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Standardizing Nonstandard Methods For Upper Banach Density Problems

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ABSTRACT. Many results in [**J1, J2**] about the addition of sets of positive upper Banach density are proven here using standard methods. These standard methods are translated from the nonstandard methods used in [**J1, J2**].

1. Introduction

There are many interesting theorems about Shnirel'man density or lower asymptotic density (see [**HR, N1, N2**]) in additive number theory. There are a few interesting results about upper Banach density (see [**B, F**]) in combinatorial number theory. However, not very many results about upper Banach density involving addition of sets can be found in literature. Recently, I obtained some results of this kind in [**J1, J2**] using nonstandard analysis. Some of these results were presented at the DIMACS workshop *Unusual Applications of Number Theory*. Since some people who are interested in the subject may not be comfortable reading the proofs using nonstandard analysis, I would like to translate them into the standard proofs in order to reach a wider range of readers.

I denote by $\mathbb{N}$ the set of all natural numbers including 0 and by $\mathbb{Z}$ the set of all integers. For any two integers $m \leqslant n$, I write $[m, n]$ exclusively for the interval of integers between m and n including m and n. The upper case letters $A, B, C, \ldots$ except X and Y, which I reserve for something else, will always denote the subsets of $\mathbb{N}$ and the lower case letters $a, b, c, \ldots$ will always denote the elements of $\mathbb{N}$ unless specified. For a finite set A, let $|A|$ be the number of elements in A. The notion $A(n)$ is an abbreviation of $|A \cap [1, n]|$ and the notion $A(m, n)$ with $m \leqslant n$ is an abbreviation of $|A \cap [m, n]|$. For any sets $A_1, A_2, \ldots, A_h$, let $\sum_{i=1}^{h} A_i$ denote the

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set

$$\{\sum_{i=1}^{h} a_i : a_i \in A_i \text{ for } i = 1, 2, \ldots, h\}.$$

If all A_i's are same, say $A_i = A$ for $i = 1, 2, \ldots, h$, then we write simply hA for $\sum_{i=1}^{h} A_i$. The Shnirel'man density $\sigma(A)$, the lower asymptotic density $\underline{d}(A)$, the upper asymptotic density $\bar{d}(A)$, and the upper Banach density $BD(A)$ of a set A are defined as the following.

$$\sigma(A) = \inf_{n \geq 1} \frac{A(n)}{n}$$

$$\underline{d}(A) = \liminf_{n \to \infty} \frac{A(n)}{n}$$

$$\bar{d}(A) = \limsup_{n \to \infty} \frac{A(n)}{n}$$

$$BD(A) = \lim_{n \to \infty} \sup_{m-l=n} \frac{A(l, m)}{m - l + 1}.$$

It is easy to see that for any set A,

$$0 \leqslant \sigma(A) \leqslant \underline{d}(A) \leqslant \bar{d}(A) \leqslant BD(A) \leqslant 1.$$

2. Sumset Phenomenon

The results in this section were proven in [**J1**] using the methods from nonstandard analysis. Here we give standard alternatives of those proofs.

A set A is called syndetic if there is a positive integer k such that $A \cap [n, n+k] \neq \emptyset$ for every $n \in \mathbb{N}$. Equivalently, a set A is syndetic iff $(A \cup \{0\}) + [0, k] = \mathbb{N}$ for some positive integer k. There is a well-known result [**F**, Proposition 3.19 (a)] in combinatorial number theory, which says that if a set A has positive upper Banach density, then $A - A = \{a - a' : a, a' \in A \text{ and } a \geqslant a'\}$ is syndetic. It is natural to ask what can be said for $A + A$. One can easily construct a set A with $BD(A) = 1$ and $A + A$ is not syndetic.

Essentially, upper Banach density is a concept of measure and syndeticity is a concept of "order-topology" on $\mathbb{N}$. So one can ask the question in a more abstract way. What kind of order-topological properties one can have for $A + A$ when A has positive measure? A result called Steinhaus's theorem in real analysis says that if two sets of real numbers, X and Y, have positive Lebesgue measure, then $X + Y$ covers a non-empty open interval. Inspired by this result, a question in nonstandard analysis was asked [**KL**, Problem 9.13] (the reader doesn't need to know the definitions of the concepts in nonstandard analysis mentioned below for reading the rest of the article).

> Whether are there any cut U in a hyperfinite Loeb probability space H for which there exist sets $A, B \subseteq H$ of positive Loeb measure such that A plus B is U–meager?

Note that a set in a topological space is called meager if it is a countable union of nowhere dense sets.

Above question was answered negatively in [**J1**].

Since a set A has upper Banach density $\geqslant \alpha$ if and only if the nonstandard copy of A has Loeb measure $\geqslant \alpha$ in a hyperfinite Loeb probability space, the set $A + A$ should bear some properties of the order structure of $\mathbb{N}$. In fact, if one translate the U–meagerness in a Loeb space into the standard world, it corresponds non-piecewise syndeticity.

A set A is called thick if A contains k consecutive positive integers for every positive integer k. A set A is called piecewise syndetic if $A+[0,k]$ is thick for some $k \in \mathbb{N}$. Thick sets, syndetic sets and piecewise syndetic sets are the objects studied in combinatorial number theory (see [**B, F**]). By deriving the consequences of the main result in [**J1**], I obtained several theorems of an interesting general phenomenon in the standard world, which says that if X and Y are large in terms of "measure", then $X+Y$ is not small in terms of "order–topology". Note that there are usually "nowhere dense" set of positive "measure". For example, Steinhaus's theorem is one of these consequences. The next theorem is another consequence.

THEOREM 2.1. *For any sets A and B, if $BD(A)>0$ and $BD(B)>0$, then $A+B$ is piecewise syndetic.*

Before proving Theorem 2.1, I would like to introduce two lemmas first.

LEMMA 2.2. *Let A be a set, $a \leqslant b$ be two positive integers, and $\alpha = \frac{A(a,b)}{b-a+1}$. For any real number ϵ between 0 and 1, there is an $a' \in [a,b]$ such that $b-a'+1 \geqslant \epsilon(b-a+1)$ and*

$$\inf_{a' \leq i \leq b} \frac{A(a',i)}{i-a'+1} > \alpha - \epsilon. \tag{2.1}$$

PROOF. Let $a'=a$ if

$$\inf_{a \leq i \leq b} \frac{A(a,i)}{i-a+1} > \alpha - \epsilon.$$

Otherwise, we construct an increasing finite sequence $a_0 < a_1 < \cdots < a_k$ inductively with $a_0 = a-1$. Suppose a_i is defined for $i \geqslant 0$. If the set

$$S = \{h \in [a_i+1, b] : \frac{A(a_i+1,h)}{h-a_i} \leqslant \alpha - \epsilon\}$$

is not empty, then let $a_{i+1} = \min S$. Otherwise the construction is terminated. Let $a' = a_k + 1$, where a_k is the last term of the sequence. Next we want to show that $b - a' + 1 \geqslant \epsilon(b-a+1)$. Since

$$\begin{aligned}
\alpha &= \frac{A(a,b)}{b-a+1} \\
&= \frac{\sum_{i=0}^{k-1} A(a_i+1, a_{i+1}) + A(a',b)}{b-a+1} \\
&= \frac{1}{b-a+1} \sum_{i=0}^{k-1} \frac{A(a_i+1,a_{i+1})}{a_{i+1}-a_i}(a_{i+1}-a_i) + \frac{A(a',b)}{b-a+1} \\
&\leqslant \alpha - \epsilon + \frac{b-a'+1}{b-a+1},
\end{aligned}$$

then $\frac{b-a'+1}{b-a+1} \geqslant \epsilon$. Hence $b-a'+1 \geqslant \epsilon(b-a+1)$. Now 2.1 is true because a_k is the last term of the sequence and $a' = a_k+1$.

□

Lemma 2.3. *Let* $A, B \subseteq \mathbb{N}$ *and* $[a,b],[c,d] \subseteq \mathbb{N}$ *such that*

$$\frac{A(a,b)}{b-a+1} = \alpha > \frac{1}{2} \text{ and } \frac{B(c,d)}{d-c+1} = \beta > \frac{1}{2}.$$

Then there is an interval $[s,t] \subseteq (A+B) \cap [a+c, b+d]$ *such that* $t-s+1 \geqslant \epsilon \cdot \min\{b-a+1, d-c+1\}$, *where* $\epsilon = \min\{\alpha - \frac{1}{2}, \beta - \frac{1}{2}\}$.

Proof. By Lemma 2.2, there are $a' \in [a,b]$ and $c' \in [c,d]$ such that $b-a'+1 \geqslant \epsilon(b-a+1)$, $d-c'+1 \geqslant \epsilon(d-c+1)$,

$$\inf_{a' \leq i \leq b} \frac{A(a',i)}{i-a'+1} > \frac{1}{2} \tag{2.2}$$

and

$$\inf_{c' \leq i \leq d} \frac{A(c',i)}{i-c'+1} > \frac{1}{2}. \tag{2.3}$$

Let $s = a' + c'$ and $t = s + \min\{b-a', d-c'\}$. Then

$$t-s+1 \geqslant \epsilon \cdot \min\{b-a+1, d-c+1\}.$$

Obviously, $[s,t] \subseteq [a+c, b+d]$. For each $m \in [s,t]$, we need to show $m \in A+B$. Note that $a' + (m-s) \leqslant b$ and $c' + (m-s) \leqslant d$. Then $A(a', a'+(m-s)) > \frac{1}{2}(m-s+1)$ by 2.2 and $B(c', c'+(m-s)) > \frac{1}{2}(m-s+1)$ by 2.3. Let

$$A' = A \cap [a', a'+(m-s)]$$

and

$$B' = \{m-i : i \in B \cap [c', c'+(m-s)]\}.$$

Then $A', B' \subseteq [a', a'+(m-s)]$. Also $|A'| > \frac{1}{2}(m-s+1)$ and

$$|B'| = B(c', c'+(m-s)) > \frac{1}{2}(m-s+1).$$

So $A' \cap B' \neq \emptyset$. Let $k \in A' \cap B'$. Then $k = x = m-y$ for some $x \in A$ and $y \in B$. Hence $m = x+y \in A+B$.

□

Now we prove Theorem 2.1.

Proof. Assume the theorem is not true. We need to derive a contradiction. First define two numbers α and β such that

$$\alpha = \sup\{BD(X) : X \subseteq \mathbb{N} \text{ and there is a } Y \subseteq \mathbb{N} \text{ such that } BD(Y) > 0 \text{ and } X+Y \text{ is not piecewise syndetic.}\}$$

and

$$\beta = \sup\{BD(Y) : Y \subseteq \mathbb{N} \text{ and there is a } X \subseteq \mathbb{N} \text{ such that } BD(X) > (23/24)\alpha \text{ and } X+Y \text{ is not piecewise syndetic.}\}.$$

By the assumption that the theorem is not true, we have $\beta > 0$.

Claim 2.1.1: $\beta \leqslant \alpha$ and $\beta < \frac{7}{12}$.

Proof It is obvious that $\beta \leqslant \alpha$ due to the way they are defined. Suppose $\beta \geqslant \frac{7}{12}$. Then $\alpha \geqslant \frac{7}{12}$. Choose A and B such that $BD(A) > (23/24)\alpha$ and $BD(B) > (23/24)\beta$. Then $BD(A) > (23/24) \cdot (7/12) > \frac{1}{2}$ and $BD(B) > \frac{1}{2}$. Let

$\langle [a_n, b_n] : n \in \mathbb{N} \rangle$ and $\langle [c_n, d_n] : n \in \mathbb{N} \rangle$ be two sequences of intervals such that $\lim_{n\to\infty}(b_n - a_n) = \infty$, $\lim_{n\to\infty}(d_n - c_n) = \infty$,

$$\frac{A(a_n, b_n)}{b_n - a_n + 1} \geqslant \frac{1}{2} + \epsilon,$$

and

$$\frac{B(c_n, d_n)}{d_n - c_n + 1} \geqslant \frac{1}{2} + \epsilon$$

for some fixed positive number ϵ. Let M be any positive number and choose an n such that $\epsilon(b_n - a_n + 1) > M$ and $\epsilon(d_n - c_n + 1) > M$. By Lemma 2.3, there is an interval $[s, t] \subseteq A + B$ such that $t - s + 1 > M$. This showed that $A + B$ is thick, which contradicts that $A + B$ is not piecewise syndetic. $\square$(Claim 2.1.1)

Let's fix two sets A and B such that $BD(A) > \frac{23}{24}\alpha$, $BD(B) > \frac{23}{24}\beta$, and $A + B$ is not piecewise syndetic. For each $k \in \mathbb{N}$, clearly

$$(A + B) + [0, k] = A + (B + [0.k])$$

is not piecewise syndetic. By the supremality of β, we have

$$BD(B + [0, k]) \leqslant \beta$$

for every $k \in \mathbb{N}$. Let $\langle [a_n, b_n] : n \in \mathbb{N} \rangle$ and $\langle [c_n, d_n] : n \in \mathbb{N} \rangle$ be two sequences of intervals such that for each $n \in \mathbb{N}$,

$$\frac{A(a_n, b_n)}{b_n - a_n + 1} > \frac{23}{24}\alpha,$$
$$\frac{B(c_n, d_n)}{d_n - c_n + 1} > \frac{23}{24}\beta,$$
$$\frac{(B + [0, 2n])(c_n, d_n)}{d_n - c_n + 1} < \frac{25}{24}\beta,$$

and $d_n - c_n + 1 > n^2$. For each $n > 40$, let

$$I_n = \{[c_n + in, c_n + (i + 1)n - 1] : i = 0, 1, \ldots, l_n\}$$

where $l_n = [\frac{d_n - c_n + 1}{n}]$. If

$$|\{i \in [0, l_n - 1] : B(c_n + in, c_n + (i + 1)n - 1) = \emptyset\}| \leqslant \frac{1}{3}l_n,$$

which means that there are at least two third of the intervals in I_n containing elements from B, then

$$\begin{aligned}|(B + [0, 2n])(c_n, d_n)| &\geqslant (2/3)l_n n - n \\ &\geqslant (2/3)(d_n - c_n + 1 - n) - n \\ &\geqslant (2/3)(d_n - c_n + 1) - 2n.\end{aligned}$$

So

$$\frac{(B + [0, 2n])(c_n, d_n)}{d_n - c_n + 1} \geqslant \frac{2}{3} - \frac{2}{n} > \frac{2}{3} - \frac{1}{20} = \frac{37}{60}.$$

On the other hand, we have

$$\frac{(B + [0, 2n])(c_n, d_n)}{d_n - c_n + 1} < \frac{25}{24}\beta \leqslant \frac{25}{24} \cdot \frac{7}{12} < \frac{37}{60}.$$

So we have a contradiction. Hence, for every $n > 40$,

$$|\{i \in [0, l_n - 1] : B(c_n + in, c_n + (i + 1)n - 1) = \emptyset\}| > \frac{1}{3}l_n$$

Claim 2.1.2: For each $n > \max\{40, \frac{12}{\beta}\}$, There exists an $i_n \in [0, l_n]$ such that

$$\frac{B(c_n + i_n n, c_n + (i_n + 1)n - 1)}{n} \geqslant \frac{13}{12}\beta.$$

Proof Suppose the claim is not true. Then

$$\begin{aligned}\frac{B(c_n, d_n)}{d_n - c_n + 1} &= \frac{\sum_{i=0}^{l_n-1} B(c_n + in, c_n + (i+1)n - 1) + B(c_n + l_n n, d_n)}{d_n - c_n + 1} \\ &= \frac{n}{d_n - c_n + 1} \sum_{i=0}^{l_n-1} \frac{B(c_n + in, c_n + (i+1)n - 1)}{n} + \frac{B(c_n + l_n n, d_n)}{d_n - c_n + 1} \\ &\leqslant \frac{n}{d_n - c_n + 1} \cdot l_n \cdot \frac{2}{3} \cdot \frac{13}{12}\beta + \frac{n}{d_n - c_n + 1} \\ &\leqslant (26/36)\beta + \frac{1}{n} < (29/36)\beta.\end{aligned}$$

This contradicts that

$$\frac{B(c_n, d_n)}{d_n - c_n + 1} > \frac{23}{24}\beta > \frac{29}{36}\beta.$$

□(Claim 2.1.2)

Following from Claim 2.1.2, we have $BD(B) \geqslant \frac{13}{12}\beta$. This contradicts the supremality of β.

□

Remark 2.4. (1) For each positive real number $r < 1$, it is not hard to construct a set A such that $BD(A) > r$ and A is not piecewise syndetic.

(2) The reader who is interested in nonstandard analysis should read the original proof in [**J1**]. By translating the nonstandard proof into the standard proof here, certain degree of intuition and motivation are lost. The reader may consider some steps of the proof here to be tedious and awkward. But these steps make perfect sense when one is working in a nonstandard setting.

Theorem 2.1 has two corollaries which are worthy of mentioning. They are proven in [**J1**].

Corollary 2.5. *If $BD(A) > 0$ and $\gcd(A - a_0) = 1$, where a_0 is the least element in A, then there is an $h \in \mathbb{N}$ such that hA is thick.*

Given an increasing sequence $\langle x_n : n \in \mathbb{N}\rangle$ of natural numbers, let

$$FS(x_n)_{n=0}^{\infty} = \{\sum_{n \in F} x_n : F \text{ is a non-empty finite subset of } \mathbb{N}\}.$$

Corollary 2.6. *If $BD(A) > 0$ and $BD(B) > 0$, then $A + B$ contains a set of form $k + FS(x_n)_{n=0}^{\infty}$.*

Remark 2.7. It is mentioned in [**B**] that for each positive real number $r < 1$, there is a set A such that $BD(A) > r$ and A does not contain any set of form $k + FS(x_n)_{n=0}^{\infty}$.

3. Buy One Get One Free Scheme

In this section, we show a method that allows one to efficiently derive a theorem about upper Banach density parallel to an existing theorem about Shnirel'man density or lower asymptotic density. Hence, whenever one obtains a result about Shnirel'man density or lower asymptotic density, one can get a result about upper Banach density for free.

Among four densities introduced at the beginning of this article, Shnirel'man density is traditionally the most interesting one to number theorists. There are many classical results about Shnirel'man density such as Mann's theorem [**HR**, page 5], Erdös-Landau's theorem [**HR**, page 10], Plünnecke's theorem [**N2**, page 225], etc. It often happens that after a result about Shnirel'man density is obtained, people like to generalize the result to lower asymptotic density or derive a parallel result about lower asymptotic density. For example, Kneser's theorem [**HR**, page 57] is a parallel theorem to Mann's theorem. Rohrbach's theorem [**HR**, page 45] is parallel to Erdös-Landau's theorem, and Plünnecke's theorem is also true if one replaces σ by $\underline{d}$. From these examples, one can see that the behavior of Shnirel'man density and the behavior of lower asymptotic density are very similar. What about the behavior of upper asymptotic density and the behavior of upper Banach density? Can one derive some parallel results about upper asymptotic density or upper Banach density to the existing results about Shnirel'man density or lower asymptotic density? Since lower asymptotic density and upper asymptotic density are usually introduced as a pair in many books and papers, it is natural for people to consider upper asymptotic density first. However, not many theorems about upper asymptotic density have been proven so far. The reason for that might be because the behavior of upper asymptotic density is quite different from that of lower asymptotic density or Shnirel'man density. For example, one can construct two sets A and B such that $\bar{d}(A) = \bar{d}(B) = \bar{d}(A + B + [0.k]) = \frac{1}{2}$ for every $k \in \mathbb{N}$. Hence there is no hope to find a theorem about upper asymptotic density parallel to Mann's theorem. According to the order of these densities, upper Banach density seems farther away from Shnirel'man density or lower asymptotic density than upper asymptotic density does. Therefore, the behavior of upper Banach density might be more different from the behavior of Shnirel'man density or lower asymptotic density. However, this is not the case! In fact, the behavior of upper Banach density bears extreme resemblance with the behavior of Shnirel'man density or lower asymptotic density.

In [**J2**], a general method was developed using nonstandard analysis, which allows us to efficiently derive a theorem about upper Banach density parallel to an existing theorem about Shnirel'man density or lower asymptotic density. The main idea is the following.

A set A has upper Banach density $\geqslant \alpha$ iff there is a copy of the standard set $\mathbb{N}$ inside the nonstandard version of $\mathbb{N}$ such that the nonstandard version of A has lower asymptotic density $\geqslant \alpha$ inside the copy of $\mathbb{N}$. The above statement is also true with lower asymptotic density replaced by Shnirel'man density. So when a set A with $BD(A) = \alpha$ is given, one can apply the existing theorem about Shnirel'man density or lower asymptotic density to that remote copy of $\mathbb{N}$ to obtain a result about the nonstandard version of A. By pushing down the nonstandard result to the standard world, one can obtain a parallel theorem about upper Banach density.

By following the same idea in the first section, one can find that it is no harder than the proof of Theorem 2.1 for translating each nonstandard proof in [**J2**] to a standard proof. One only needs to finitize every hyperfinite argument. But by doing so, one has to dissect the proof and get into those hard-core finite combinatorial arguments of each existing theorem about Shnirel'man density or lower asymptotic density. Those arguments are difficult and very different from theorem to theorem. So this retail style translation would force us to deal with each theorem individually. Therefore, it is not very efficient and usually obscures the general picture.

Recently, I found that the ergodic methods developed in [**F**] can play the same role as the nonstandard methods on this subject. The ergodic methods also offer a wholesale style treatment for deriving parallel theorems about upper Banach density. So I would like to use ergodic methods here as the standard alternatives of the nonstandard proofs in [**J2**].

Theorem 3.20 of [**F**] almost establishes the needed relation between upper Banach density and lower asymptotic density. It says that if $BD(A) > 0$, then there is a set R such that $\bar{d}(R) = \underline{d}(R) > 0$ and for every n, there is an m such that $m + R \cap [0, n] \subseteq A$. In order to use this idea for our purpose, we need to make a little adjustment. We would like to prove the following.

THEOREM 3.1. *Suppose $BD(A) = \alpha$. Then one can find a set R with $\bar{d}(R) = \underline{d}(R) = \alpha$ that for every n, there is an m such that $m + R \cap [0, n] = A \cap [m, m+n]$.*

Let's first prove a corollary of Theorem 3.1, which relates upper Banach density to Shnirel'man density.

COROLLARY 3.2. *Suppose $BD(A) = \alpha$ and $\epsilon > 0$. Then one can find a set Q with $\sigma(Q) \geqslant \alpha - \epsilon$ that for every n, there is an m such that $m + Q \cap [1, n] = A \cap [m+1, m+n]$.*

PROOF. Let R be the set produced in Theorem 3.1. Similar to the proof of Lemma 2.2, we construct an increasing sequence $\langle a_n : n = 0, 1, \ldots, k\rangle$.

Let the sequence to be an empty sequence and $Q = R$ if

$$\inf_{n \geq 1} \frac{R(n)}{n} \geqslant \alpha - \epsilon.$$

Then the proof is finished because $\sigma(Q) \geqslant \alpha - \epsilon$.

Otherwise, we construct an increasing sequence $a_0 < a_1 < \cdots < a_k$ inductively with $a_0 = 0$. Suppose a_i is defined for $i \geqslant 0$. If the set

$$S = \{h > a_i : \frac{R(a_i + 1, h)}{h - a_i} \leqslant \alpha - \epsilon\}$$

is not empty, then let $a_{i+1} = \min S$. Otherwise the construction is terminated.

Claim 3.2.1: The sequence has finite length.

Proof Suppose the sequence is never terminated. Then for each n one has

$$\frac{R(a_n)}{a_n} \leqslant \alpha - \epsilon,$$

which contradicts the fact that $\underline{d}(R) = \alpha$. $\square$(Claim 3.2.1)

Let a_k be the last element of the sequence and let

$$Q = \{m - a_k : m \in R \text{ and } m \geqslant a_k\}.$$

Clearly, $\sigma(Q) \geqslant \alpha - \epsilon$. For each n, there exists an m' such that

$$m' + R \cap [0, a_k + n] = A \cap [m', m' + a_k + n].$$

Hence,

$$\begin{aligned} m' + a_k + Q \cap [1, n] &= m' + R \cap [a_k + 1, a_k + n] \\ &= A \cap [m' + a_k + 1, m' + a_k + n]. \end{aligned}$$

Now let $m = m' + a_k$ and this ends the proof.

□

Now we prove Theorem 3.1

Proof. All ideas of the proof of Theorem 3.1 are from [**F**] except Claim 3.1.1 below.

Let A be a set with $BD(A) = \alpha$. Let $\Omega = \{0, 1\}^{\mathbb{Z}}$ be the set of all functions from $\mathbb{Z}$ to $\{0, 1\}$ with the product topology (consider $\{0, 1\}$ as a two-point discrete space). Consider A as a subset of $\mathbb{Z}$ and let ϕ_A be the characteristic function of A. Then $\phi_A \in \Omega$. Let T be a bijection from Ω to Ω such that for every $\omega \in \Omega$, $T(\omega)(x) = \omega(x + 1)$ for every $x \in \mathbb{Z}$. Let T^0 be the identity map and for every positive integer k, let $T^k = T \circ T^{k-1}$ and let T^{-k} be the inverse function of T^k. Let

$$X = \overline{\{T^k(\phi_A) : k \in \mathbb{Z}\}},$$

the orbit closure of ϕ_A under T. Then X is a separable compact topological subspace of Ω. Let $C(X)$ be the set of all real-valued continuous functions on X. Then $C(X)$ is a separable Banach space under the supreme norm $\| \cdot \|$, *i.e.* $\|f\| = \sup\{|f(x)| : x \in X\}$.

Now we choose a sequence of intervals $\langle [a_n, b_n] : n \in \mathbb{N} \rangle$ such that $\lim_{n\to\infty}(b_n - a_n) = \infty$ and $\lim_{n\to\infty} \frac{A(a_n, b_n)}{b_n - a_n + 1} = \alpha$. Without loss of generality, we can assume that for every $f \in C(X)$, the limit

$$\lim_{n\to\infty} \frac{1}{b_n - a_n + 1} \sum_{a_n \leq k \leq b_n} f(T^k(\phi_A)) \tag{3.1}$$

exists. The way to do this is the following. Let C_0 be a countable dense subset of $C(X)$. Using Cantor's diagonal argument, one can replace the sequence of the intervals mentioned above by a subsequence so that the limit in 3.1 exists for every $f \in C_0$. Since every function in $C(X)$ is a uniform limit of a function sequence in C_0, then the limit in 3.1 also exists for every $f \in C(X)$.

Now we define a positive linear functional L on $C(X)$. For each $f \in C(X)$, let

$$L(f) = \lim_{n\to\infty} \frac{1}{b_n - a_n + 1} \sum_{a_n \leq k \leq b_n} f(T^k(\phi_A)).$$

It is not hard to verify that L is a well-defined linear functional, $|L(f)| \leqslant \|f\|$, and $L(f) \geqslant 0$ whenever $f \geqslant 0$. By Riesz Representation Theorem, there is a finite Borel measure μ on X such that for every $f \in C(X)$,

$$L(f) = \int_X f d\mu.$$

Since

$$\mu(X) = \int_X 1 d\mu = L(1)$$
$$= \lim_{n\to\infty} \frac{1}{b_n - a_n + 1} \sum_{a_n \le k \le b_n} 1(T^k(\phi_A)) = 1,$$

then μ is a probability measure. It is also easy to verify that $L(f \circ T) = L(f)$ for every $f \in C(X)$. So T is a μ–measure preserving transformation from X to X. Let

$$E = \{\omega \in X : \omega(0) = 1\}.$$

Then E is a clopen subset of X. Hence the characteristic function Φ_E of E is an element in $C(X)$. Since for each $k \in \mathbb{N}$

$$\Phi_E(T^k(\phi_A)) = 1 \iff T^k(\phi_A) \in E$$
$$\iff T^k(\phi_A)(0) = 1$$
$$\iff \phi_A(k) = 1$$
$$\iff k \in A,$$

then

$$\mu(E) = \int_X \Phi_E d\mu = L(\Phi_E)$$
$$= \lim_{n\to\infty} \frac{1}{b_n - a_n + 1} \sum_{a_n \le k \le b_n} \Phi_E(T^k(\phi_A))$$
$$= \lim_{n\to\infty} \frac{A(a_n, b_n)}{b_n - a_n + 1} = \alpha.$$

Since Φ_E is a μ–measurable function and T is a μ–measure–preserving transformation, by Birkhoff ergodic theorem [**F**, page 59], there exists a measurable function $\bar{f} \in L^1(X)$ such that for μ–almost all $\omega \in X$

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \Phi_E(T^k(\omega)) = \bar{f}(\omega).$$

Claim 3.1.1: For μ–almost all $\omega \in X$, $\bar{f}(\omega) = \alpha$.

Proof: Suppose that there is an $\omega \in X$ such that

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \Phi_E(T^k(\omega)) = \beta > \alpha.$$

Let $R = \{m \in \mathbb{N} : \omega(m) = 1\}$. Then

$$k \in R \iff \omega(k) = 1$$
$$\iff T^k(\omega)(0) = 1$$
$$\iff T^k(\omega) \in E$$
$$\iff \Phi_E(T^k(\omega)) = 1.$$

Hence,

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \Phi_E(T^k(\omega)) = \lim_{n\to\infty} \frac{R(0,n)}{n+1} = \beta.$$

So $\underline{d}(R) = \beta$. Since $\omega \in X$, then for every n, there is a k_n such that $\omega(i) = T^{k_n}(\phi_A)(i)$ for $i = 0, 1, \ldots, n$. So

$$k_n + R \cap [0, n] = A \cap [k_n, k_n + n].$$

Since

$$\lim_{n\to\infty} \frac{R(0,n)}{n+1} = \beta,$$

then

$$\lim_{n\to\infty} \frac{A(k_n, k_n+n)}{n+1} = \beta.$$

This shows that $BD(A) \geqslant \beta > \alpha = BD(A)$, a contradiction. So $\bar{f}(\omega) \leqslant \alpha$ is true for μ–almost all $\omega \in X$.

On the other hand, since T is μ–invariant, then

$$\begin{aligned}
\int_X \bar{f} d\mu &= \int_X (\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \Phi_E(T^k(\omega))) d\mu \\
&= \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \int_X \Phi_E(T^k(\omega)) d\mu \\
&= \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \int_X \Phi_E(\omega) d\mu \\
&= \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \mu(E) \\
&= \mu(E) = \alpha.
\end{aligned}$$

Hence $\int_X (\alpha - \bar{f}) d\mu = 0$. Since $\alpha - \bar{f}$ is μ–almost surely non–negative, then we conclude that $\bar{f}(\omega) = \alpha$ for μ–almost all $\omega \in X$. □(Claim 3.1.1)

Now let's choose an $\omega \in X$ such that

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \Phi_E(T^k(\omega)) = \alpha$$

and let $R = \{m \in \mathbb{N} : \omega(m) = 1\}$. Then $\underline{d}(R) = \alpha$. For every $n \in \mathbb{N}$, since $\omega \in X$, there is a $k_n \in \mathbb{Z}$ such that $\omega(i) = T^{k_n}(\phi_A)(i) = \phi_A(k_n + i)$ for $i = 0, 1, \ldots, n$. So

$$k_n + R \cap [0, n] = A \cap [k_n, k_n + n].$$

Without loss of generality, we can assume that $0 \in R$ (or simply replace R by $R - r_0$, where r_0 is the least element in R). Then it is easy to see that $m = k_n \geqslant 0$ for every n. This ends the proof. □

With Theorem 3.1 and Corollary 3.2 in our hands, we can now derive parallel theorems about upper Banach density. We derive four theorems in this section for a demonstration of the general method of the applications of Theorem 3.1 and Corollary 3.2.

We first derive a theorem parallel to Plünnecke's Theorem. A set B is called a basis of order h if $hB = \mathbb{N}$. Plünnecke's Theorem says that if B is a basis of order h, then for every set A, $\sigma(A + B) \geqslant \sigma(A)^{1-\frac{1}{h}}$.

THEOREM 3.3. *If B is a basis of order h, then for every set A,*

$$BD(A+B) \geqslant BD(A)^{1-\frac{1}{h}}. \tag{3.2}$$

PROOF. Let $BD(A) = \alpha$. For each $\epsilon > 0$, let Q be the set obtained in Corollary 3.2 such that $\sigma(Q) \geqslant \alpha - \epsilon$. By Plünnecke's Theorem, one has

$$\sigma(Q+B) \geqslant \sigma(Q)^{1-\frac{1}{h}} = (\alpha-\epsilon)^{1-\frac{1}{h}}.$$

For every n, since there is a k such that $k + Q \cap [1, n] = A \cap [k+1, k+n]$, then $k + (Q+B) \cap [1, n] \subseteq (A+B) \cap [k+1, k+n]$. This shows that

$$\begin{aligned} BD(A+B) &\geqslant \lim_{n\to\infty} \sup_{k\geq 0} \frac{(A+B)(k+1,k+n)}{n} \\ &\geqslant \inf_{n\geq 1} \frac{(Q+B)(1,n)}{n} \\ &= \sigma(Q+B) \geqslant (\alpha-\epsilon)^{1-\frac{1}{h}}. \end{aligned}$$

Let $\epsilon \to 0$ and we have 3.2.

□

Next we want to derive a theorem parallel to Erdös–Landau's Theorem and to Rohrbach's Theorem. Let B be a basis of order h. The real number h^* is called the average order of B if

$$h^* = \sup_{n\geq 1} \frac{1}{n} \sum_{k=1}^{n} h(k),$$

where $h(k) = \min\{h' : k \in h'B\}$. Clearly, $h^* \leqslant h$. Erdös-Landau's Theorem says that if B is a basis of average order h^*, then for every set A,

$$\sigma(A+B) \geqslant \sigma(A) + \frac{1}{2h^*}\sigma(A)(1-\sigma(A)).$$

A set B is called an asymptotic basis of order h if there is an $m \in \mathbb{N}$ such that $hB \cup [0, m] = \mathbb{N}$. Let B be an asymptotic basis of order h and m be the number mentioned above. The real number h^* is called an average order of an asymptotic basis B if

$$h^* = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=m+1}^{n} h(k).$$

Again $h^* \leqslant h$. Rohrbach's Theorem says that if B is an asymptotic basis of average order h^*, then

$$\underline{d}(A+B) \geqslant \underline{d}(A) + \frac{1}{2h^*}\underline{d}(A)(1-\underline{d}(A)).$$

Our parallel theorem is the following.

THEOREM 3.4. *If B is an asymptotic basis of average order h^*, then for every set A,*

$$BD(A+B) \geqslant BD(A) + \frac{1}{2h^*}BD(A)(1-BD(A)). \tag{3.3}$$

PROOF. Let $BD(A) = \alpha$. and let R be the set obtained in Theorem 3.1 with $\underline{d}(R) = \alpha$. By Rohrbach's Theorem, we have

$$\begin{aligned}\underline{d}(R+B) &\geqslant \underline{d}(R) + \frac{1}{2h^*}\underline{d}(R)(1-\underline{d}(R)) \\ &= \alpha + \frac{1}{2h^*}\alpha(1-\alpha).\end{aligned}$$

For every n, since there is a k such that

$$k + R \cap [0,n] = A \cap [k, k+n],$$

then there is a k such that

$$k + (R+B) \cap [0,n] \subseteq (A+B) \cap [k, k+n].$$

This shows that

$$\begin{aligned}BD(A+B) &\geqslant \liminf_{n\to\infty} \frac{(A+B)(k,k+n)}{n+1} \\ &\geqslant \underline{d}(R+B) \\ &\geqslant \alpha + \frac{1}{2h^*}\alpha(1-\alpha).\end{aligned}$$

Hence, 3.3 is true. □

REMARK 3.5. In Theorem 3.3, the basis B of order h can be replaced by a piecewise basis of order h. Also in Theorem 3.4, the asymptotic basis of average order h^* can be replaced by a piecewise asymptotic basis of average order h^*. The definitions of these concepts can be found in [**J2**]. The reason why we don't want to make these replacements here is that if we do, then we have to, without nonstandard analysis, dissect the proofs of Plünnecke's Theorem, Erdös–Landau's Theorem, and Rohrbach's Theorem and get into the finite combinatorial arguments.

Next we derive a parallel theorem to Mann's Theorem. Mann's Theorem says that if $0 \in A \cap B$, then $\sigma(A+B) \geqslant \min\{1, \sigma(A) + \sigma(b)\}$.

THEOREM 3.6. *For any two sets A and B,*

$$BD(A+B+\{0,1\}) \geqslant \min\{1, BD(A) + BD(B)\}.$$

The proof of Theorem 3.6 in [**J2**] uses Besicovitch's Theorem [**HR**, page 6]. Here we would like to prove Theorem 3.6 using next theorem parallel to Kneser's Theorem.

REMARK 3.7.
(1) The term $\{0,1\}$ can be replaced by any two consecutive positive integers $\{a, a+1\}$. However, it can't be removed. For example, if one let A and B both be the set of all even numbers, then $BD(A+B) = BD(A) = BD(B) = \frac{1}{2}$.
(2) For each real number α, let $\lceil \alpha \rceil$ be the least integer greater than or equal to α. By Mann's Theorem, a set A with $\sigma(A) > 0$ and containing 0 is a basis of order at most $\lceil \frac{1}{\sigma(A)} \rceil$. Let's call a set B a Banach basis of order h if hB is thick. By Theorem 3.6, a set A with $BD(A) > 0$ and containing two consecutive numbers is a Banach basis of order at most $2\lceil \frac{1}{BD(A)} \rceil - 1$. This result is also optimal. For example, let

$$A = \{10n : n \in \mathbb{N}\} \cup \{10n+1 : n \in \mathbb{N}\}.$$

Then $BD(A) = \frac{1}{5}$. So A is a Banach basis of order 9. Since for every n, $10n + 9$ is not in $8A$, then A is not a Banach basis of order 8.

(3) Let

$$A = \bigcup_{n=1}^{\infty} [2^{2^{2n}}, 2^{2^{2n}+1} - 1] \text{ and} \tag{3.4}$$

$$B = \bigcup_{n=1}^{\infty} [2^{2^{2n+1}}, 2^{2^{2n+1}+1} - 1]. \tag{3.5}$$

Then $\bar{d}(A) = \bar{d}(B) = \bar{d}(A + B + [0, k]) = \frac{1}{2}$ for any $k \in \mathbb{N}$. So Mann's Theorem has no nice parallel theorem about upper asymptotic density. But if one considers $2A$ instead of $A + B$, then there is a result by G. A. Frieman, which says that if A is a set such that $\gcd(A - a_0) = 1$, where a_0 is the least element in A, then $\bar{d}(A) < \frac{1}{2}$ implies $\bar{d}(2A) \geqslant \frac{3}{2}\bar{d}(A)$ and $\bar{d}(A) \geqslant \frac{1}{2}$ implies $\bar{d}(2A) \geqslant \frac{1+\bar{d}(A)}{2}$. This result is a corollary of [**N2**, Theorem 1.15], one of Frieman's theorems in [**N2**]. On the other hand, for any α and β with $0 \leqslant \alpha \leqslant \frac{1}{2}$ and $\frac{1}{2} < \beta \leqslant 1$, it is not hard to construct a set A and a set B similar to 3.4 or 3.5 such that $\bar{d}(A) = \alpha$, $\bar{d}(2A) = \frac{3}{2}\alpha$ and $\bar{d}(B) = \beta$, $\bar{d}(2B) = \frac{1+\beta}{2}$.

Next we derive a theorem, which is not in [**J2**], parallel to Kneser's Theorem. Kneser's Theorem says that for any $k+1$ sets $A_0, A_1, \ldots, A_k$, if

$$\underline{d}(\sum_{i=0}^{k} A_i) < \liminf_{n\to\infty} \sum_{i=0}^{k} \frac{A_i(n)}{n},$$

then there is a positive integer g and a set $E \subseteq \sum_{i=0}^{k} A_i$ such that E is essentially the union of congruence classes modulo g, *i.e.* there is a positive integer m such that

$$\{e + ng : e \in E \text{ and } n \in \mathbb{Z}\} \cap \mathbb{N} \smallsetminus [0, m] = E \smallsetminus [0, m],$$

and

$$\underline{d}(E) = \bar{d}(E) \geqslant (\liminf_{n\to\infty} \sum_{i=0}^{k} \frac{A_i(n)}{n}) - \frac{k}{g}.$$

Note that since

$$\sum_{i=0}^{k} \underline{d}(A_i) \leqslant \liminf_{n\to\infty} \sum_{i=0}^{k} \frac{A_i(n)}{n},$$

then Kneser's Theorem is also true if one replaces

$$\liminf_{n\to\infty} \sum_{i=0}^{k} \frac{A_i(n)}{n}$$

by $\sum_{i=0}^{k} \underline{d}(A_i)$. We would like to use this replacement instead of the original form of Kneser's Theorem in the next theorem because we want to avoid introducing too much notation to obscure the main idea.

Theorem 3.8. *Let $A_0, A_1, \ldots, A_k$ be $k+1$ sets. If*

$$BD(\sum_{i=0}^{k} A_i) < \sum_{i=0}^{k} BD(A_i), \tag{3.6}$$

then there are $g, l \in \mathbb{N}$, a sequence of intervals $\langle [a_n, b_n] : n = 1, 2, \ldots \rangle$ with $b_n - a_n + 1 = gn$, and a set $E \subseteq \sum_{i=0}^{k} A_i$ such that $E \cap [a_n, b_n]$ is the union of l arithmetic sequences of length n with common difference g and

$$BD(\sum_{i=0}^{k} A_i) \geqslant BD(E) \geqslant \frac{l}{g} \geqslant \sum_{i=0}^{k} BD(A_i) - \frac{k}{g}.$$

Proof. Let $BD(A_i) = \alpha_i$ and $\sum_{i=0}^{k} \alpha_i = \alpha$. Suppose $BD(\sum_{i=0}^{k} A_i) < \alpha$. For each i let R_i be the set obtained in Theorem 3.1 corresponding to A_i. Then $\underline{d}(R_i) = \alpha_i$. Without loss of generality, let's assume $0 \in R_i$.

Claim 3.8.1 $\underline{d}(\sum_{i=0}^{k} R_i) < \alpha$.

Proof: Suppose $\underline{d}(\sum_{i=0}^{k} R_i) \geqslant \alpha$. For each n, let m_i be such that

$$m_i + R_i \cap [0, n] = A_i \cap [m_i, m_i + n]$$

and let $M = \sum_{i=0}^{k} m_i$. Then

$$M + (\sum_{i=0}^{k} R_i) \cap [0, n] \subseteq (\sum_{i=0}^{k} A_i) \cap [M, M + n].$$

Hence

$$\frac{(\sum_{i=0}^{k} A_i)(M, M + n)}{n + 1} \geqslant \frac{(\sum_{i=0}^{k} R_i)(0, n)}{n + 1}.$$

This shows

$$BD(\sum_{i=0}^{k} A_i) \geqslant \underline{d}(\sum_{i=0}^{k} R_i) \geqslant \alpha,$$

a contradiction to 3.6. □(Claim 3.8.1)

By Kneser's Theorem, there exist $g, l, m \in \mathbb{N}$ and a set $R_E \subseteq \sum_{i=0}^{k} R_i$ such that

$$R_E \smallsetminus [0, m] = \{e + gn : e \in R_E \text{ and } n \in \mathbb{Z}\} \cap \mathbb{N} \smallsetminus [0, m]$$

and

$$\underline{d}(R_E) = \frac{l}{g} \geqslant \alpha - \frac{k}{g}.$$

Now we define $[a_n, b_n]$ and E_n by induction on n. Let $[c_1, d_1]$ be such that $d_1 - c_1 + 1 = g$ and $c_1 > m$. Then there exists an $m_{1,i}$ for each $i = 0, 1, \ldots, k$ such that

$$m_{1,i} + R_i \cap [0, d_1] = A_i \cap [m_{1,i}, m_{1,i} + d_1].$$

So

$$\sum_{i=0}^{k} m_{1,i} + R_E \cap [0, d_1] \subseteq (\sum_{i=0}^{k} A_i) \cap [\sum_{i=0}^{k} m_{1,i}, \sum_{i=0}^{k} m_{1,i} + d_1].$$

Let

$$[a_1, b_1] = [\sum_{i=0}^{k} m_{1,i} + c_1, \sum_{i=0}^{k} m_{1,i} + d_1]$$

and let

$$E_1 = \sum_{i=0}^{k} m_{1,i} + R_E \cap [c_1, d_1].$$

Suppose we have found $[a_{n-1}, b_{n-1}]$ and E_{n-1}. Let $c_n > b_{n-1}$ and $d_n = c_n + gn - 1$. Then we do the exactly same as above with $a_1, b_1, c_1, b_1, m_{1,i}$ being replaced by $a_n, b_n, c_n, d_n, m_{n,i}$, and define

$$E_n = \sum_{i=0}^{k} m_{n,i} + R_E \cap [c_n, d_n].$$

Finally, we define $E = \bigcup_{n=1}^{\infty} E_n$. Note that the intervals $[a_n, b_n]$'s are pairwise disjoint. Now it is easy to check that the numbers g, l, the set E, and the sequence $\langle [a_n, b_n] : n = 1, 2, \ldots \rangle$ satisfy all the requirements.

□

Next we use Theorem 3.8 to prove Theorem 3.6.

PROOF. Let $BD(A) = \alpha$ and $BD(B) = \beta$. The theorem is obviously true if $BD(A + B) \geqslant \min\{1, \alpha + \beta\}$. Suppose $BD(A + B) < \min\{1, \alpha + \beta\}$. By Theorem 3.8, there are g, l, the set E, and the sequence $\langle [a_n, b_n] : n = 1, 2, \ldots \rangle$ such that $b_n - a_n + 1 = gn$, $E \cap [a_n, b_n]$ is the union of l arithmetic sequences of length n with common difference g, and $\frac{l}{g} \geqslant \alpha + \beta - \frac{1}{g}$. Clearly, $l < g$ because $BD(A + B) < 1$. Hence $(E + \{0, 1\}) \cap [a_n, b_n]$ contains one more arithmetic sequence of length n or $n - 1$ with common difference g, which is not in E_n. This shows that

$$\begin{aligned} BD(A + B + \{0, 1\}) &\geqslant BD(E + \{0, 1\}) \\ &= \frac{l + 1}{g} = \frac{l}{g} + \frac{1}{g} \geqslant \alpha + \beta \\ &= BD(A) + BD(B). \end{aligned}$$

□

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