap \Omega^{\star_\alpha} = \emptyset$. By Maharam Theorem and Lemma 3.1 there is a measure-preserving isomorphism Φ from Loeb algebra $\bar{\mathcal{B}}$ to the algebra $\bar{\mathcal{B}}(\{0,1\}^\lambda)$. For each $\alpha < (2^{\aleph_0})^+$ let $U_\alpha \subseteq \{0,1\}^\lambda$ be a Baire set such that $\Phi(\bar{C}_\alpha) = \bar{U}_\alpha$ and let $w_\alpha \subseteq \lambda$ be the countable support of U_α. By Δ-System Lemma there is a subset $E \subseteq (2^{\aleph_0})^+$ such that $card(E) = (2^{\aleph_0})^+$ and $\{w_\alpha : \alpha \in E\}$ forms a Δ-system with common root $r \subseteq \lambda$. For each $\alpha \in E$ let

$$W_\alpha = \{h \in \{0,1\}^r : \nu_{(\lambda \smallsetminus r)}(\{g \restriction (\lambda \smallsetminus r) : g \in U_\alpha \wedge g \restriction r = h\}) > 0\}.$$

Then $\nu_r(W_\alpha) > 0$ by Fubini Theorem. Since there are at most $2^{\aleph_0}$ Baire subsets of $\{0,1\}^r$ we can assume that for any $\alpha, \beta \in E$ the sets W_α and W_β are same modulo a

Loeb measure zero set. It is easy to check that for any finite $E_0 \subseteq E$ one has

$$\nu_\lambda(\bigcap_{\alpha\in E_0} U_\alpha) > 0.$$

Hence pulling back by Φ one has that

$$L_\mu(\bigcap_{\alpha\in E_0} C_\alpha) > 0.$$

This shows that the family $\{C_\alpha : \alpha \in E\}$ has the finite intersection property. But since E is unbounded in $(2^{\aleph_0})^+$, then clearly one has

$$\bigcap_{\alpha\in E} C_\alpha \subseteq \bigcap_{\alpha\in E}(\Omega \setminus \Omega^{* V_\alpha}) = \emptyset.$$

This contradicts the assumption that $\mathcal{C}$ is a compact family. $\square$

Corollary 6.2. *Suppose CH holds and λ is a strong limit cardinal with $cf(\lambda) = \aleph_1$. Then there is a nonstandard universe $^*V$ such that $^*V$ satisfies $IP_{\aleph_1}$, $\Xi_{^*V} = \lambda$ and no Loeb space in $^*V$ is compact.*

Remark Corollary 6.2 is given solely for a clear comparison with Theorem 3.9 of [8]. Note that if $^*V$ satisfies $SMA_{\aleph_1}$ and $cf(\Xi_{^*V}) = \aleph_1$, then $\Xi_{^*V}$ is a singular strong limit cardinal. In fact, for any singular strong limit cardinal λ with $cf(\lambda) = \aleph_1$ there is a $^*V$ satisfying $SMA_{\aleph_1}$ such that $\Xi_{^*V} = \lambda$. In that $^*V$, assuming CH, every Loeb space is compact.

7. Existence of Ergodic Transformations

Recall that in [5] it is proven that for any Loeb space $(\Omega, \mathcal{B}, L_\mu)$ there is no ergodic transformation T on Ω such that the graph of T is a Borel subset of $\Omega \times \Omega$. It is also asked in Problem 2.3 of [5] whether it is ever possible for a Loeb space to have an ergodic transformation. In this section we obtain an easy application of $SMA_{\aleph_1}$, which partially answers above question.

Theorem 7.1. *Suppose $^*V$ satisfies $SMA_{\aleph_1}$. Then every Loeb space in $^*V$ has a ergodic transformation.*

Proof: Let $(\Omega, \mathcal{B}, L_\mu)$ be any Loeb space in $^*V$. Let $\bar{\mathcal{B}}$, $\bar{\mathcal{B}}(\{0,1\}^{\lambda\times\mathbb{Z}})$, Φ, F $\hat{F}$ and $\hat{T}$ be same as in the proof of Theorem 3.2. By $SMA_{\aleph_1}$ there is a measure-preserving transformation T on Ω, which induces $\hat{T}$. Obviously T is a ergodic (strong mixing) transformation on the Loeb space $(\Omega, \mathcal{B}, L_\mu)$. $\square$

Remark: In fact, assuming $SMA_{\aleph_1}$ any property of a measure-preserving transformation of $\{0,1\}^\lambda$, which is invariant under measure algebra isomorphism could be realized by a measure-preserving transformation of $(\Omega, \mathcal{B}, L_\mu)$.

Theorem 7.2. *Given any $\aleph_1$-saturated nonstandard universe ${}^*V$, for every Loeb space $(\Omega, \mathcal{B}, L_\mu)$ there is a complete subalgebra $\mathcal{C} \subseteq \mathcal{B}$ and an internal permutation T of Ω such that $(\Omega, \mathcal{C}, L_\mu \restriction \mathcal{C})$ is an atomless probability space and T is an ergodic transformation on $(\Omega, \mathcal{C}, L_\mu \restriction \mathcal{C})$.*

Proof: Suppose $\Phi : \bar{\mathcal{B}} \mapsto \bar{\mathcal{B}}(\{0,1\}^\lambda)$ is a measure-preserving isomorphism. Consider a complete subalgebra $\bar{\mathcal{D}} \subseteq \bar{\mathcal{B}}(\{0,1\}^\lambda)$ generated by all $\bar{A}_i$'s for $i \in \omega$, where

$$A_i = \{f \in \{0,1\}^\lambda : f(i) = 0\}.$$

Let F_0 be an ergodic transformation of the space $(\{0,1\}^\omega, \mathcal{B}(\{0,1\}^\omega), \nu_\omega)$. Define a map $F : \{0,1\}^\lambda \mapsto \{0,1\}^\lambda$ such that

$$F(f) \restriction \omega = F_0(f \restriction \omega) \text{ and } F(f) \restriction \lambda \setminus \omega = f \restriction \lambda \setminus \omega.$$

Then F is an ergodic transformation on $(\{0,1\}^\lambda, \mathcal{D}, \nu_\lambda \restriction \mathcal{D})$, which is an atomless probability space. Let $\hat{F} : \bar{\mathcal{D}} \mapsto \bar{\mathcal{D}}$ be induced by F. Let $\bar{\mathcal{C}} = \Phi^{-1}[\bar{\mathcal{D}}]$ and let

$$\hat{T} = \Phi^{-1} \circ \hat{F} \circ \Phi \restriction \bar{\mathcal{C}}.$$

Then $\hat{T} : \bar{\mathcal{C}} \mapsto \bar{\mathcal{C}}$ is an automorphism. Clearly, $\bar{\mathcal{C}}$ is completely generated by $\{\Phi^{-1}(\bar{A}_i) : i \in \omega\}$. Let $\mathcal{C}$ be the complete subalgebra of $\mathcal{B}$ generated by $\bigcup \bar{\mathcal{C}}$. Then $\mathcal{C}$ is completely generated by a countable subalgebra $\mathcal{C}_0$ of internal subsets of Ω. Obviously, $\mathcal{C}$ is atomless. For each $C \in \mathcal{C}_0$ let's choose an internal set $D_C \subseteq \Omega$ such that $\hat{T}(\bar{C}) = \bar{D}_C$. For each finite $\sigma \subseteq \mathcal{C}_0$ and each $n \in \mathbb{N}$ let

$$\mathcal{F}_{n,\sigma} = \{p : p \text{ is an internal permutation of } \Omega \text{ such that}$$
$$\text{for each } C \in \sigma \text{ one has } \mu(p[C] \Delta D_C) < \frac{1}{n}.\}.$$

Then $\mathcal{F}_{n,\sigma}$ is internal. It is easy to see that the set

$$\{\mathcal{F}_{n,\sigma} : n \in \mathbb{N} \text{ and } \sigma \subseteq \mathcal{C}_0 \text{ is finite.}\}$$

has the finite intersection property. By $\aleph_1$-saturation there is an internal permutation T in the intersection of all those $\mathcal{F}_{n,\sigma}$'s. It is now routine to check that T is an ergodic transformation on $(\Omega, \mathcal{C}, L_\mu \restriction \mathcal{C})$. $\square$

Remark Again T above could be made to be a strong mixing transformation.

Problem 7.3. *Could a Loeb space have an ergodic transformation, without assuming $SMA_{\aleph_1}$?*

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