Density Versions of Plünnecke Inequality – Epsilon-Delta Approach

Renling Jin

Abstract

We discuss whether Plünnecke's inequality for Shnirel'man density with respect to Shnirel'man basis can be generalized to other densities with respect to other concepts of basis. We show behavioral disparities between lower densities and upper densities on Plünnecke's inequality. We provide elementary proofs for the generalizations of Plünnecke's inequality to lower asymptotic density and upper Banach density. These two results are slightly more general than two theorems proved by the author using nonstandard analysis. A similar generalization to upper asymptotic density is not true. We also provide an elementary proof of a new generalization to lower Banach density with respect to **upper** Banach basis. In the last section we present a simplified proof of Plünnecke's inequality for Shnirel'man density with respect to Shnirel'man basis without introducing the impact function.

1 Introduction

In this paper we prove four theorems – Theorem 4, Theorem 6, Theorem 7, and Theorem 2. Theorem 7 is new while Theorem 4 and Theorem 6 are slightly more general than two theorems proved by the author in [8]. However, the proofs in [8] employs methods from nonstandard analysis and many people who are interested in the subject may not be familiar with nonstandard analysis. In order to reach more readers, we provide elementary proofs

---

College of Charleston, Charleston, South Carolina, USA

Mathematics Subject Classification 2010 Primary 11B05, 11B13

Keywords: Plünnecke's inequality, additive basis, Shnirel'man density, lower asymptotic density, upper asymptotic density, lower Banach density, upper Banach density

The research was supported in part by College of Charleston 2011 Summer Research Grant. The author would also like to express his gratitude to Professor Georges Grekos and his colleagues in Laboratoire des Mathématiques Unifiées de Saint-Etienne (LaMUSE) for their hospitality and interesting discussions on the subject of this paper during his visit there in the summer of 2010.

of Theorems 4 and Theorem 6 by a standard ϵ-δ approach. We also supply brief comments in the beginning of each proof to motivate why some steps should be taken and what are the purposes of these steps. Theorem 2 is due to Plünnecke. We present a simplified proof of Theorem 2 in the last section based on Plünnecke's inequality for truncated additive graph (see Theorem 3). The proof does not rely on impact function as does in [10, 9].

For any two sets A, B of numbers, let $A \pm B \doteq \{a+b : a \in A \text{ and } b \in B\}$. If a is a number, let $A \pm a \doteq A \pm \{a\}$.

Definition 1. Let A be a subset of $\mathbb{N} = \{0, 1, 2 \ldots\}$. Then

- the *Shnirel'man density* of A is defined by $\sigma(A) = \inf_{n \geqslant 1} \dfrac{A(n)}{n}$;
- the *lower asymptotic density* of A is defined by $\underline{d}(A) = \liminf_{n \to \infty} \dfrac{A(n)}{n}$;
- the *upper asymptotic density* of A is defined by $\overline{d}(A) = \limsup_{n \to \infty} \dfrac{A(n)}{n}$;
- the *lower Banach density* of A is defined by $\underline{u}(A) = \lim_{n \to \infty} \inf_{k \in \mathbb{N}} \dfrac{A(k, k+n)}{n+1}$;
- the *upper Banach density* of A is defined by $\overline{u}(A) = \lim_{n \to \infty} \sup_{k \in \mathbb{N}} \dfrac{A(k, k+n)}{n+1}$.

where $A(a, b)$ counts the number of elements in $A \cap \{a, a+1, \ldots, b\}$ and $A(n) \doteq A(1, n)$.

Clearly, for any set $A \subseteq \mathbb{N}$, we have that

$$0 \leqslant \begin{matrix} \sigma(A) \\ \underline{u}(A) \end{matrix} \leqslant \underline{d}(A) \leqslant \overline{d}(A) \leqslant \overline{u}(A) \leqslant 1.$$

Notice that σ and $\underline{u}$ are not comparable. We use the symbol u for Banach density because Banach density sometimes is also called uniform density [6].

Definition 2. A set $B \subseteq \mathbb{N}$ is called a *Shnirel'man basis of order* h if

$$hB = \underbrace{B + B + \cdots + B}_{h} = \mathbb{N}.$$

Notice that for $h \geqslant 2$, B is a Shnirel'man basis of order h if and only if $\sigma(hB) = 1$.

Definition 3. A set $B \subseteq \mathbb{N}$ is called

- a *lower asymptotic basis of order* h if $\underline{d}(hB) = 1$,
- an *upper asymptotic basis of order* h if $\overline{d}(hB) = 1$,

- a *lower Banach basis of order* h if $\underline{u}(hB) = 1$, and
- an *upper Banach basis of order* h if $\overline{u}(hB) = 1$.

In 1937 Erdős proved the following theorem to show that if B is a Shnirel'man basis of order h, then $\sigma(A + B)$ is strictly greater than $\sigma(A)$ for any set A as long as $0 < \sigma(A) < 1$ is satisfied. In fact, Erdős gave a specific lower bound for $\sigma(A + B)$ in terms of $\sigma(A)$ and h.

Theorem 1. *If B is a Shnirel'man basis of order h, then*

$$\sigma(A+B) \geqslant \sigma(A) + \frac{1}{2h} \cdot \sigma(A)\left(1 - \sigma(A)\right). \tag{1}$$

Landau made an improvement later by replacing h with h^* – the average order of the Shnirel'man basis B, in (1) ([7, page 10]). In 1938 Rohrbach proved a lower asymptotic density version of Theorem 1 with respect to asymptotic basis of average asymptotic order h^* [7, page 45]. A set $B \subseteq \mathbb{N}$ is called an *asymptotic basis of order* h if hB contains all but finitely many positive integers. Clearly, a Shnirel'man basis of order h must be an asymptotic basis of order $\leqslant h$ and an asymptotic basis of order h must be a lower asymptotic basis of order $\leqslant h$. Converses are not true. The definition of *average asymptotic order* h^* of an asymptotic basis can be found in [7].

In 1970 Plünnecke obtained following significant improvement of Theorem 1 in [10]. A proof of Theorem 2 can also be found in [9, page 225] and [11].

Theorem 2 (Plünnecke, 1970). *If B is a Shnirel'man basis of order h, then for every $A \subseteq \mathbb{N}$*

$$\sigma(A+B) \geqslant \sigma(A)^{1-\frac{1}{h}}. \tag{2}$$

It is mentioned in [8] that the order h in Theorem 2 cannot be replaced by the average order h^*. Notice that $x^{1-\frac{1}{h}}$ is greater if h is smaller for a fixed $x \in (0, 1)$.

Differing from Erdős' combinatorial approach, Plünnecke's approach is graph theoretic. By analyzing the relations among the minimum growth rates of Plünnecke graph at different levels, Plünnecke was able to prove a powerful inequality, which leads to Theorem 2. In this paper we need only the following special case of the inequality for the setting of a truncated additive graph of sumsets.

Theorem 3 (Plünnecke, 1957). *Let $A_0, B \subseteq \mathbb{N}$ and $h, n \in \mathbb{N}$ be such that $A_0(0, n) \neq 0$. For each $1 \leqslant i \leqslant h$ let*

$$D_{n,i} = \min\left\{\frac{(A' + iB)(0,n)}{A'(0,n)} : \emptyset \neq A' \subseteq A_0 \cap [0,n]\right\}. \tag{3}$$

Then

$$D_{n,1} \geqslant (D_{n,2})^{1/2} \geqslant \cdots \geqslant (D_{n,h})^{1/h}. \tag{4}$$

The proof of the general Plünnecke inequality, which implies (4), as well as related concepts, can be found in [9, Chater 7] or [11, Chapter 1]. Notice that $D_{n,i}$ depends on A_0 and B. We will mention which sets A_0 and B are used when a confusion may arise in applications of (4).

Although Shnirel'man basis has been used in many classical results, some important basis-like sets are not Shnirel'man bases and some other important basis-like sets are Shnirel'man bases with unnecessarily large order. For example, the set P of all prime numbers is not a Shnirel'man basis but is a lower asymptotic basis of order 3 (see [2, 14, 4]). It is not known whether P is an asymptotic basis of order 3 (Goldbach Conjecture). For another example, the set C of all cubes of non-negative integers is a Shnirel'man basis of order 9, an asymptotic basis of order 7, and a lower asymptotic basis of order 4 (see [3]). It is not known whether C is an asymptotic basis of order 4, 5, or 6. Therefore, it is interesting to explore whether Theorem 2 is still true when Shnirel'man density is replaced by lower asymptotic density and Shnirel'man basis is replaced by lower asymptotic basis. In [11, Theorem 7.2] it is proved that for any $A, B \subseteq \mathbb{N}$ we have that $\underline{d}(A+B) \geqslant \underline{d}(A)^{1-1/h}\sigma(hB)^{1/h}$. We will improve this by substituting $\sigma(hB)^{1/h}$ with $\underline{d}(hB)^{1/h}$.

When we discover some results for lower asymptotic density, it is logical to ask whether the same is true for upper asymptotic density. After lower and upper asymptotic densities are considered, the next logical step may be to explore whether the results for lower and upper asymptotic densities are also true for other densities such as lower Banach density and upper Banach density. Upper Banach density is popular among some combinatorial number theorists [1, 5].

In Section 2 we prove Theorem 4. We show that for any $A, B \subseteq \mathbb{N}$, $\underline{d}(A+B) \geqslant \underline{d}(A)^{1-1/h}\underline{d}(hB)^{1/h}$. A reference will be given that there are $A, B \subseteq \mathbb{N}$ such that $\overline{d}(A+B) \geqslant \overline{d}(A)^{1-1/h}\overline{d}(hB)^{1/h}$ is not true. In Section 3 we prove Theorem 6 and Theorem 7. We show that $\overline{u}(A+B) \geqslant \overline{u}(A)^{1-1/h}\overline{u}(hB)^{1/h}$ and $\underline{u}(A+B) \geqslant \underline{u}(A)^{1-1/h}\overline{u}(hB)^{1/h}$ for any $A, B \subseteq \mathbb{N}$. Notice that the second inequality is interesting because the last term $\overline{u}(hB)$ instead of $\underline{u}(hB)$ is used. In Section 4 we present a simplified proof of Theorem 2.

2 Lower/upper asymptotic densities

Theorem 4. *For any $A, B \subseteq \mathbb{N}$ and integer $h \geqslant 1$ we have that*

$$\underline{d}(A+B) \geqslant \underline{d}(A)^{1-\frac{1}{h}}\underline{d}(hB)^{\frac{1}{h}}. \tag{5}$$

Proof. Let $\underline{d}(A) = \alpha$ and $\underline{d}(hB) = \beta$.

First we explain the idea of the proof. Let's check what obstacles need to be overcome when we try to derive (5) from (4). By (4) with A_0 being replaced by A we have that

$$\begin{aligned}\frac{(A+B)(0,n)}{A(0,n)} &\geqslant D_{n,1} \geqslant D_{n,h}^{1/h} \\ &= \left(\frac{(A'+hB)(0,n)}{A'(0,n)}\right)^{1/h} = \left(\frac{(A'+hB)(z,n)}{A'(z,n)}\right)^{1/h}\end{aligned}$$

for some non-empty set $A' \subseteq A \cap [z,n]$ with $z = \min A'$. Hence

$$\frac{(A+B)(0,n)}{n+1} \geqslant \frac{A(0,n)}{n+1}\left(\frac{(A'+hB)(z,n)}{A'(z,n)}\right)^{1/h}.$$

Now $(A+B)(0,n)/(n+1)$ is close to $\underline{d}(A+B)$ and $A(0,n)/(n+1)$ is close to α. If we can show that $(A'+hB)(z,n)/A'(z,n)$ is less than or close to β/α, then the right side of the inequality above is greater than or close to $\alpha^{1-1/h}\beta^{1/h}$, which is the right side of (5).

Let's take a closer look at the term

$$\frac{(A'+hB)(z,n)}{A'(z,n)} = \frac{(A'+hB)(z,n)/(n-z+1)}{A'(z,n)/(n-z+1)}.$$

Notice that we do not have too much information on A'. Since

$$(A'+hB)(z,n) \geqslant (hB)(0,n-z+1) \text{ and } \underline{d}(hB) = \beta,$$

we have that $(A'+hB)(z,n)/(n-z+1)$ is greater than or close to β if $n-z$ is sufficiently large. So we need to make sure that the least element z in A' is sufficiently less than n, which may not be true if, for example, $A' = \{n\}$. Hence we want to make sure that the situation such as $A' = \{n\}$ won't happen. Also we want $A'(z,n)/(n-z+1)$ to be less than or close to α, which may not be true either if $A' = A \cap [z,n]$ is very dense. Thus we need to trim the set A so that the set $A' \subseteq A \cap [0,n]$ is forced to be thin.

We now present the formal proof using an ϵ-δ argument. Without loss of generality, we can assume that $0 < \alpha < 1$ and $\beta > 0$ because otherwise (5) is trivially true.

Given any $\epsilon > 0$, it suffices to find an N such that

$$\frac{(A+B)(0,n)}{n+1} \geqslant \alpha^{1-\frac{1}{h}}\beta^{\frac{1}{h}} - \epsilon$$

for any $n \geqslant N$. By a limit argument we can find a $\delta \in (0,1)$ such that

$$(\alpha - 2\delta)\left(\frac{\beta-\delta}{\alpha+\delta}\right)^{1/h} > \alpha^{1-\frac{1}{h}}\beta^{\frac{1}{h}} - \epsilon.$$

By the conditions given we can find $N_1 \in \mathbb{N}$ such that if $n \geqslant N_1$, then $A(0,n)(0,n)/(n+1) > \alpha - \delta$ and $(hB)(0,n)/(n+1) > \beta - \delta$. Let

$$N = \lfloor (2N_1/\delta)^2 \rfloor + 1.$$

We will show that N is what we want.

Given any $n \geqslant N$, let $k = n - \lfloor \sqrt{n} \rfloor - 1$. Notice that $n - k > \sqrt{n} > N_1$. We now trim the set $A \cap [0, n]$. Let $C_0 = A \cap [0, k]$. Then

$$\frac{C_0(0,n)}{n+1} = \frac{A(0,n) - A(k+1,n)}{n+1} \geqslant \alpha - \delta - \frac{2}{\sqrt{n}} \geqslant \alpha - 2\delta.$$

We now make further modifications of C_0 by an inductive process. For each $i = 0, 1, \ldots, k$ let

$$C_{i+1} = \begin{cases} C_i & \text{if } \frac{C_i(k-i,n)}{n-k+i+1} < \alpha + \delta, \\ C_i \smallsetminus \{k-i\} & \text{otherwise.} \end{cases}$$

Let $A_0 = C_{k+1}$. Notice that $C_0 \supseteq C_1 \supseteq \cdots \supseteq C_{k+1}$ and for any $0 \leqslant z \leqslant k$ we have that $A_0(z,n)/(n-z+1) \leqslant \alpha + \delta$.

We now want to show that $C_i(0,n)/(n+1) \geqslant \alpha - 2\delta$ for any $i \in [0, k+1]$ by induction on i. For $i = 0$ we have had that $C_0(0,n)/(n+1) \geqslant \alpha - 2\delta$. Suppose that $C_i(0,n)/(n+1) \geqslant \alpha - 2\delta$ for some $i \leqslant k$. If

$$\frac{C_i(k-i,n)}{n-k+i+1} < \alpha + \delta,$$

then $C_{i+1} = C_i$, which implies that $C_{i+1}(0,n)/(n+1) \geqslant \alpha - 2\delta$. Hence we can assume that

$$\frac{C_i(k-i,n)}{n-k+i+1} \geqslant \alpha + \delta.$$

Notice that

$$\frac{C_{i+1}(0,n)}{n+1} = \frac{A(0,k-i-1)}{n+1} + \frac{C_i(k-i,n)}{n+1} - \frac{1}{n+1}.$$

If $k - i > N_1$, then

$$\begin{aligned} &\frac{C_{i+1}(0,n)}{n+1} \\ &\geqslant (\alpha - \delta)\frac{k-i}{n+1} + (\alpha + \delta)\frac{n-k+i+1}{n+1} - \frac{1}{n+1} \\ &\geqslant (\alpha - \delta)\frac{k-i}{n+1} + \alpha\frac{n-k+i+1}{n+1} \geqslant \alpha - 2\delta \end{aligned}$$

by the choice of $n \geqslant N$ and if $k - i \leqslant N_1$, then

$$\begin{aligned} \frac{C_{i+1}(0,n)}{n+1} &\geqslant \frac{C_i(k-i,n)}{n+1} - \frac{1}{n+1} \\ &\geqslant (\alpha + \delta)\frac{n-k+i+1}{n+1} - \frac{1}{n+1} \end{aligned}$$

$$\geqslant (\alpha+\delta)\left(1-\frac{k-i}{n+1}\right)-\delta \geqslant \alpha\left(1-\frac{N_1}{n+1}\right)-\delta$$
$$\geqslant \alpha(1-\delta)-\delta \geqslant \alpha-\delta-\delta=\alpha-2\delta.$$

This completes the induction. Therefore, we have that $A_0(0,n)/(n+1) \geqslant \alpha-2\delta$ because $A_0 = C_{k+1}$.

Now we apply Theorem 3 and have that

$$\frac{(A+B)(0,n)}{n+1} \geqslant \frac{A_0(0,n)}{n+1} * \frac{(A_0+B)(0,n)}{A_0(0,n)}$$
$$\geqslant (\alpha-2\delta)D_{n,1} \geqslant (\alpha-2\delta)D_{n,h}^{1/h}$$
$$\geqslant (\alpha-2\delta)\left(\frac{(A'+hB)(0,n)}{A'(0,n)}\right)^{1/h} = (\alpha-2\delta)\left(\frac{(A'+hB)(z,n)}{A'(z,n)}\right)^{1/h}$$
$$\geqslant (\alpha-2\delta)\left(\frac{(hB)(0,n-z)/(n-z+1)}{A_0(z,n)/(n-z+1)}\right)^{1/h} \geqslant (\alpha-2\delta)\left(\frac{\beta-\delta}{\alpha+\delta}\right)^{1/h}$$
$$\geqslant \alpha^{1-1/h}\beta^{1/h}-\epsilon$$

where $\emptyset \neq A' \subseteq A_0$ and $z = \min A'$. Recall that $n-z \geqslant n-k > N_1$, which is used in one of the steps above. This completes the proof.

Corollary 1. *(1) If B is a lower asymptotic basis of order h, then $\underline{d}(A+B) \geqslant \underline{d}(A)^{1-1/h}$ for any $A \subseteq \mathbb{N}$.*

(2) Let P be the set of all prime numbers and C be the set of all cubes of non-negative integers. Then $\underline{d}(3P)=1$ and $\underline{d}(4C)=1$. Hence by Theorem 4 we have that for any $A \subseteq \mathbb{N}$,

$$\underline{d}(A+P) \geqslant \underline{d}(A)^{2/3} \text{ and } \underline{d}(A+C) \geqslant \underline{d}(A)^{3/4}.$$

[11, Theorem 7.2] shows that $\underline{d}(A+B) \geqslant \underline{d}(A)^{1-1/h}\sigma(hB)^{1/h}$. Hence Theorem 4 improves [11, Theorem 7.2]. Notice that $\sigma(hP)=0$ for any h and $\sigma(hC)<1$ for each $h<9$. When $\underline{d}(A)$ is small, I. Z. Ruzsa obtained much better lower bound of $\underline{d}(A+B)$ for B being the set of prime numbers or the set of all k-th powers of non-negative integers [12, 13].

We state the following theorem for the disparity between lower and upper asymptotic densities.

Theorem 5. *There are $A, B \subseteq \mathbb{N}$ with $\overline{d}(A)=\frac{1}{2}$ and B being an upper asymptotic basis of order 2, i.e., $\overline{d}(2B)=1$ such that*

$$\overline{d}(A+B)=\overline{d}(A).$$

Since the proof of Theorem 5 in [8] is already standard, we do not repeat it here.

The reader might be curious about whether Theorem 4 is still true if $\underline{d}(hB)$ in (5) is replaced by $\overline{d}(hB)$. The following example show that it isn't.

Example 1. We want to construct a set A with $\underline{d}(A) = 1/2$ and an upper asymptotic basis B of order 1 such that $\underline{d}(A+B) = 1/2 = \underline{d}(A)$. Let

$$A = \bigcup_{n=0}^{\infty} \left[2^{2^n}, \left(2^{2^{n+1}} + 2^{2^n}\right)/2\right] \text{ and}$$

$$B = \bigcup_{n=0}^{\infty} \left[2^{2^{2n-1}}, 2^{2^{2n}}\right].$$

It is easy to see that $\underline{d}(A) = 1/2$. It is also easy to see that $\overline{d}(B) = 1$. Thus B is an upper Banach basis of order 1. Notice that

$$(A+B) \cap \left[0, 2^{2^{2n+1}} - 1\right] \subseteq \left[0, \left(2^{2^{2n+1}} + 2^{2^{2n}}\right)/2 + 2^{2^{2n}}\right].$$

Hence we have that $1 - 1/h = 0$, $1/h = 1$, and

$$\underline{d}(A+B) \leqslant \lim_{n\to\infty} \left(\left(2^{2^{2n+1}} + 2^{2^{2n}}\right)/2 + 2^{2^{2n}}\right)/2^{2^{2n+1}} = \frac{1}{2} < 1 = \underline{d}(A)^0 \overline{d}(B).$$

3 Lower/upper Banach densities

It is easy to show that $\overline{u}(A) = \alpha$ if and only if α is the greatest number such that there exists a sequence of intervals $[a_n, b_n] \subseteq \mathbb{N}$ with $b_n - a_n \to \infty$ and $A(a_n, b_n)/(b_n - a_n + 1) \to \alpha$.

It is also easy to show that $\underline{u}(A) = \alpha$ if and only if α is the least number such that there exists a sequence of intervals $[a_n, b_n] \subseteq \mathbb{N}$ with $b_n - a_n \to \infty$ and $A(a_n, b_n)/(b_n - a_n + 1) \to \alpha$.

We would like to mention the following two trivial but useful propositions, which give us some epsilon-delta type equivalent forms of upper Banach density and lower Banach density, respectively.

Proposition 1. *For any set $C \subseteq \mathbb{N}$ and $\alpha > 0$, $\overline{u}(C) \geqslant \alpha$ if and only if for every $\epsilon > 0$ and every $N \in \mathbb{N}$ there exists an interval $[a, b] \subseteq \mathbb{N}$ such that $b - a \geqslant N$ and $C(a, b)/(b - a + 1) > \alpha - \epsilon$.*

From Proposition 1 one can see intuitively that if $\overline{u}(C) = \alpha$ and $[a, b]$ is a sufficiently long interval such that $C(a, b)/(b - a + 1) > \alpha - \epsilon$ for some very small positive real number ϵ, then asymptotically the elements in $C \cap [a, b]$ should be very evenly distributed. This is because if $C \cap [a, b]$ is more dense in a subinterval $[a', b']$ with $b' - a'$ not too small, then $\overline{u}(C)$ would eventually be strictly greater than α, a contradiction to $\overline{u}(C) = \alpha$.

Proposition 2. *For any set $C \subseteq \mathbb{N}$ and $\alpha > 0$, $\underline{u}(C) \geqslant \alpha$ if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for any $a, b \in \mathbb{N}$, if $b - a \geqslant N$, then $C(a,b)/(b-a+1) > \alpha - \epsilon$.*

Now we prove two main theorems of this section.

Theorem 6. *For any $A, B \subseteq \mathbb{N}$ and any integer $h \geqslant 1$.*

$$\overline{u}(A+B) \geqslant \overline{u}(A)^{1-\frac{1}{h}} \overline{u}(hB)^{\frac{1}{h}}.$$

Proof. Let $\overline{u}(A) = \alpha$ and $\overline{u}(hB) = \beta$.

First we explain the ideas in the proof. We need to prove that $\overline{u}(A+B) \geqslant \alpha^{1-1/h}\beta^{1/h}$, which means that for a given $\epsilon > 0$ we need to find sufficiently long interval $[a,b] \subseteq \mathbb{N}$ such that $(A+B)(a,b)/(b-a+1) > \alpha^{1-1/h}\beta^{1/h} - \epsilon$. Clearly, we want to pick an interval $[a,b]$ so that $A(a,b)$ is as large as possible, i.e., $A(a,b)/(b-a+1)$ is close to α.

In order to apply (4) we need to translate the interval $[a,b]$ to $[0,b-a]$, translate the set $(A+B) \cap [a,b]$ to $(A+B-a) \cap [0,b-a]$, and consider $(A+B-a)(0,b-a)/(b-a+1)$ instead. Since the upper Banach density of $(h(B-a)) \cap \mathbb{N}$ may be different from β, we keep B unchanged and only translate $A \cap [a,b]$ to $A_0 = (A-a) \cap [0,b-a]$. Then we apply Theorem 3 with $n = b - a$.

By (4) we have that

$$\frac{(A_0+B)(0,n)}{n+1} \geqslant \frac{A_0(0,n)}{n+1} \left(\frac{(A'+hB)(0,n)}{A'(0,n)} \right)^{1/h}$$

for some non-empty set $A' \subseteq A_0$. Since $A_0(0,n)/(n+1)$ is close to α we need only to show that $(A'+hB)(0,n)/A'(0,n)$ is greater than or close to β/α.

Since $\overline{u}(hB) = \beta$, we can find sufficiently long intervals $[m, m+k]$ such that $(hB)(m,m+k)/(k+1)$ is close to β. We choose n much larger than $m+k$.

Ideally we hope that A' is very discrete so that any two elements in it have a distance greater than k. If this is the case and A' is forced to be in $[0, n-m-k]$, then $(A'+hB)(0,n) \supseteq (A'+((hB)\cap[m,m+k]))(0,n)$ is greater than or equal to $(hB)(m,m+k)A'(0,n)$, which is close to $(k+1)\beta A'(0,n)$. Hence the ratio $(A'+hB)(0,n)/A'(0,n)$ is greater than $(k+1)\beta$. If k is large enough, then $(k+1)\beta \geqslant \beta/\alpha$.

What can we do if A' is not that discrete? Since A is rather evenly distributed in $[a,b]$ we can assume that $A'(x,x+k)/(k+1)$ is less than or close to α for any $x \in [0, n-m-2k]$. However, the elements x' in $A' \cap [x, x+k]$ generate a set $x' + ((hB) \cap [m, m+k])$ in $(A'+hB) \cap [0,n]$. Hence the ratio of the size of the set generated by $A' \cap [x,x+k]$ in $A'+hB$ over $A'(x,x+k)$ is greater than or close to β/α. Now we partition the interval $[0, n-m-2k]$ into subintervals of the length k and combine all the ratio of the size of the set in $(A'+hB) \cap [0,n]$ generated by each subinterval over the size of that subinterval we can have that the ratio $(A'+hB)(0,n)/A'(0,n)$ is greater than or close to β/α, which is exactly what we need.

Now we present the formal proof. Again we can assume, without loss of generality, that $0<\alpha<1$ and $\beta>0$.

Given any $\epsilon>0$ and $N\in\mathbb{N}$, it suffices to find an interval $[a,b]\subseteq\mathbb{N}$ such that $b-a\geqslant N$ and

$$\frac{(A+B)(a,b)}{b-a+1}>\alpha^{1-\frac{1}{h}}\beta^{\frac{1}{h}}-\epsilon$$

by Proposition 1. Let $\delta\in(0,1)$ be such that

$$(\alpha-2\delta)\left(\frac{\beta-4\delta}{\alpha+\delta}\right)^{1/h}>\alpha^{1-\frac{1}{h}}\beta^{\frac{1}{h}}-\epsilon.$$

Since $\overline{u}(hB)=\beta$, for each $k\in\mathbb{N}$ we can fix $c_k,d_k\in\mathbb{N}$ such that $d_k>2k/\delta$ and $(hB)(c_k,c_k+2d_k-1)/(2d_k)\geqslant\beta-\delta$.

Claim There exist $k,a,b\in\mathbb{N}$ such that $k>\max\{N,1/\delta\}$, $b-a>(c_k+d_k)^2$, $A(a,b)/(b-a+1)>\alpha-\delta$,

$$\frac{A(a+id_k,a+(i+1)d_k-1)}{d_k}<\alpha+\delta \tag{6}$$

for $i=0,1,\ldots,\lfloor(b-a+1)/d_k\rfloor-1$, and

$$\frac{(hB)(x,x+d_k-1)}{d_k}>\beta-4\delta \tag{7}$$

for any $x\in[c_k,c_k+d_k]$.

Proof of Claim: Suppose that the claim is not true. For any $k>\max\{N,1/\delta\}$ we can choose $a_k,b_k\in\mathbb{N}$ such that $b_k-a_k>(c_k+d_k)^2$ and $A(a_k,b_k)/(b_k-a_k+1)>\alpha-\delta$ by Proposition 1. Since the claim is not true, we have that either (6) is false for some $i=i_k\in[0,\lfloor(b_k-a_k+1)/d_k\rfloor-1]$ or (7) is false for some $x=x_k\in[c_k,c_k+d_k]$. If (6) fails for infinitely many k, we can find a sequence of intervals $[a_k+i_kd_k,\ a_k+(i_k+1)d_k-1]$ for infinitely many $k>\max\{N,1/\delta\}$ such that $A(a_k+i_kd_k,a_k+(i_k+1)d_k-1)/d_k\geqslant\alpha+\delta$. This implies that $\overline{u}(A)\geqslant\alpha+\delta$, which contradicts the assumption that $\overline{u}(A)=\alpha$. Hence we can assume that (7) fails for all sufficiently large k, i.e., there exists an $M\in\mathbb{N}$ such that for all $k\geqslant M$ we have that $(hB)(x_k,x_k+d_k-1)/d_k\leqslant\beta-4\delta$ for some $x_k\in[c_k,c_k+d_k]$. We now want to derive a contradiction by showing that $\overline{u}(hB)\geqslant\beta+\delta$.

Let $k\geqslant M$. Notice that $(hB)(c_k,c_k+2d_k-1)/(2d_k)\geqslant\beta-\delta$ and $(hB)(x_k,x_k+d_k-1)/d_k\leqslant\beta-4\delta$.

If both intervals $[c_k,x_k-1]$ and $[x_k+d_k,c_k+2d_k-1]$ are longer that k, $(hB)(c_k,x_k-1)/(x_k-c_k)<\beta+\delta$, and $(hB)(x_k+d_k,c_k+2d_k-1)/(c_k+$

$d_k - x_k) < \beta + \delta$, then

$$\begin{aligned}
&(hB)(c_k, c_k + 2d_k - 1) \\
&\leqslant (hB)(c_k, x_k - 1) + (hB)(x_k, x_k + d_k - 1) + (hB)(x_k + d_k, c_k + 2d_k - 1) \\
&< (\beta + \delta)(x_k - c_k) + (\beta - 4\delta)d_k + (\beta + \delta)(c_k + d_k - x_k) \\
&= (\beta + \delta)d_k + (\beta - 4\delta)d_k = (2\beta - 3\delta)d_k < (\beta - \delta)(2d_k),
\end{aligned}$$

which contradicts the choice of c_k and d_k. Hence we can assume that if both intervals $[c_k, x_k - 1]$ and $[x_k + d_k, c_k + 2d_k - 1]$ are longer that k, then there is an interval $[u_k, v_k]$ such that $v_k - u_k > k$ and $(hB)(u_k, v_k)/(v_k - u_k + 1) \geqslant \beta + \delta$.

If $x_k - c_k \leqslant k$ and $(hB)(x_k + d_k, c_k + 2d_k - 1) < (\beta + \delta)(c_k + d_k - x_k)$, then by the fact that $d_k > 2k/\delta$ we have that $c_k + d_k - x_k > k$ and

$$\begin{aligned}
&(hB)(c_k, c_k + 2d_k - 1) \\
&\leqslant k + (hB)(x_k, x_k + d_k - 1) + (hB)(x_k + d_k, c_k + 2d_k - 1) \\
&< k + (\beta - 4\delta)d_k + (\beta + \delta)(c_k + d_k - x_k) \\
&\leqslant k + (\beta - 4\delta)d_k + (\beta + \delta)d_k \\
&= (\beta - \delta)(2d_k) + k - \delta d_k < (\beta - \delta)(2d_k),
\end{aligned}$$

which again contradicts the choice of c_k and d_k. Hence we can find an interval $[u_k, v_k] = [x_k + d_k, c_k + 2d_k - 1]$ longer than k such that $(hB)(u_k, v_k)/(v_k - u_k + 1) \geqslant \beta + \delta$.

If $c_k + d_k - x_k \leqslant k$, then by the same argument as in the last paragraph we can find an interval $[u_k, v_k] = [c_k, x_k - 1]$ longer than k such that $(hB)(u_k, v_k)/(v_k - u_k + 1) \geqslant \beta + \delta$.

Therefore, the existence of the intervals $[u_k, v_k]$ for all $k > M$ implies that $\overline{u}(hB) \geqslant \beta + \delta$, which contradicts the assumption $\overline{u}(hB) = \beta$. This completes the proof of the claim.

We now fix a $k > \max\{N, 1/\delta\}$ and an interval $[a, b]$ according to the claim such that $b - a > (c_k + d_k)^2$, $A(a, b)/(b - a + 1) > \alpha - \delta$, and satisfies (6) and (7).

Let $A_0 = (A \cap [a, b - c_k - d_k]) - a$ and $n = b - a$. Then we have that

$$\frac{A_0(0, n)}{n + 1} \geqslant \frac{A(a, b)}{b - a + 1} - \frac{c_k + d_k}{b - a + 1} \geqslant \alpha - \delta - \frac{1}{c_k + d_k} \geqslant \alpha - 2\delta.$$

For each $i = 0, 1, \ldots, \lfloor (n + 1)/d_k \rfloor - 1$ let $I_i = [id_k, (i + 1)d_k - 1]$. Then $|A_0 \cap I_i| < (\alpha + \delta)|I_i| = (\alpha + \delta)d_k$ by (6).

Now we apply Theorem 3. Let $\emptyset \neq A' \subseteq A_0$ be such that $D_{n,h} = (A' + hB)(0, n)/A'(0, n)$. For each $x \in A'$ we have that $x \leqslant n - c_k - d_k$. If $x' \in I_i \cap A'$, then let $t = (i + 1)d_k - x'$ and we have that $t \leqslant d_k$. Hence

$$\begin{aligned}
&(A' + hB)((i + 1)d_k + c_k, (i + 1)d_k + c_k + d_k - 1) \\
&\geqslant |x' + ((hB) \cap [c_k + t, c_k + t + d_k - 1])| > \beta - 4\delta
\end{aligned}$$

by (7). Let

$$\mathcal{I} = \{i \in [0, \lfloor (n+1)/d_k \rfloor - 1] : I_i \cap A' \neq \emptyset\}.$$

Then we have that $(A'+hB)(0,n) \geqslant (\beta-4\delta)d_k|\mathcal{I}|$ and $A'(0,n) \leqslant (\alpha+\delta)d_k|\mathcal{I}|$. Hence

$$\begin{aligned}
&\frac{(A+B)(a,b)}{b-a+1} \geqslant \frac{(A_0+B)(0,n)}{n+1} \\
&\geqslant \frac{A_0(0,n)}{n+1} D_{n,1} \geqslant (\alpha-2\delta) D_{n,h}^{1/h} = (\alpha-2\delta)\left(\frac{(A'+hB)(0,n)}{A'(0,n)}\right)^{1/h} \\
&\geqslant (\alpha-2\delta)\left(\frac{(\beta-4\delta)d_k|\mathcal{I}|}{(\alpha+\delta)d_k|\mathcal{I}|}\right)^{1/h} \geqslant (\alpha-2\delta)\left(\frac{\beta-4\delta}{\alpha+\delta}\right)^{1/h} \\
&> \alpha^{1-\frac{1}{h}}\beta^{\frac{1}{h}} - \epsilon.
\end{aligned}$$

This completes the proof.

Theorem 7. *For any $A, B \subseteq \mathbb{N}$ and an integer $h \geqslant 1$,*

$$\underline{u}(A+B) \geqslant \underline{u}(A)^{1-\frac{1}{h}}\,\overline{u}(hB)^{\frac{1}{h}}.$$

Proof. The proof of Theorem 7 combines the ideas from the proofs of both Theorem 4 and Theorem 6. Let $\underline{u}(A) = \alpha$ and $\overline{u}(hB) = \beta$. Without loss of generality we can assume that $0 < \alpha < 1$ and $\beta > 0$.

Here is the idea of the proof. Given $\epsilon > 0$, we want to show that if $[a,b]$ is any sufficiently long interval, then $(A+B)(a,b)/(b-a+1) > \alpha^{1-1/h}\beta^{1/h} - \epsilon$. We can translate $A \cap [a,b]$ to $A_0 = (A \cap [a,b]) - a$ and consider $(A_0+B)(0, b-a)/(b-a+1)$ instead. Let $n = b-a$. By (4) we have that

$$\frac{(A_0+B)(0,n)}{n+1} \geqslant \frac{A_0(0,n)}{n+1}\left(\frac{(A'+hB)(0,n)}{A'(0,n)}\right)^{1/h}$$

for some $\emptyset \neq A' \subseteq A_0$. The first term $A_0(0,n)/(n+1)$ at the right side is greater than or close to α. For the term inside the parentheses we can use the same idea as in the proof of Theorem 6 by partitioning $[0,n]$ into subintervals of length d_k, which is sufficiently long but relatively small with respect to n. We want to force A' to be as sparse as possible. Therefore, we use the same idea as in the proof of Theorem 4 by trimming down the set A_0. However, we have to be careful not to trim too much to make $A_0(0,n)/(n+1)$ significantly less than α.

Now we present the formal proof. Given any $\epsilon > 0$, we want to find N so that for any $a, b \in \mathbb{N}$, if $b - a \geqslant N$, then $(A+B)(a,b)/(b-a+1) > \alpha^{1-1/h}\beta^{1/h} - \epsilon$.

Let $\delta \in (0,1)$ be such that

$$(\alpha - 2\delta)\left(\frac{\beta - 4\delta}{\alpha + \delta}\right)^{1/h} > \alpha^{1-\frac{1}{h}}\beta^{\frac{1}{h}} - \epsilon.$$

Since $\overline{u}(hB) = \beta$, for each $k \in \mathbb{N}$, by the same argument as in the claim of the proof of Theorem 6, there exist $c_k, d_k \in \mathbb{N}$ such that $d_k > k$ and $(hB)(x, x + d_k - 1) > (\beta - 4\delta)d_k$ for every $x \in [c_k, c_k + d_k - 1]$.

Since $\underline{u}(A) = \alpha$, there is an $N_1 \in \mathbb{N}$ such that for any interval $[a, b]$ of length $\geqslant N_1$ we have that $A(a, b)/(b - a + 1) > \alpha - \delta$ by Proposition 2. Fix $k \geqslant \max\{N_1, \lfloor 1/\delta \rfloor + 1\}$. Let $N = \lfloor 2(c_k + d_k)^2/\delta \rfloor + 1$. We show that the number N is what we want.

Given any $a, b \in \mathbb{N}$ with $b - a \geqslant N$, let $A_1 = (A \cap [a, b - c_k - d_k]) - a$ and $n = b - a$. For each $i = 0, 1, \ldots, \lfloor (n + 1)/d_k \rfloor - 1$ let $I_i = [id_k, (i + 1)d_k - 1]$. Notice that I_i has a length d_k. Since $d_k > N_1$, we have that $|A_1 \cap I_i| > \alpha - \delta$ for each $i \leqslant \lfloor (n - c_k - 2d_k)/d_k \rfloor - 1$ by the choice of $k \geqslant N_1$. By deleting some elements if necessary we can find $C_i \subseteq A_1 \cap I_i$ for each $i = 0, 1, \ldots, \lfloor (n - c_k - 2d_k)/d_k \rfloor - 1$ such that $\alpha - \delta < |C_i|/d_k < \alpha + \delta$. Let

$$A_0 = \bigcup_{i=0}^{\lfloor (n - c_k - 2d_k)/d_k \rfloor - 1} C_i.$$

Then

$$\frac{A_0(0, n)}{n + 1} = \frac{1}{n + 1}\left(\sum_{i=0}^{\lfloor (n - c_k - 2d_k)/k \rfloor - 1} |C_i|\right) \geqslant (\alpha - \delta) - \frac{c_k + d_k}{n + 1} \geqslant \alpha - 2\delta.$$

Now we apply Theorem 3. Let $\emptyset \neq A' \subseteq A_0$ be such that $D_{n,h} = (A' + hB)(0, n)/A'(0, n)$. Notice that since $|A_0 \cap I_i|/d_k < \alpha + \delta$ and $A' \subseteq A_0$, we have that $|A' \cap I_i|/d_k < \alpha + \delta$. Let

$$\mathcal{I} = \{i \in [0, \lfloor (n - c_k - 2d_k)/d_k \rfloor - 1] : A' \cap I_i \neq \emptyset\}.$$

Then $|A'| < (\alpha + \delta)d_k|\mathcal{I}|$. Again if $I_i \cap A' \neq \emptyset$, then

$$(A' + hB)((i + 1)d_k + c_k, (i + 1)d_k + c_k + d_k - 1) \geqslant (\beta - 4\delta)d_k.$$

Hence $(A' + hB)(0, n) \geqslant (\beta - 4\delta)d_k|\mathcal{I}|$. Combining all these arguments together we have that

$$\begin{aligned}
\frac{(A + B)(a, b)}{b - a + 1} &\geqslant \frac{(A_0 + B)(0, n)}{n + 1} \\
&\geqslant \frac{A_0(0, n)}{n + 1} D_{n,1} \geqslant (\alpha - 2\delta) D_{n,h}^{1/h} = (\alpha - 2\delta)\left(\frac{(A' + hB)(0, n)}{A'(0, n)}\right)^{1/h} \\
&\geqslant (\alpha - 2d)\left(\frac{(\beta - 4\delta)d_k|\mathcal{I}|}{(\alpha + \delta)d_k|\mathcal{I}|}\right)^{1/h} \geqslant (\alpha - 2d)\left(\frac{\beta - 4\delta}{\alpha + \delta}\right)^{1/h}
\end{aligned}$$

$$> \alpha^{1-\frac{1}{h}} \beta^{\frac{1}{h}} - \epsilon.$$

This completes the proof.

In Theorem 7 we use upper Banach basis for the setting of lower Banach density. This is an interesting contrast to Example 1 in the lower/upper asymptotic density setting.

Corollary 2. *If $B \subseteq \mathbb{N}$ is an upper Banach basis of order h, then*

$$\overline{u}(A+B) \geqslant \overline{u}(A)^{1-\frac{1}{h}} \text{ and } \underline{u}(A+B) \geqslant \underline{u}(A)^{1-\frac{1}{h}}$$

for any $A \subseteq \mathbb{N}$.

4 Plünnecke's Theorem revisited

In this section we follow the similar style from previous sections to reprove Theorem 2. The main accomplishment here is that our proof does not rely on so called the impact function or Wirkungsfunktion $\phi(\xi, B)$ in [10, 9]. We feel that the introduction of the impact function in [10, 9] makes the idea less transparent. Notice that our proof in this section does not involve any ϵ-δ argument and is purely combinatorial.

Proof of Theorem 2. Suppose that $A \subseteq \mathbb{N}$ and B is a Shnirel'man basis of order h. Given any $n \geqslant 1$, it suffices to prove that $(A+B)(n)/n \geqslant \sigma(A)^{1-1/h}$.

Let $[a,b] \subseteq \mathbb{N}$. We say that A has a *minimal forward ratio* on $[a,b]$ if

$$\frac{A(a,b)}{b-a+1} = \min\left\{\frac{A(a,z)}{z-a+1} : z \in [a,b]\right\}.$$

Notice that if A has a minimal forward ratio on $[a,b]$, then for any $z \in [a,b]$, $A(z,b)/(b-z+1) \leqslant A(a,b)/(b-a+1)$.

Lemma 1. *Suppose that $[a,b] \subseteq \mathbb{N}$, A has a minimal forward ratio on $[a,b]$, and B is a Shnirel'man basis of order h. Then*

$$(A+B)(a,b) \geqslant (b-a+1)\left(\frac{A(a,b)}{b-a+1}\right)^{1-\frac{1}{h}}.$$

Proof. Let $A_0 = A - a$. Notice that $(A+B)(a,b) \geqslant (A_0+B)(0,b-a)$. Notice also that if $A' \subseteq A_0 \cap [z, b-a]$ for some $0 \leqslant z \leqslant b-a$, then $A'(z,b-a)/(b-a-z+1) \leqslant A_0(0,b-a)/(b-a+1)$ by the minimality of the forward ratio of A on $[a,b]$. Hence by (4) we have that

$$\frac{(A+B)(a,b)}{A(a,b)} \geqslant \frac{(A_0+B)(0,b-a)}{A_0(0,b-a)}$$

$$\geqslant \left(\frac{(A'+hB)(z,b-a)}{A'(z,b-a)}\right)^{1/h} \geqslant \left(\frac{1}{A'(z,b-a)/(b-a-z+1)}\right)^{1/h}$$
$$\geqslant \left(\frac{1}{A_0(0,b-a)/(b-a+1)}\right)^{1/h} \geqslant \left(\frac{1}{A(a,b)/(b-a+1)}\right)^{1/h}$$

for some $\emptyset \neq A' \subseteq A_0 \cap [0, b-a]$ with $z = \min A_0$. Now we have that

$$(A+B)(a,b) \geqslant A(a,b)\left(\frac{1}{A(a,b)/(b-a+1)}\right)^{1/h} = (b-a+1)\left(\frac{A(a,b)}{b-a+1}\right)^{1-\frac{1}{h}}.$$

This completes the proof of the lemma.

We now construct a finite sequence $n_0 = 1 < n_1 < \cdots < n_k = n+1$ such that A has a minimal forward ratio on $[n_{i-1}, n_i - 1]$ for $i = 1, 2, \ldots, k$. This can be done by induction on i such that if $n_{i-1} \leqslant n$, then let

$$\alpha_i = \min\left\{\frac{A(n_{i-1}, z)}{z - n_{i-1} + 1} : z \in [n_{i-1}, n]\right\}$$

and $n_i \in [n_{i-1}+1, n+1]$ be the greatest such that $A(n_{i-1}, n_i - 1)/(n_i - n_{i-1}) = \alpha_i$. Clearly, A has a minimal forward ratio on each interval $[n_{i-1}, n_i - 1]$ for $i = 1, 2, \ldots, k$. It is also easy to see that $\sigma(A) \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_k$.

By applying Lemma 1 to the second term in the inequalities below we have that

$$\frac{(A+B)(n)}{n} = \frac{1}{n}\sum_{i=1}^{k}(A+B)(n_{i-1}, n_i - 1)$$
$$\geqslant \frac{1}{n}\sum_{i=1}^{k}(n_i - n_{i-1})\left(\frac{A(n_{i-1}, n_i - 1)}{n_i - n_{i-1}}\right)^{1-\frac{1}{h}}$$
$$\geqslant \sum_{i=1}^{k}\frac{n_i - n_{i-1}}{n}\alpha_i^{1-\frac{1}{h}} \geqslant \sum_{i=1}^{k}\frac{n_i - n_{i-1}}{n}\sigma(A)^{1-\frac{1}{h}} = \sigma(A)^{1-\frac{1}{h}}.$$

Since $n \geqslant 1$ is arbitrary, the proof is completed.

References

1. V. Bergelson, *Ergodic Ramsey Theory–an Update, Ergodic Theory of* $\mathbb{Z}^d$ *Actions* (edited by M. Pollicott and K. Schmidt), London Math. Soc. Lecture Note Series 228 (1996), 1–61.
2. N. G. Chudakov, Doklady Akademii Nauk SSSR 17 (1937), 335–338.
3. H. Davenport, *On Waring's problem for cubes*, Acta Mathematica, **71** (1939), 123–143.
4. T. Estermann, *On Goldbach's problem: proof that almost all even positive integers are sums of two primes*, Proceedings of London Mathematical Society, 44 (1938), 307–314.

5. H. Furstenberg, **Recurrence in Ergodic Theory and Combinatorial Number Theory**, Princeton University Press, 1981.
6. G. Grekos, V. Toma, and J. Tomanová, *A note on uniform or Banach density*, Annales Mathématiques Blaise Pascal, **17** (2010), 153–163.
7. H. Halberstam and K. F. Roth, **Sequences**, Oxford University Press, 1966
8. R. Jin, *Plünnecke's Theorem for other densities*, The Transactions of American Mathematical Society, **363** (2011), 5059–5070
9. M. B. Nathanson, **Additive Number Theory–Inverse Problems and the Geometry of Sumsets**, Springer, 1996.
10. H. Plünnecke, *Eine zahlentheoretische Anwendung der Graphentheorie*, Journal für die reine und angewandte Mathematik, **234** (1970), 171–183.
11. I. Z. Ruzsa, *Sumsets and Structure*, Combinatorial number theory and additive group theory, Advanced courses in mathematics, CRM Barcelona, 87–210, Birkhäuser, Basel – Boston – Berlin, 2009.
12. I. Z. Ruzsa, *An additive property of squares and primes*. Acta Arithmetica, 49 (1988), 281–289.
13. I. Z. Ruzsa, *An additive problem for powers of primes*. Journal of Number Theory, 33 (1989), 71–82.
14. J. G. Van der Corput, *Sur l'hypothse de Goldbach*. Proc. Akad. Wet. Amsterdam 41 (1938), 76-80.