Existence of Some Sparse Sets of Nonstandard Natural Numbers [1]

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Abstract

Answers are given to two questions concerning the existence of some sparse subsets of $\mathcal{H} = \{0, 1, \ldots, H-1\} \subseteq {}^*\mathbb{N}$, where H is a hyperfinite integer. In §1, we answer a question of Kanovei by showing that for a given cut U in $\mathcal{H}$, there exists a countably determined set $X \subseteq \mathcal{H}$ which contains exactly one element in each U-monad, if and only if $U = a \cdot \mathbb{N}$ for some $a \in \mathcal{H} \smallsetminus \{0\}$. In §2, we deal with a question of Keisler and Leth in [6]. We show that there is a cut $V \subseteq \mathcal{H}$ such that for any cut U, (i) there exists a U-discrete set $X \subseteq \mathcal{H}$ with $X + X = \mathcal{H}\ (mod\ H)$ provided $U \subsetneqq V$, (ii) there does not exist any U-discrete set $X \subseteq \mathcal{H}$ with $X + X = \mathcal{H}\ (mod\ H)$ provided $U \supsetneqq V$. We obtain some partial results for the case $U = V$.

0 Notation and Definition

We always work within a fixed ω_1-saturated nonstandard universe ${}^*V$ in the sense of [1]. The reader is assumed to be familiar with the basic definitions and facts about nonstandard universe and nonstandard analysis. Those definitions and facts could be found in §4.4 of [1], [2] or [7]. We write $\alpha, \beta, \ldots$ for ordinals and $\kappa, \lambda, \ldots$ for cardinals. We write ω for the first infinite ordinal which is also the set of all standard natural numbers. For any $n \leqslant \omega$ we write ${}^n2$ for the set of all functions from $\{k \in \omega : k < n\}$ to 2. We write also ${}^{<\omega}2$ for the set $\bigcup_{n\in\omega} {}^n2$. Let ${}^*\mathbb{N}$ denote the set of all natural numbers and ${}^*\mathbb{Z}$ denote the set of all integers in ${}^*V$. For any real number r in ${}^*V$, let $[r]$ denote the greatest integer in ${}^*\mathbb{Z}$ which is less than or equal to r. For any set S, let $card(S)$ denote the set theoretic cardinality of S. If S is an internal set, then let $|S|$ denote the internal cardinality of S in ${}^*V$. For any $a, b \in {}^*\mathbb{Z}$ we write $[a, b]$ *exclusively* for the interval of integers, *i.e.* $[a, b] = \{x \in {}^*\mathbb{Z} : a \leqslant x \leqslant b\}$. Similarly for $[a, b)$, $(a, b]$ and (a, b). In the cases of $[a, b)$, $(a, b]$ or (a, b) we allow the numbers at the open-ends being non-integer real numbers. For example, $(-\frac{5}{2}, \frac{5}{2}) = \{-2, -1, 0, 1, 2\}$.

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Note that $[a, b] = \emptyset$ when $a > b$. Throughout this paper, let H be a fixed hyperfinite integer, *i.e.* $H \in {}^*\mathbb{N} \smallsetminus \mathbb{N}$, and let $\mathcal{H} = \{0, 1, \ldots, H-1\}$.

An infinite initial segment U of ${}^*\mathbb{N}$ is called a cut if U is closed under addition. For example, $\mathbb{N}$, the set of all standard natural numbers, is a cut. Let U be a cut. If $a \in {}^*\mathbb{N} \smallsetminus \{0\}$, then the set

$$a \cdot U = \{x \in {}^*\mathbb{N} : \exists u \in U \ (x < a \cdot u)\}$$

is a cut. If $a \in {}^*\mathbb{N} \smallsetminus U$, then the set

$$a/U = \{x \in {}^*\mathbb{N} : \forall u \in U \ (x < a/u)\}$$

is also a cut. For any $a \in {}^*\mathbb{N}$ the set

$$U\!-\!monad(a) = \{x \in {}^*\mathbb{N} : |x - a| \in U\}$$

is called the U-monad of a. Note that the set of all U-monads forms a partition of ${}^*\mathbb{N}$. Given a cut U, let

$$cf(U) = \min\{card(S) : S \subseteq U \text{ and } \forall u \in U \ \exists s \in S \ (u < s)\}.$$

A set $X \subseteq \mathcal{H}$ (or ${}^*\mathbb{N}$) is called countably determined iff there exists a sequence $\langle A_\sigma : \sigma \in {}^{<\omega}2\rangle$ of internal subsets of $\mathcal{H}$ (or ${}^*\mathbb{N}$), and there exists a set $I \subseteq {}^{\omega}2$ such that

$$X = \bigcup_{f \in I} \bigcap_{n \in \omega} A_{f \restriction n}.$$

Without loss of generality, we always assume that the sequence $\langle A_\sigma : \sigma \in {}^{<\omega}2\rangle$ is decreasing, *i.e.* $A_\sigma \subseteq A_\tau$ when $\tau \subseteq \sigma$. For any function f from ω to 2, let A_f denote the set $\bigcap_{n \in \omega} A_{f \restriction n}$.

Let's give definitions of some sparse subsets of ${}^*\mathbb{N}$. Suppose $U \subseteq \mathcal{H}$ is a cut. A set X (in $\mathcal{H}$ or in ${}^*\mathbb{N}$) is called a *U-choice set* iff X contains exactly one element in each U-monad (in $\mathcal{H}$ or in ${}^*\mathbb{N}$, respectively). A set $X \subseteq \mathcal{H}$ is called, according to [6], *U-discrete* iff for any $x \in X$ there is a $d \in \mathcal{H} \smallsetminus U$ such that $[x - d, x + d] \cap X = \{x\}$. A set $X \subseteq \mathcal{H}$ is called, according also to [6], *uniformly U-discrete* iff there is an $a \in \mathcal{H} \smallsetminus U$ such that for any $x, y \in X$ one has $x \neq y$ implies $|x - y| \geqslant a$. A U-discrete set is a subset of some U-choice set. Let $\oplus$ denote the addition on $\mathcal{H}$ modulo H, *i.e.* for any $a, b \in \mathcal{H}$, $a \oplus b = a + b$ if $a + b < H$ and $a \oplus b = a + b - H$ if $a + b \geqslant H$. For any two subsets X, Y of $\mathcal{H}$, let $X \oplus Y$ be the set $\{x \oplus y : x \in X, \ y \in Y\}$.

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1 Countably determined U-choice set

During the week-long International Symposium of Nonstandard Analysis and Its Applications held in Edinburgh, Scotland in August, 1996, Vladimir Kanovei raised the following question: Does there exist a countably determined $\mathbb{N}$-choice set $X \subseteq \mathcal{H}$? Before answering the question let's look at some background information and motivation of the question. Parallel to the classical descriptive set theory over the real line, one can develop a descriptive set theory over a hyperfinite set. Starting from the algebra of all internal subsets of $\mathcal{H}$, one can have a smallest σ-algebra, called "Borel" algebra, generated by those internal sets. With those Borel sets, one can generate all projective sets using projection operator. Beyond the algebra of all projective sets, there is a bigger algebra of countably determined sets. Countably determined sets are introduced in [3]. Note that all projective sets are countably determined. Note also that every subset of the real line is countably determined if one replaces internal subsets of $\mathcal{H}$ by open subsets of the real line in the definition. Hence the concept of countably determined sets over $\mathcal{H}$ has no interesting analogue in the classical setting. In order to show that countably determined sets are far from exhausting all subsets of $\mathcal{H}$ it is interesting to find some nice but non-countably determined subsets of $\mathcal{H}$.

First of all, Proposition 5.6 of [6] shows that there is no analytic U-choice set for any cut U. In fact, the proof shows that there is no Loeb measurable U-choice set for any cut U. I don't know if there exists a cut U such that there is a projective U-choice set of $\mathcal{H}$ with complexity higher than analytic or co-analytic. Hence, a U-choice set may not be simple in terms of descriptive complexity. Suppose X is an $H/\mathbb{N}$-choice set. By Corollary 2.6 of [5], one can show that X is not countably determined. In Proposition 5.4 of [6], above result was generalized to that if a cut U has an uncountable cofinality, then it is impossible to have a countably determined U-choice set in $\mathcal{H}$. The condition of $cf(U)$ being uncountable is used in an essential way in the proof so that it cannot easily be generalized to the case of $cf(U)$ being countable. To complete the picture, it is natural to explore the cases when $cf(U) = \omega$. Kanovei's question is merely the first testing case for $U = \mathbb{N}$.

In this section we are going to complete the picture. For convenience, we consider countably determined subsets of ${}^*\mathbb{N}$ instead of $\mathcal{H}$. It is easy to see that if X is a countably determined U-choice set of ${}^*\mathbb{N}$ and $U \subseteq \mathcal{H}$, then $X \cap \mathcal{H}$ or $(X \cap \mathcal{H}) \cup \{H-1\}$ is a countably determined U-choice set of $\mathcal{H}$. On the other hand, if X is an internal subset of $\mathcal{H}$i, then $\bigcup\{X + nH : n \in {}^*\mathbb{N}\}$ is also internal. Hence, if X is a countably determined U-choice set of $\mathcal{H}$ and $a \in X \cap (U - \text{monad}(H))$, then $\bigcup\{(X \smallsetminus \{a\}) + nH : n \in {}^*\mathbb{N}\}$ is a countably determined U-choice set of ${}^*\mathbb{N}$.

Theorem 1.1 *There exists a countably determined* $\mathbb{N}$*-choice set* $X \subseteq {}^*\mathbb{N}$.

Proof: For every $n \in \mathbb{N}$ and $\sigma \in {}^n2$ we define

$$A_\sigma = \{2^n \cdot K + \sum_{i<n} 2^i \cdot \sigma(i) : K \in {}^*\mathbb{N}\}.$$

Obviously, $\langle A_\sigma : \sigma \in {}^{<\omega}2\rangle$ is a decreasing sequence of internal subsets of ${}^*\mathbb{N}$. Let $I \subseteq {}^\omega 2$ be maximal such that for any $f, g \in I$

(1) f is not eventually 1, *i.e.* there are infinitely many $n \in \omega$ such that $f(n) = 0$,

(2) if $f \neq g$, then f and g are not eventually same, *i.e.* there are infinitely many $n \in \omega$ such that $f(n) \neq g(n)$.

Now our desired countably determined set is

$$X = \bigcup_{f \in I} \bigcap_{n \in \omega} A_{f \restriction n}.$$

Claim 1.1.1 For any $f \in I$ and for any $x, y \in A_f$, if $x \neq y$, then $|x - y| \notin \mathbb{N}$.

Proof of Claim 1.1.1: Let x, y be two different elements in A_f. It suffices to show that for any $m \in \mathbb{N}$, one has $|x - y| > m$. Since $x, y \in A_{f \restriction m}$ and $x \neq y$, there are different $K_1, K_2 \in {}^*\mathbb{N}$ such that

$$x = 2^m \cdot K_1 + \sum_{i<m} 2^i \cdot f(i) \text{ and}$$

$$y = 2^m \cdot K_2 + \sum_{i<m} 2^i \cdot f(i).$$

Hence $|x - y| = 2^m \cdot |K_1 - K_2| \geqslant 2^m > m$. $\square$(Claim 1.1.1)

Claim 1.1.2 Given any $f, g \in I$ with $f \neq g$, if $x \in A_f$ and $y \in A_g$, then $|x - y| \notin \mathbb{N}$.

Proof of Claim 1.1.2: Without loss of generality, we assume that $x \geqslant y$. By the overspill principle, there is a hyperfinite integer K such that $x \in A_{^*f \restriction K}$ and $y \in A_{^*g \restriction K}$, where $^*f$ and $^*g$ are nonstandard versions of f and g. Let $K_1, K_2 \in {}^*\mathbb{N}$ be such that

$$x = 2^K \cdot K_1 + \sum_{i<K} 2^i \cdot {}^*f(i) \text{ and}$$

$$y = 2^K \cdot K_2 + \sum_{i<K} 2^i \cdot {}^*g(i).$$

It is easy to see that $K_1 \geqslant K_2$. Without loss of generality, we assume that $K_1 \neq K_2$ (otherwise we pick the largest $i_0 < K$ such that ${}^*f(i_0) \neq {}^*g(i_0)$, and then replace K by i_0 and replace K_1, K_2 by ${}^*f(i_0)$ and ${}^*g(i_0)$. Note that i_0 is not standard because f and g are not eventually same). Since g is not eventually 1, there is a hyperfinite integer L with $L < K$ such that ${}^*g(L) = 0$. Hence

$$x - y = 2^K \cdot (K_1 - K_2) + \sum_{i<K} 2^i \cdot ({}^*f(i) - {}^*g(i))$$

$$\geqslant 2^K - (\sum_{i=0}^{L-1} 2^i + \sum_{i=L+1}^{K-1} 2^i) \geqslant 2^L.$$

This shows that $|x - y| \notin \mathbb{N}$. $\square$(Claim 1.1.2)

Combining above two claims, we conclude that any two different elements in X are in different $\mathbb{N}$-monads.

Claim 1.1.3 For any $x \in {}^*\mathbb{N}$ there is an $f \in I$ and there is a $y \in A_f$ such that $|x - y| \in \mathbb{N}$.

Proof of Claim 1.1.3: By the transfer property, one can find a $K \in {}^*\mathbb{N}$ and an internal function $F \in {}^K 2$ such that

$$x = \sum_{i<K} 2^i \cdot F(i).$$

For convenience, let $F(K) = 0$. Let $g = F \restriction \mathbb{N}$. Without loss of generality, we can assume that g is not eventually 1. The reason is the following. Suppose g is eventually 1. Then there is $a \in \mathbb{N}$ and $b \in {}^*\mathbb{N} \setminus \mathbb{N}$ with $b < K$ such that $F \restriction [a, b] \equiv 1$ and $F(b+1) = 0$. So one has

$$x = \sum_{i<K} 2^i \cdot F(i) = \sum_{i=0}^{a-1} 2^i \cdot F(i) + \sum_{i=a}^{b} 2^i + \sum_{i=b+2}^{K} 2^i \cdot F(i)$$

$$= \sum_{i=0}^{a-1} 2^i \cdot F(i) + 2^{b+1} - 2^a + \sum_{i=b+2}^{K} 2^i \cdot F(i).$$

Define an internal function G such that $G \restriction [0, b] \equiv 0$, $G(b+1) = 1$ and

$$G \restriction [b+2, K] \equiv F \restriction [b+2, K].$$

Let

$$x' = \sum_{i \leqslant K} 2^i \cdot G(i).$$

It is easy to see that

$$|x' - x| \leqslant \sum_{i=0}^{a-1} 2^i \cdot F(i) + 2^a \in \mathbb{N}.$$

Now replace x by x' and replace g by $G \restriction \mathbb{N}$.

Since I is maximal, then there exists an $f \in I$ such that f and g are eventually same. Let $m \in \mathbb{N}$ be such that $f(n) = g(n)$ for all $n > m$. By the overspill principle, one can find a hyperfinite integer $L < K$ such that

$${}^*f \restriction [m, L] = {}^*g \restriction [m, L] = F \restriction [m, L].$$

Let's define an internal function $F' \in {}^{K+1}2$ such that

$$F' \restriction [0, L] = {}^*f \restriction [0, L] \text{ and } F' \restriction [L+1, K] = F \restriction [L+1, K].$$

Let

$$y = \sum_{i<K} 2^i \cdot F'(i).$$

Then $y \in A_f$ and $|x - y| \leqslant \sum_{i=0}^{m-1} 2^i$. Hence one has $|x - y| \in \mathbb{N}$. □(Claim 1.1.3)

Claim 1.1.3 implies that X meets every $\mathbb{N}$-monad. Hence the theorem follows from above three claims. □

Corollary 1.2 *If $a \in {}^*\mathbb{N} \setminus \{0\}$ and $U = a \cdot \mathbb{N}$, then there is a countably determined U-choice set $Y \subseteq {}^*\mathbb{N}$.*

Proof: Let X be the countably determined set constructed in Theorem 1.1 and define

$$Y = a{\cdot}X = \{a{\cdot}x : x \in X\}.$$

Obviously, Y is countably determined. It is easy to see that any U-monad contains at most one element in Y. Given $b \in {}^*\mathbb{N}$, let $c = [b/a]$. Then there is an $x \in X$ such that $|x - c| \in \mathbb{N}$. So $a{\cdot}x \in Y$ and

$$|a{\cdot}x - b| \leqslant |a{\cdot}x - a{\cdot}c| + |a{\cdot}c - b|$$

$$\leqslant a{\cdot}|x - c| + a = a{\cdot}(|x - c| + 1) \in a{\cdot}\mathbb{N} = U.$$

Hence Y contains exactly one element in each U-monad. $\square$

Now the only case left is when U has countable cofinality but $U \neq a{\cdot}\mathbb{N}$ for any $a \in {}^*\mathbb{N} \smallsetminus \{0\}$.

Theorem 1.3 *Let U be a cut such that $cf(U) = \omega$ and $U \neq a\mathbb{N}$ for any $a \in {}^*\mathbb{N} \smallsetminus \{0\}$. Then there does not exist a countably determined U-choice set $X \subseteq {}^*\mathbb{N}$.*

Proof: The idea of the proof is from Proposition 1.5 of [8]. Since $cf(U)$ is countable and $U \neq a{\cdot}\mathbb{N}$ for any $a \in {}^*\mathbb{N} \smallsetminus \{0\}$, one can find an increasing sequence $\langle a_n : n \in \omega\rangle$ cofinal in U such that for all $n \in \omega$, $a_{n+1} \notin a_n\mathbb{N}$. Suppose the theorem is not true. Let X be a countably determined U-choice set in ${}^*\mathbb{N}$. Let $I \subseteq {}^\omega 2$ and let $\langle A_\sigma : \sigma \in {}^{<\omega}2\rangle$ be a decreasing sequence of internal subsets of ${}^*\mathbb{N}$ such that

$$X = \bigcup_{f \in I} \bigcap_{n \in \omega} A_{f \upharpoonright n}.$$

For each $m \in \mathbb{N}$ and $Y \subseteq {}^*\mathbb{N}$, let

$$Y + [-a_m, a_m] = \{x \in {}^*\mathbb{N} : \exists y \in Y\ (y - a_m \leqslant x \leqslant y + a_m)\}.$$

Then $X_m = X + [-a_m, a_m]$ is countably determined and ${}^*\mathbb{N} = \bigcup_{m \in \omega} X_m$. In fact

$$X_m = \bigcup_{f \in I} \bigcap_{n \in \omega} (A_{f \upharpoonright n} + [-a_m, a_m]).$$

Claim 1.3.1 For any $f \in I$ and any $m \in \omega$, there exists a $\sigma_{f,m} \in {}^{<\omega}2$, $\sigma_{f,m} \subseteq f$ such that if a and b are two different elements of $A_{\sigma_{f,m}}$, then $|a - b| \geqslant a_{m+1}$.

Proof of Claim 1.3.1: Apply the overspill principle. $\square$(Claim 1.3.1)

For each $m \in \omega$ let $\Sigma_m = \{\sigma_{f,m} : f \in I\}$. Note that Σ_m is at most countable. For any $m \in \omega$ one has

$$X \subseteq \bigcup_{\sigma \in \Sigma_m} A_\sigma.$$

Hence

$$X_m \subseteq \bigcup_{\sigma \in \Sigma_m} (A_\sigma + [-a_m, a_m]).$$

So one has

$${}^*\mathbb{N} = \bigcup_{m \in \omega} \bigcup_{\sigma \in \Sigma_m} (A_\sigma + [-a_m, a_m]).$$

By ω_1-saturation there exists an $M \in \omega$ and a finite $\Sigma_m^0 \subseteq \Sigma_m$ for each $m < M$ such that

$${}^*\mathbb{N} = \bigcup_{m < M} \bigcup_{\sigma \in \Sigma_m^0} (A_\sigma + [-a_m, a_m]).$$

But by induction one can find a sequence of intervals

$$[c_{M-1}, d_{M-1}] \supseteq [c_{M-2}, d_{M-2}] \supseteq \ldots \supseteq [c_1, d_1] \supseteq [c_0, d_0]$$

such that $d_m - c_m \notin a_m \cdot \mathbb{N}$ and

$$[c_m, d_m] \cap (\bigcup_{\sigma \in \Sigma_m^0} (A_\sigma + [-a_m, a_m]) = \emptyset$$

for each $m < M$, which contradicts that $[c_0, d_0] \neq \emptyset$. $\square$

Remarks: (1) Combining above results and Proposition 5.4 of [6], we can conclude that for a given cut U, there exists a countably determined U-choice set X if and only if $U = a \cdot \mathbb{N}$ for some $a \in {}^*\mathbb{N} \setminus \{0\}$.

(2) Let X be the countably determined set constructed in Theorem 1.1 and let $X' = X \cap \mathcal{H}$. Let L_μ be the Loeb measure generated by a normalized counting measure on $\mathcal{H}$. Then X' is not Loeb measurable. This is because $(X' + n) \cap (X' + m) = \emptyset$ for $n, m \in \mathbb{Z}, n \neq m$ and $\bigcup_{n \in \mathbb{Z}} (X' + n)$ has Loeb measure one.

(3) Using X' in (2), one can construct a countably determined bijection from $\mathcal{H}$ to $\{0, 1, \ldots, 2(H-1)\}$. Without loss of generality, let $0 \in X'$ and $H - 1 \in X'$. For each $n \in \mathbb{Z}$ let F_{2n} be a function from $X' + 2n$ such that for each $a \in X'$, $F_{2n}(a + 2n) = a + n$ and let F_{2n+1} be a function from $X' + (2n+1)$ such that for each $a \in X'$ $F_{2n+1}(a + 2n + 1) = H + a + n$. Obviously, each F_n is countably

determined. Hence $F = \bigcup_{n\in\mathbb{Z}} F_n$ is countably determined. It is easy to see that F is the desired bijection. Using the same argument, it is not hard to construct a countably determined bijection from $\mathcal{H}$ to $\{0, 1, \ldots, m(H-1)\}$ for a given $m \in \mathbb{N}$. This offers an alternative construction of a similar function in [5]. See also [4] for motivation.

2 U-discrete set X with $X \oplus X = \mathcal{H}$

Problem 9.14 of [6] stated that: For which cut $U \subseteq \mathcal{H}$ does there exists a U-discrete set $X \subseteq \mathcal{H}$ such that $X \oplus X = \mathcal{H}$? The authors of [6] pointed out there that there is such an X when $U = \mathbb{N}$ and there is no such an X when $X = H/\mathbb{N}$. In this section, we answer the question for all cuts U except one. The methods of the proofs here are basically cardinality arguments.

Throughout this section, let's fix the following notation. Let $\lambda = card(\mathcal{H})$, let

$$W_\lambda = \{x \in \mathcal{H} : card([0, x]) < \lambda\}$$

and let $V = H/W_\lambda$. Given $a, b \in \mathcal{H}$ such that b is infinitely greater than a, *i.e.* $b \notin a{\cdot}\mathbb{N}$, and given any interval $I = [x, y-1] \subseteq \mathcal{H}$ such that $y - x = b$, let's fix a collection of intervals

$$\mathcal{I}(I, a) = \{[x + l{\cdot}a, x + (l+1){\cdot}a) : l = 0, 1, \ldots, [b/a] - 1\}.$$

It is easy to see that $card(\mathcal{I}(I, a)) = card([0, [b/a]))$.

Lemma 2.1 *The set W_λ is a cut. Hence V is also a cut.*

Proof: Obviously, $\mathbb{N} \subseteq W_\lambda$. For any $a, b \in W_\lambda \smallsetminus \{0, 1\}$, one has

$$card([0, a+b)) = card([0, a)) + card([0, b))$$

$$\leqslant card([0, a))card([0, b)) \leqslant \max\{card([0, a)), card([0, b)), \omega\} < \lambda.$$

Hence $a + b, ab \in W_\lambda$. $\square$

Remark: Above proof shows that W_λ is a cut closed under multiplication. So W_λ can never be $H/\mathbb{N}$. This means $\mathbb{N} \subsetneqq V$ is always true.

Lemma 2.2 *Let $a \in \mathcal{H}$ and $L = [(H-1)/a]$. Then either $L \in \mathbb{N}$ or $card([0,L]) = card([0,H/a))$. Hence $a \in V$ iff $L \notin W_\lambda$.*

Proof: Assume $L \notin \mathbb{N}$. Since $L - 1 \leqslant [H/a] \leqslant L+1$, then

$$card([0,L]) = card([0,L-1])$$

$$\leqslant card([0,H/a)) \leqslant card([0,L+1]) = card([0,L]). \quad \square$$

Theorem 2.3 *Suppose $U \subsetneqq V$. Then there exists a U-discrete set $X \subseteq \mathcal{H}$ such that $X \oplus X = \mathcal{H}$.*

Proof: Let's fix an enumeration $\{a_\alpha : \alpha < \lambda\}$ of $\mathcal{H}$. Since U is a proper subset of V, we can find an $a \in V \smallsetminus U$. Let

$$L = \max\{x \in \mathcal{H} : a \cdot x \leqslant H\}.$$

By Lemma 2.2, one has $card([0, L/m)) = \lambda$ for any $m \in \mathbb{N}$. We construct a set

$$X = \{x_\alpha : \alpha < \lambda\} \cup \{y_\alpha : \alpha < \lambda\},$$

and a collection of intervals

$$\{I_\alpha : I_\alpha = [x_\alpha - a, x_\alpha + a),\ \alpha < \lambda\} \cup \{J_\alpha : J_\alpha = [y_\alpha - a, y_\alpha + a),\ \alpha < \lambda\}$$

inductively on $\alpha < \lambda$ such that for each $\alpha < \lambda$,

(1) $x_\alpha \oplus y_\alpha = a_\alpha$,

(2) $x_\alpha \notin (\bigcup_{\beta<\alpha} I_\beta) \cup (\bigcup_{\beta<\alpha} J_\beta)$ and $y_\alpha \notin (\bigcup_{\beta\leqslant\alpha} I_\beta) \cup (\bigcup_{\beta<\alpha} J_\beta)$.

The theorem follows from the construction because by (2), any two different elements in X are at least a apart, and by (1), one has $X \oplus X = \mathcal{H}$.

Suppose $\{x_\beta : \beta < \alpha\} \cup \{y_\beta : \beta < \alpha\}$ and $\{I_\beta : \beta < \alpha\} \cup \{J_\beta : \beta < \alpha\}$ have been constructed so that (1) and (2) are satisfied for every $\beta < \alpha$.

Case 1: $a_\alpha \geqslant H/2$. Let

$$\mathcal{I} = \{[l \cdot a, (l+1) \cdot a) : \frac{a_\alpha}{2a} + 1 < l < \frac{a_\alpha}{a} - 1\}.$$

Then $\mathcal{I}$ is a collection of disjoint intervals and $card(\mathcal{I}) = \lambda$. Besides, each interval in $\mathcal{I}$ has length a. For each $\beta < \alpha$ there are at most twelve intervals $T_i^\beta \in \mathcal{I}$ for $i < 12$ such that for each $x \in (a, \frac{a_\alpha}{2} - a)$

$$x \in I_\beta \cup J_\beta \text{ implies } a_\alpha - x \in \bigcup_{i<6} T_i^\beta,$$

and

$$(I_\beta \cup J_\beta) \cap (\frac{a_\alpha}{2} + a, a_\alpha - a) \subseteq \bigcup_{6 \leqslant i < 12} T_i^\beta.$$

Since $card(\mathcal{I}) = \lambda$ there is an $I \in \mathcal{I} \smallsetminus \{T_i^\beta : \beta < \alpha,\ i < 12\}$. Choose any $y_\alpha \in I$ and let $x_\alpha = a_\alpha - y_\alpha$. Form intervals I_α and J_α accordingly. It is easy to see that the conditions (1) and (2) are satisfied.

Case 2: $a_\alpha < H/2$.

The proof is similar to Case 1. Just replace a_α by $H + a_\alpha$ and find y_α in the interval $(\frac{H+a_\alpha}{2} + a, H-1]$. □

Remarks: (1) Since $\mathbb{N} \subsetneqq V$, then there exists always an $\mathbb{N}$-discrete set $X \subseteq \mathcal{H}$ such that $X \oplus X = \mathcal{H}$.

(2) The set X constructed above is uniformly U-discrete.

Theorem 2.4 *Given any cut $U \subseteq \mathcal{H}$ such that $U \supsetneqq V$, then there does not exist a U-discrete set $X \subseteq \mathcal{H}$ such that $X \oplus X = \mathcal{H}$.*

Proof: The proof is a simple cardinality argument. Let $a \in U \smallsetminus V$. Suppose such X exists. Then any two different elements in X have distance greater than a because $a \in U$. Hence X contains at most one element in each interval $[l{\cdot}a, (l+1){\cdot}a)$, where $l \leqslant L = \max\{k \in \mathcal{H} : a{\cdot}k < H\}$. By Lemma 2.2, one has $card(X) \leqslant card([0, L+1]) < \lambda$. So

$$\lambda = card(\mathcal{H}) = card(X \oplus X) \leqslant (card(X))^2 = card(X) < \lambda,$$

a contradiction. □

Remarks: The proof actually shows that there does not exist a U-choice set $X \subseteq \mathcal{H}$ such that $X \oplus X = \mathcal{H}$.

Theorem 2.5 *Assume $U = V$. If $cf(W_\lambda) = cf(\lambda) < \lambda$,*

$$\sup\{card([0,a]) : a \in W_\lambda\} = \lambda,$$

and the nonstandard universe is $cf(\lambda)$-saturated, then there is a U-discrete set $X \subseteq \mathcal{H}$ such that $X \oplus X = \mathcal{H}$.

Proof: Let $\eta = cf(\lambda)$ and let $\langle \kappa_\alpha : \alpha < \eta \rangle$ be a strictly increasing sequence of infinite regular cardinals cofinal in λ. By the fact that $\sup\{card([0,a]) : a \in W_\lambda\} = \lambda$, one can find an increasing sequence $\langle a_\alpha \in W_\lambda : \alpha < \eta \rangle$ cofinal in W_λ such that for each $\alpha < \lambda$, one has

$$\kappa_\alpha < card([0,a_\alpha]) < card([0,a_{\alpha+1}]).$$

Since $card(\mathcal{H}) = \lambda$ one can enumerate $\mathcal{H}$ such that $\mathcal{H}_\alpha = \{b^\alpha_\beta : \beta < \kappa_\alpha\}$ and $\mathcal{H} = \bigcup_{\alpha<\eta} X_\alpha$. We want to construct a set

$$X_\alpha = \{x^\alpha_\beta : \beta < \kappa\} \cup \{y^\alpha_\beta : \beta < \kappa\}$$

and a collection of intervals

$$\mathcal{I}_\alpha = \{I^\alpha_\beta : I^\alpha_\beta = [x^\alpha_\beta - [H/a_{\alpha+1}], x^\alpha_\beta + [H/a_{\alpha+1}]],\ \beta < \kappa\} \cup$$

$$\{J^\alpha_\beta : J^\alpha_\beta = [y^\alpha_\beta - [H/a_{\alpha+1}], y^\alpha_\beta + [H/a_{\alpha+1}]],\ \beta < \kappa\}$$

for each $\alpha < \eta$. We do that by induction on both α and β such that for each $\alpha < \eta$ and $\beta < \kappa_\alpha$

(1) $x^\alpha_\beta \oplus y^\alpha_\beta = b^\alpha_\beta$,

(2) $x^\alpha_\beta \notin (\bigcup_{\gamma<\alpha} \bigcup \mathcal{I}_\gamma) \cup (\bigcup_{\delta<\beta}(I^\alpha_\delta \cup J^\alpha_\delta))$ and $y^\alpha_\beta \notin (\bigcup_{\gamma<\alpha} \bigcup \mathcal{I}_\gamma) \cup (\bigcup_{\delta<\beta}(I^\alpha_\delta \cup J^\alpha_\delta)) \cup I^\alpha_\beta$.

The theorem follows from the construction. Let $X = \bigcup_{\alpha<\eta} X_\alpha$. By (1) one has $X \oplus X = \mathcal{H}$. By (2) X is U-discrete because for each $x \in X_\alpha$ there is an interval I, where $I = I^\alpha_\beta$ if $x = x^\alpha_\beta$ and $I = J^\alpha_\beta$ if $x = y^\alpha_\beta$, such that $|I| = 2[H/a_{\alpha+1}] + 1 \notin U$, x is the middle point of I and $I \cap X = \{x\}$.

Suppose for some $\alpha < \eta$ and $\beta < \kappa_\alpha$ the set

$$(\bigcup_{\gamma<\alpha} X_\gamma) \cup \{x^\alpha_\delta : \delta < \beta\} \cup \{y^\alpha_\delta : \delta < \beta\}$$

and the collection of intervals

$$(\bigcup_{\gamma<\alpha} \mathcal{I}_\gamma) \cup \{I^\alpha_\delta : \delta < \beta\} \cup \{J^\alpha_\delta : \delta < \beta\}$$

have been constructed. We need to find $x^\alpha_\beta, y^\alpha_\beta, I^\alpha_\beta$ and J^α_β.

Case 1: $b^\alpha_\beta \geqslant H/2$.

We first construct two sequences of intervals $\langle T^i_\gamma : \gamma \leqslant \alpha \rangle$ for $i = 0, 1$ such that

(a) $T_0^0 \cap T_0^1 = \emptyset$,

(b) $T_\gamma^i \subseteq T_\delta^i$ whenever $\delta < \gamma \leqslant \alpha$,

(c) $|T_\gamma^i| = 2[H/a_{\gamma+1}] + 1$ for each $\gamma \leqslant \alpha$,

(d) $\forall x < b_\beta^\alpha$ ($x \in T_\gamma^0$ iff $b_\beta^\alpha - x \in T_\gamma^1$) for each $\gamma \leqslant \alpha$,

(e) $(T_\gamma^0 \cup T_\gamma^1) \cap \bigcup \mathcal{I}_\gamma = \emptyset$ for each $\gamma < \alpha$,

(f) $(T_\alpha^0 \cup T_\alpha^1) \cap \bigcup_{\delta<\beta}(I_\delta^\alpha \cup J_\delta^\alpha) = \emptyset$.

Suppose T_δ^i is obtained for each $\delta < \gamma$ and $i = 0, 1$ (note that γ could be 0). If γ is a successor of γ', let $S_\gamma^i = T_{\gamma'}^i$. If γ is a limit ordinal, let S_γ^i be defined as the following. Since the nonstandard universe is η-saturated, the set $\bigcap_{\gamma'<\gamma} T_{\gamma'}^i$ is not empty. Moreover, since $a_\gamma > a_{\gamma'}$ for every $\gamma' < \gamma$, there exists an interval $S_\gamma^0 \subseteq \bigcap_{\gamma'<\gamma} T_{\gamma'}^0$ such that

$$|S_\gamma^0| = 2[H/a_\gamma] + 1.$$

Let $S_\gamma^1 = \{b_\beta^\alpha - x : x \in S_\gamma^0\}$. Then by (d), one has

$$S_\gamma^1 \subseteq \bigcap_{\gamma'<\gamma} T_{\gamma'}^1 \text{ and}$$

$$x \in S_\gamma^0 \text{ iff } b_\beta^\alpha - x \in S_\gamma^1.$$

Suppose we have had S_γ^i as above for γ being either a successor or a limit ordinal. Let $\mathcal{I}_0 = \mathcal{I}(S_\gamma^0, [H/a_{\gamma+1}])$ be the collection of subintervals of S_γ^0 defined before Lemma 2.1, and let

$$\mathcal{I}_1 = \{J_I : I \in \mathcal{I}_0,\ J_I = \{b_\beta^\alpha - x : x \in I\}\}.$$

Then $\mathcal{I}_i$ is a collection of disjoint subintervals of S_γ^i for $i = 0, 1$, and

$$card(\mathcal{I}_0) = card(\mathcal{I}_1) \geqslant card([0, a_{\gamma+1}/a_\gamma)).$$

Since

$$card([0, a_{\gamma+1})) = card([0, a_\gamma))card([0, a_{\gamma+1}/a_\gamma))$$

and

$$card([0, a_{\gamma+1})) > card([0, a_\gamma)),$$

one has that

$$card([0, a_{\gamma+1})) = card([0, a_{\gamma+1}/a_\gamma)).$$

Hence

$$card(\mathcal{I}_i) = card([0, a_{\gamma+1})) > \kappa_\gamma.$$

By a cardinality argument similar to the one used in the proof of Theorem 2.3 and by the facts that if $\gamma < \alpha$, then

$$card(\{I^\gamma_\delta : \delta < \kappa_\gamma\} \cup \{J^\gamma_\delta : \delta < \kappa_\gamma\}) < card(\mathcal{I}_i),$$

and if $\gamma = \alpha$, then

$$card(\{I^\gamma_\delta : \delta < \beta\} \cup \{J^\gamma_\delta : \delta < \beta\}) < card(\mathcal{I}_i),$$

one can find $T^0_\gamma \in \mathcal{I}_0$ and $T^1_\gamma \in \mathcal{I}_1$ such that the sequence $\langle T^i_\delta : \delta \leqslant \gamma\rangle$ satisfies the conditions (a)–(f). This ends the construction of Case 1.

Now let $I^\alpha_\beta = T^0_\alpha$ and $J^\alpha_\beta = T^1_\alpha$. Let x^α_β be the middle point of I^α_β and let y^α_β be the middle point of J^α_β.

Case 2: $b^\alpha_\beta < H/2$. The proof is similar to the proof of Case 1. Just replace b^α_β by $H + b^\alpha_\beta$, and require $T^i_0 \subseteq \mathcal{H}$.

This ends the construction of X_α and $\mathcal{I}_\alpha$. It is easy to check that the sequences constructed satisfy the requirements (1) and (2). $\square$

Remark: We don't know if the theorem is still true without assuming $cf(\lambda)$-saturation. In the theorem, λ is a singular cardinal. If we assume λ is a successor cardinal, then $\sup\{card([0,a]) : a \in W_\lambda\} < \lambda$. If we assume λ is an inaccessible cardinal, then the proof still works as long as the nonstandard universe is $cf(\lambda)$-saturated. But since $cf(\lambda) = \lambda$, then the case become trivial because one will have $W_\lambda = \mathbb{N}$ and $V = H/\mathbb{N}$.

Theorem 2.6 *Assume $U = V$. If (1) $cf(W_\lambda) < cf(\lambda)$, or (2) $cf(W_\lambda) = cf(\lambda) < \lambda$ and there is a $\kappa < \lambda$ such that $\sup\{card([0,a]) : a \in W_\lambda\} \leqslant \kappa$, then there does not exist any U-discrete set $X \subseteq \mathcal{H}$ such that $X \oplus X = \mathcal{H}$.*

Proof: Suppose the theorem is not true. Let X be such a set. Since X is U-discrete, then for each $x \in X$ there is a $b_x \in \mathcal{H} \smallsetminus U$ such that $[x - b_x, x + b_x] \cap X = \{x\}$. Let $\eta = cf(W_\lambda)$ and let $\langle a_\alpha : \alpha < \eta\rangle$ be an increasing sequence cofinal in W_λ. For each $\alpha < \eta$ let

$$X_\alpha = \{x \in X : b_x \geqslant H/a_\alpha\}.$$

It is easy to see that $X = \bigcup_{\alpha<\eta} X_\alpha$. For each $\alpha < \eta$, one has in (1) that $card(X_\alpha) < \lambda$ by Lemma 2.2, and in (2) that $card(X_\alpha) \leqslant \kappa$ again by Lemma 2.2. Hence in both (1) and (2) one has

$$\lambda = card(\mathcal{H}) = card(X \oplus X) = (card(X))^2$$

$$= card(X) = card(\bigcup_{\alpha<\eta} X_\alpha) = \sup_{\alpha<\eta}(card(X_\alpha)) < \lambda,$$

which is a contradiction. $\square$

Remarks: (1) If $W_\lambda \neq \mathbb{N}$, then $V \subsetneqq H/\mathbb{N}$. If $W_\lambda = \mathbb{N}$, then either $cf(W_\lambda) < cf(\lambda)$ or $\sup\{card([0,a]) : a \in W_\lambda\} = \omega < \lambda$. By Theorem 2.4 and Theorem 2.6 there does not exist an $X \subseteq \mathcal{H}$ such that X is $H/\mathbb{N}$-discrete and $X \oplus X = \mathcal{H}$. Hence our results so far cover both cases of $U = \mathbb{N}$ and $U = H/\mathbb{N}$. By the way, the direct proof of the fact that every $H/\mathbb{N}$-discrete subset of $\mathcal{H}$ is at most countable, is not hard at all.

(2) We don't know the complete answer for the case $U = V$.

When $\lambda = 2^\omega$ we have the following two results for U-choice sets. The first result gives a "most sparse" set $X \subseteq \mathcal{H}$ with $X \oplus X = \mathcal{H}$ while the second gives a "least sparse" but countably determined set $X \subseteq \mathcal{H}$ with $X \oplus X = \mathcal{H}$.

Theorem 2.7 *Suppose $card(\mathcal{H}) = 2^\omega$. Then (1) there exists a $H/\mathbb{N}$-choice set $X \subseteq \mathcal{H}$ such that $X \oplus X = \mathcal{H}$, (2) there exists a countably determined $\mathbb{N}$-choice set $X \subseteq \mathcal{H}$ such that $X \oplus X = \mathcal{H}$.*

Proof: The proof of (1) is almost identical to the proof of Theorem 2.3. Just replace $\mathcal{I}$ by the collection of all $H/\mathbb{N}$-monads. Note that there are exactly 2^ω $H/\mathbb{N}$-monads in $\mathcal{H}$.

Let's prove (2). The idea of the proof is also similar to the proof of Theorem 2.3. This time we use the sets $A_f \cap \mathcal{H}$ instead of intervals or monads. Let A_σ be the set constructed in Theorem 1.1. For convenience, we write A_σ here for the set $A_\sigma \cap \mathcal{H}$. Let $\mathcal{H} = \{a_\alpha : \alpha < 2^\omega\}$ be an enumeration. We construct an increasing sequence of subsets of ${}^\omega 2$, say $\langle Z_\alpha : \alpha < 2^\omega \rangle$, such that for every $\alpha < 2^\omega$

(1) $card(Z_\alpha) = card(\alpha) < 2^\omega$,

(2) none of functions in Z_α is eventually 1,

(3) for any $f, g \in Z_\alpha$, $f \neq g$ implies f and g are not eventually same,

(4) for every $\beta < \alpha$ there are $f, g \in Z_\alpha$ such that there exist $x \in A_f \cap \mathcal{H}$ and $y \in A_g \cap \mathcal{H}$ with $x \oplus y = a_\beta$,

Claim 2.7.1 The theorem follows from the construction.

Proof of Claim 2.7.1: Let $Z \supseteq \bigcup_{\alpha<2^\omega} Z_\alpha$ be maximal such that (2) and (3) hold for Z replacing Z_α, and let

$$X = \bigcup_{f \in Z} \bigcap_{n \in \omega} A_{f \restriction n}.$$

Obviously, X is countably determined. By (4), one has $X \oplus X = \mathcal{H}$, and by (2), (3) and the proof of Theorem 1.1, one has that X in an $\mathbb{N}$-choice set. $\Box$(Claim 2.7.1)

First let's define two equivalence relations on $\mathcal{H}$. For any $x, y \in \mathcal{H}$, let $x \sim_0 y$ iff there exists an $f \in {}^\omega 2$ with $x, y \in A_f$, and let $x \sim_1 y$ iff there exist two functions $f, g \in {}^\omega 2$, which are eventually same, such that $x \in A_f$ and $y \in A_g$.

Claim 2.7.2 (1) For any $x, y \in \mathcal{H}$, $x \sim_0 y$ iff there exists a hyperfinite integer K such that 2^K divides $|x - y|$.

(2) Let $x \in A_f$, where $f \in {}^\omega 2$ is not eventually constant. For any $y \in \mathcal{H}$, one has $x \sim_1 y$ iff there exists a standard integer k and a hyperfinite integer K such that 2^K divides $|x - y| + k$.

Proof of Claim 2.7.2: (1) "$\Rightarrow$": Let $f \in {}^\omega 2$ be such that $x, y \in A_f$. Then there exist $K_1, K_2 \in {}^*\mathbb{N}$ and $K \in {}^*\mathbb{N} \smallsetminus \mathbb{N}$ such that

$$x = 2^K \cdot K_1 + \sum_{i<K} 2^i \cdot {}^*f(i) \text{ and}$$

$$y = 2^K \cdot K_2 + \sum_{i<K} 2^i \cdot {}^*f(i).$$

Hence $|x - y| = 2^K \cdot |K_1 - K_2|$. This shows that 2^K divides $|x - y|$.

"$\Leftarrow$": Suppose $x \leqslant y$ and 2^K divides $y - x$ for some hyperfinite integer K. Let $f \in {}^\omega 2$, $k \in {}^*\mathbb{N}$ and $L \in {}^*\mathbb{N} \smallsetminus \mathbb{N}$ be such that $L < K$ and

$$x = 2^L \cdot k + \sum_{i<L} 2^i \cdot {}^*f(i).$$

Then there exists a $k' \in {}^*\mathbb{N}$ such that

$$y = 2^K \cdot k' + x = 2^L \cdot (k + 2^{K-L} \cdot k') + \sum_{i<L} 2^i \cdot {}^*f(i).$$

So $x, y \in A_f$.

(2) "$\Rightarrow$": Let $g \in {}^\omega 2$ be eventually same as f such that $y \in A_g$. Suppose $m \in \mathbb{N}$ such that $f \upharpoonright \mathbb{N} \smallsetminus [0, m) = g \upharpoonright \mathbb{N} \smallsetminus [0, m)$. Then there exist $K_1, K_2 \in {}^*\mathbb{N}$ and $K \in {}^*\mathbb{N} \smallsetminus \mathbb{N}$ such that

$$x = 2^K \cdot K_1 + \sum_{i<K} 2^i \cdot {}^*f(i) \text{ and}$$

$$y = 2^K \cdot K_2 + \sum_{i<K} 2^i \cdot {}^*g(i).$$

Hence

$$2^K \cdot |K_1 - K_2| - \sum_{i<m} 2^i \leqslant |x - y| \leqslant 2^K \cdot |K_1 - K_2| + \sum_{i<m} 2^i.$$

This shows that there is a standard integer k such that 2^K divides $|x - y| + k$.

"$\Leftarrow$" Without loss of generality let's assume $x \leqslant y$ and 2^K divides $y - x + k$ for some hyperfinite integer K and standard integer k. Let $k' \in {}^*\mathbb{N}$ and $L \in {}^*\mathbb{N} \smallsetminus \mathbb{N}$ be such that $L < K$ and

$$x = 2^L \cdot k' + \sum_{i<L} 2^i \cdot {}^*f(i).$$

Then there exists a $k'' \in {}^*\mathbb{N}$ such that

$$y = 2^K \cdot k'' + x - k = 2^L \cdot (k' + 2^{K-L} \cdot k'') + \sum_{i<L} 2^i \cdot {}^*f(i) - k.$$

Let $g \in {}^\omega 2$ be such that $y \in A_g$. By the facts that k is standard and f is not eventually constant, one has that g and f are eventually same. $\square$(Claim 2.7.2)

We now construct those Z_α's by induction on $\alpha < 2^\omega$. Let

$$\mathcal{N}_0 = \{f \in {}^\omega 2 : f \text{ is eventually constant.}\}.$$

One has $card(\mathcal{N}_0) = \omega$. Suppose we have had Z_β for all $\beta < \alpha$. Let's assume first that $a_\alpha \geqslant H/2$. For any $x, y < a_\alpha$, one has, by Claim 2.7.2, that $x \sim_0 y$ iff $a_\alpha - x \sim_0 a_\alpha - y$. So for each $f \in {}^\omega 2$ there is a unique $g_f \in {}^\omega 2$ such that $x \in A_f$ iff $a_\alpha - x \in A_{g_f}$. Note that the map $f \mapsto g_f$ is a bijection from ${}^\omega 2$ to ${}^\omega 2$. We need to find two functions f and g in ${}^\omega 2$ such that f and g are not eventually same, neither f nor g is eventually constant, neither f nor g is eventually same as any element in $\bigcup_{\gamma<\alpha} Z_\gamma$, and $g = g_f$.

Without loss of generality, we assume that a_α is even. Similar to the proof of Claim 2.7.2, one can show that for any even numbers $z_0, z_1 < a_\alpha$,

$$z_0 \sim_0 z_1 \text{ iff } (a_\alpha - z_0)/2 \sim_0 (a_\alpha - z_1)/2.$$

So for any $h \in {}^\omega 2$ there is a function $p^h \in {}^\omega 2$ such that for any even number $z < a_\alpha$ one has that

$$z \in A_h \text{ iff } (a_\alpha - z)/2 \in A_{p^h}.$$

Let

$$\bar{Z} = (\bigcup_{\gamma<\alpha} Z_\gamma) \cup \mathcal{N}_0 \cup \{g_f : f \in (\bigcup_{\gamma<\alpha} Z_\gamma) \cup \mathcal{N}_0\} \cup \{p^h : h \in \mathcal{N}_0\}.$$

Then $card(\bar{Z}) = card(\alpha) < 2^\omega$. Since there are 2^ω equivalence classes of eventually same functions from ω to 2, one can always find a $g \in {}^\omega 2$ such that g is not eventually same as any functions in $\bar{Z}$. Now pick $y < a_\alpha/2$ such that $y \in A_g$ and let $f \in {}^\omega 2$ be such that $x = a_\alpha - y \in A_f$. Then $g = g_f$. Since $g \notin \bar{Z}$, neither f nor g is eventually constant,

Claim 2.7.3 f is not eventually same as any function in $\bigcup_{\gamma<\alpha} Z_\gamma$.

Proof of Claim 2.7.3: Suppose the claim is not true. Let $f' \in \bigcup_{\gamma<\alpha} Z_\gamma$ be such that f and f' are eventually same. By Claim 2.7.2 (2), one can show that $g = g_f$ and $g_{f'}$ are eventually same. But this contradicts the choice of g. □(Claim 2.7.3)

Claim 2.7.4 f and g are not eventually same.

Proof of Claim 2.7.4: Suppose f and g are eventually same. Then there is a unique $h \in {}^\omega 2$, which is eventually constant, such that $|x - y| \in A_h$. But

$$|x - y| = |a_\alpha - x - x| = |a_\alpha - 2y| = a_\alpha - 2y = z \in A_h$$

implies

$$y = (a_\alpha - z)/2 \in A_{p^h}.$$

So $g = p^h$. This contradicts that $g \notin \bar{Z}$. □(Claim 2.7.4)

Now let $Z_\alpha = (\bigcup_{\gamma<\alpha} Z_\gamma) \cup \{f, g\}$, and this ends the construction. □

Remark: We don't know if one can find a cut $U \subseteq \mathcal{H}$ such that there exists a countably determined U-discrete set $X \subseteq \mathcal{H}$ with $X \oplus X = \mathcal{H}$. However, if such a U does exist, then it can't be equal to or smaller than $[\sqrt{H}]\cdot\mathbb{N}$.

Theorem 2.8 *Let $U \subseteq \mathcal{H}$ be a cut such that $[\sqrt{H}]\cdot\mathbb{N} \subsetneqq U$. Then there does not exist a countably determined set $X \subseteq \mathcal{H}$ such that X is U-discrete and $X \oplus X = \mathcal{H}$.*

Proof: Fix an $a = [\sqrt{H}]\cdot K \in U \setminus [\sqrt{H}]\cdot\mathbb{N}$ for some hyperfinite integer K. Suppose there is a countably determined U-discrete set $X \subseteq \mathcal{H}$ with $X \oplus X = \mathcal{H}$. Let $I \subseteq {}^{\omega}2$ and $\langle A_\sigma : \sigma \in {}^{<\omega}2\rangle$ be a decreasing sequence of internal subsets of $\mathcal{H}$ such that

$$X = \bigcup_{f\in I}\bigcap_{n\in\omega} A_{f\restriction n}.$$

Given a function $f \in I$, then A_f is also U-discrete. In particular, any two different elements in A_f have distance greater than a. By ω_1-saturation, there is a $\sigma_f \in {}^{<\omega}2$ such that $\sigma_f \subseteq f$ and any two different elements in A_{σ_f} have distance greater than a. Let $\Sigma = \{\sigma_f : f \in I\}$. Then Σ is finite or countable because $\Sigma \subseteq {}^{<\omega}2$. Obviously,

$$X \subseteq \bigcup_{\sigma\in\Sigma} A_\sigma.$$

Since $X \oplus X = \mathcal{H}$, one has

$$\bigcup_{\sigma,\sigma'\in\Sigma} (A_\sigma \oplus A_{\sigma'}) = \mathcal{H}.$$

Again by ω_1-saturation there is a finite subset $\Sigma_0 \subseteq \Sigma$ such that

$$\bigcup_{\sigma,\sigma'\in\Sigma_0} (A_\sigma \oplus A_{\sigma'}) = \mathcal{H}.$$

Since $|A_\sigma| \leqslant (H/a) + 1$ for every $\sigma \in \Sigma_0$, then one has

$$|A_\sigma \oplus A_{\sigma'}| \leqslant (H+a)^2/a^2 \leqslant 4H^2/(\sqrt{H}-1)^2K^2 \leqslant 16H/K^2.$$

Hence we have the following contradiction

$$|\mathcal{H}| = H \leqslant \sum_{\sigma,\sigma'\in\Sigma_0} |A_\sigma \oplus A_{\sigma'}| \leqslant H\frac{16|\Sigma_0|^2}{K^2} < H. \quad \square$$

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