

# Plünnecke's Theorem for Asymptotic Densities

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Abstract

Plünnecke proved that if  $B \subseteq \mathbb{N}$  is a basis of order  $h > 1$ , i.e.,  $\sigma(hB) = 1$ , then  $\sigma(A+B) \geq \sigma(A)^{1-\frac{1}{h}}$  where  $A$  is an arbitrary subset of  $\mathbb{N}$  and  $\sigma$  represents Shnirel'man density. In this paper we explore whether  $\sigma$  can be replaced by other asymptotic densities. We show that Plünnecke's inequality above is true if  $\sigma$  is replaced by lower asymptotic density  $\underline{d}$  or by upper Banach density  $BD$  but not by upper asymptotic density  $\bar{d}$ . The result about  $\underline{d}$  has some interesting consequences such as the inequality  $\underline{d}(A+P) \geq \underline{d}(A)^{2/3}$  for any  $A \subseteq \mathbb{N}$ , where  $P$  is the set of all prime numbers, and the inequality  $\underline{d}(A+C) \geq \underline{d}(A)^{3/4}$  for any  $A \subseteq \mathbb{N}$ , where  $C$  is the set of all cubes of non-negative integers. The result about  $BD$  generalizes [10, Theorem 3] by reducing the requirement of  $B$  being a piecewise basis to the requirement of  $B$  being an upper Banach basis.

## 1 Introduction and a Brief History

Let  $\mathbb{N}$  be the set of all non-negative integers and let  $A$  be a subset of  $\mathbb{N}$ . The *Shnirel'man density* of  $A$  is defined by

$$\sigma(A) = \inf_{n \geq 1} \frac{A(n)}{n}$$

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where  $A(n) = |A \cap [1, n]|$ . A set  $B \subseteq \mathbb{N}$  is called an *essential component* if  $\sigma(A + B) > \sigma(A)$  where  $A + B = \{a + b : a \in A \text{ and } b \in B\}$  for any set  $A \subseteq \mathbb{N}$  with  $0 < \sigma(A) < 1$ . Since the early time of the last century people have been interested in finding which set  $B \subseteq \mathbb{N}$  can be an essential component (cf. [6]). By Shnirel'man's Theorem [6, page 3] it can be easily seen that if  $0 \in B$  and  $\sigma(B) > 0$ , then  $B$  is an essential component. But even if  $\sigma(B) = 0$ ,  $B$  can still be an essential component. A set  $B \subseteq \mathbb{N}$  is called a *basis* of order  $h$  if

$$hB = \underbrace{B + B + \cdots + B}_h = \mathbb{N}.$$

Let  $h > 1$ . Note that  $B$  is a basis of order  $h$  iff  $\sigma(hB) = 1$ . If  $B$  is a basis of some finite order, then  $B$  is an essential component although such  $B$  may have Shnirel'man density 0. For example,  $B = \{n^2 : n \in \mathbb{N}\}$  is a basis of order 4 by Lagrange's Theorem and  $\sigma(B) = 0$ . In 1937 Erdős proved that if  $B$  is a basis of order  $h$ , then

$$\sigma(A + B) \geq \sigma(A) + \frac{1}{2h} \cdot \sigma(A) (1 - \sigma(A)). \quad (1)$$

A short time later, Landau noticed that in Erdős' proof  $h$  can be replaced by average order  $h^*$  (cf. [6, page 10]). Let  $B \subseteq \mathbb{N}$  be a basis of order  $h$ . For each  $m \in \mathbb{N}$  let  $h_B(m) = \min\{h : m \in hB\}$ . The *average order*  $h^*$  of  $B$  is defined by

$$h^* = \sup_{n \geq 1} \frac{1}{n} \sum_{m=1}^n h_B(m).$$

It is easy to see that  $h^* \leq h \leq 2h^*$  (cf. [6, page 12]). In 1938 Rohrbach proved a theorem for lower asymptotic density parallel to Erdős–Landau's result. A set  $B \subseteq \mathbb{N}$  is called an *asymptotic basis* of order  $h$  if  $hB$  contains all sufficiently large positive integers. The *average asymptotic order*  $h^*$  of an asymptotic basis is defined by

$$h^* = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n h_B(m)$$

where  $h_B(m)$  is defined to be 0 if  $m \notin hB$ . The *lower asymptotic density* of  $A$  is defined by

$$\underline{d}(A) = \liminf_{n \rightarrow \infty} \frac{A(n)}{n}.$$

Rohrbach proved (cf. [6, page 45]) that if  $B$  is an asymptotic basis of average asymptotic order  $h^*$ , then

$$\underline{d}(A + B) \geq \underline{d}(A) + \frac{1}{2h^*} \cdot \underline{d}(A) (1 - \underline{d}(A)).$$

In 1970 Plünnecke obtained following significant improvement of Erdős–Landau’s result (cf. [15] or [14, page 225]).

**Theorem 1.1 (Plünnecke, 1970)** *If  $B$  is a basis of order  $h$ , then for every  $A \subseteq \mathbb{N}$*

$$\sigma(A + B) \geq \sigma(A)^{1 - \frac{1}{h}}.$$

One can easily show (cf. [14, page 226]) that  $f(x) \geq g(x)$  for  $0 \leq x \leq 1$  by elementary differential calculus, where  $f(x) = x^{1 - \frac{1}{h}}$  and  $g(x) = x + \frac{1}{h}x(1 - x)$ . Hence Plünnecke’s Theorem implies (1) even if  $2h$  in the denominator is replaced by  $h$ . Erdős–Landau’s result is also a consequence of Theorem 1.1 because of  $h \leq 2h^*$ .

Plünnecke’s approach is completely different from that of Erdős–Landau’s. By analyzing the growth rates of Plünnecke graph at different levels, Plünnecke was able to prove a powerful inequality among these growth rates, which leads to Theorem 1.1. For simplicity we will introduce directly the Plünnecke’s inequality on truncated sumsets instead of on Plünnecke graph. Let  $A \subseteq \mathbb{N}$  and  $a, b \in \mathbb{N}$ . We denote  $A(a, b)$  for the number  $|A \cap [a, b]|$ .

**Theorem 1.2 (Plünnecke, 1957)** *Let  $A, B \subseteq \mathbb{N}$  and  $h, n \in \mathbb{N}$  be such that  $A(0, n) \neq 0$ . For each  $1 \leq i \leq h$  define*

$$D_{A,B,n,i} = \min \left\{ \frac{(A' + iB)(0, n)}{A'(0, n)} : \emptyset \neq A' \subseteq A \cap [0, n] \right\}.$$

*Then*

$$D_{A,B,n,1} \geq (D_{A,B,n,2})^{1/2} \geq \dots \geq (D_{A,B,n,h})^{1/h}.$$

Intuitively,  $D_{A,B,n,i}$  can be viewed as the minimum growth rate of  $A + iB$  over  $A$  truncated at  $n$ . The proof of Theorem 1.2 for general Plünnecke graph and the verification that a truncated additive graph is a Plünnecke graph can be found in [14, Chapter 7].

Can we replace  $h$  in Theorem 1.1 by the average order  $h^*$  similar to what Landau did to Erdős' Theorem? The following simple example dashes this hope.

**Example 1.3** *Let  $A = \{1 + 3n : n \in \mathbb{N}\}$  and  $B = \{3n : n \in \mathbb{N}\} \cup A$ . Then  $A+B = A \cup \{2+3n : n \in \mathbb{N}\}$ ,  $\sigma(A) = \underline{d}(A) = \frac{1}{3}$  and  $\sigma(A+B) = \underline{d}(A+B) = \frac{2}{3}$ . It is easy to check that  $B$  is a basis of order  $h = 2$ , average order  $h^* = \frac{3}{2}$ , and average asymptotic order  $h' = \frac{4}{3}$ . Note that*

$$\underline{d}(A)^{1-\frac{1}{h'}} = \left(\frac{1}{3}\right)^{\frac{1}{4}} \geq \sigma(A)^{1-\frac{1}{h^*}} = \left(\frac{1}{3}\right)^{\frac{1}{3}} > \frac{2}{3} = \sigma(A+B) = \underline{d}(A+B).$$

The author was informed by Georges Grekos that the above example appeared in some article in German around 1953.

Can we derive theorems for other densities parallel to Theorem 1.1 similar to what Rohrbach did to Erdős–Landau's theorem? Yes we can. In fact we can loose the requirement of  $B$  being an asymptotic basis to the requirement of  $B$  being a lower asymptotic basis when  $\sigma$  is replaced by  $\underline{d}$ . We can also derive a parallel theorem for upper Banach density. In fact there is already a theorem about upper Banach density parallel to Theorem 1.1 in [10]. However, the theorem in [10] will be significantly improved in this paper.

Let  $A \subseteq \mathbb{N}$ . The *upper Banach density* of  $A$  is defined by

$$BD(A) = \limsup_{n \rightarrow \infty} \sup_{k \in \mathbb{N}} \frac{|A(k, k+n)|}{n+1}.$$

Upper Banach density is popular among mathematicians who work on combinatorial number theory problems using ergodic methods (cf. [1, 5]). Upper Banach density is also called uniform density in [8]. Note that  $BD(A) = \alpha$

iff  $\alpha$  is the greatest real number such that there exists a sequence of interval  $\{[a_n, b_n] : n \in \mathbb{N}\}$  with

$$\lim_{n \rightarrow \infty} (b_n - a_n) = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{A(a_n, b_n)}{b_n - a_n + 1} = \alpha. \quad (2)$$

In [10] a set  $B \subseteq \mathbb{N}$  is called a *piecewise basis* of order  $h$  if there exists a sequence  $\{c_n : n \in \mathbb{N}\}$  of positive integers such that for each  $n$

$$[0, n] \subseteq h((A - c_n) \cap \mathbb{N}).$$

In [10] it is proved that if  $B$  is a piecewise basis of order  $h$ , then for any set  $A \subseteq \mathbb{N}$

$$BD(A + B) \geqslant BD(A)^{1 - \frac{1}{h}}.$$

Note that a basis of order  $h$  is a piecewise basis of order at most  $h$ . Simply take  $c_n = 0$ . However, the definition of a piecewise basis seems tedious. In this paper we will improve this result by substituting piecewise basis with upper Banach basis.

How about upper asymptotic density? For a set  $A \subseteq \mathbb{N}$  the *upper asymptotic density* is defined by

$$\bar{d}(A) = \limsup_{n \rightarrow \infty} \frac{A(n)}{n}.$$

If  $\underline{d}(A) = \bar{d}(A)$  we call  $d(A) = \underline{d}(A) = \bar{d}(A)$  the *asymptotic density* of  $A$ . Clearly, we have the following inequalities among these densities. For a set  $A \in \mathbb{N}$  we have

$$0 \leqslant \sigma(A) \leqslant \underline{d}(A) \leqslant \bar{d}(A) \leqslant BD(A) \leqslant 1. \quad (3)$$

Although  $\bar{d}$  is in the middle of  $\underline{d}$  and  $BD$  in terms of their magnitudes, the behavior of  $\bar{d}$  is very different from that of  $\underline{d}$  and  $BD$ . The reader will see the difference when we try to generalize Plünnecke's Theorem to these densities.

**Definition 1.4** Let  $B \subseteq \mathbb{N}$ .

- $B$  is called a lower asymptotic basis of order  $h$  if  $\underline{d}(hB) = 1$ .
- $B$  is called an upper asymptotic basis of order  $h$  if  $\overline{d}(hB) = 1$ .
- $B$  is called an upper Banach basis of order  $h$  if  $BD(hB) = 1$ .

Note that if  $\underline{d}(hB) = 1$ , then  $\overline{d}(hB) = 1$ . Hence the density of  $hB$  exists and  $d(hB) = 1$ . This shows that the lower asymptotic basis above should be called asymptotic basis. Unfortunately, the name “asymptotic basis” is already taken. A lower asymptotic basis is sometimes also called a weak basis or Schwarz basis in some literature. Note that a lower asymptotic basis of order  $h$  must be an asymptotic basis of order at most  $2h$ . It was proven in [7] that for every  $h$  there is a lower asymptotic basis  $B$  of order  $h$  such that  $B$  is not an asymptotic bases of order  $2h - 1$ . Note that an asymptotic basis of order  $h$  is a lower asymptotic basis of order at most  $h$ , a lower asymptotic basis of order  $h$  is an upper asymptotic basis of order at most  $h$ , and an upper asymptotic basis of order  $h$  is an upper Banach basis of order at most  $h$ . Note also that a piecewise basis of order  $h$  is an upper Banach basis of order at most  $h$ .

Following are the theorems we will prove in this paper.

**Theorem 1.5** *Let  $A, B \subseteq \mathbb{N}$  and  $B$  be a lower asymptotic basis of order  $h$ . Then*

$$\underline{d}(A + B) \geq \underline{d}(A)^{1 - \frac{1}{h}}.$$

We would like to mention three consequences of Theorem 1.5.

Let  $P$  be the set of all prime numbers. By a result proved independently by Estermann [4], Chudakov, and van der Corput,  $P$  is a lower asymptotic basis of order 3.<sup>1</sup> Hence Theorem 1.5 implies that  $\underline{d}(A + P) \geq \underline{d}(A)^{2/3}$  for every  $A \subseteq \mathbb{N}$ .

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<sup>1</sup> $P$  is an asymptotic basis of order 4 by Vinogradov’s Three Primes Theorem and an asymptotic basis of order 3 if Goldbach Conjecture is true.

Let  $S$  be the set of all squares of integers. Then  $S$  is a basis of order 4 by a well-known theorem of Lagrange. Hence Theorem 1.5 implies that  $\underline{d}(A + S) \geq \underline{d}(A)^{3/4}$  for every  $A \subseteq \mathbb{N}$ .

More interestingly, let  $C$  be the set of all cubes of non-negative integers. Then  $C$  is a lower asymptotic basis of order 4 by a result of Davenport [3]<sup>2</sup>. Hence Theorem 1.5 implies that  $\underline{d}(A + C) \geq \underline{d}(A)^{3/4}$  for every  $A \subseteq \mathbb{N}$ .

**Theorem 1.6** *There are  $A, B \subseteq \mathbb{N}$  with  $\bar{d}(A) = \frac{1}{2}$  and  $\bar{d}(2B) = 1$  such that*

$$\bar{d}(A + B) = \bar{d}(A).$$

**Theorem 1.7** *Let  $A, B \subseteq \mathbb{N}$  and  $B$  be a upper Banach basis of order  $h$ . Then*

$$BD(A + B) \geq BD(A)^{1 - \frac{1}{h}}.$$

Since a piecewise basis of order  $h$  is an upper Banach basis of order at most  $h$  but not vice versa, Theorem 1.7 is clearly an improvement of [10, Theorem 3].

**Theorem 1.8** *There are  $A, B \subseteq \mathbb{N}$  with  $\underline{d}(A) = \frac{1}{2}$  and  $\bar{d}(2B) = 1$  such that*

$$\underline{d}(A + B) = \underline{d}(A).$$

We can define essential component in terms of densities besides  $\sigma$ . For example, we call a set  $B$  an *essential component in terms of  $\bar{d}$*  if  $\bar{d}(A + B) > \bar{d}(A)$  for any  $A \subseteq \mathbb{N}$  with  $0 < \bar{d}(A) < 1$ .

**Theorem 1.9** *A lower asymptotic basis of a finite order is an essential component in terms of upper asymptotic density.*

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<sup>2</sup> $C$  is an asymptotic basis of order 7. The question whether  $C$  can be an asymptotic basis of order less than 7 is still open.

As done before, we would like to use tools from nonstandard analysis in the proofs of Theorem 1.5, Theorem 1.7, and Theorem 1.9. The proofs of Theorem 1.6 and Theorem 1.8 do not involve nonstandard methods. Nonstandard tools have been proven very useful and efficient in, for example, [2, 10, 11, 12]. The reader is recommended to consult one of [2, 9, 10, 13] for the basic notation, ideas, and principles in nonstandard analysis. Other introductory texts for Robinsonian style nonstandard analysis should also be sufficient. If we work within a nonstandard universe, we always assume that the nonstandard universe is countably saturated.

Before starting the proofs we introduce some notation dealing with infinitesimals. Let  $r, s \in {}^*\mathbb{R}$ . By  $r \approx s$  we mean that  $r$  is infinitesimally close to  $s$ , *i.e.*,  $|r - s|$  is less than any positive standard real numbers. By  $r \ll s$  we mean that  $r < s$  but  $r \not\approx s$ . By  $r \lesssim s$  we mean that  $r < s$  or  $r \approx s$ . We define  $r \gg s$  and  $r \gtrsim s$  by a symmetric way.

## 2 Proof of Theorem 1.5

We first prove a nonstandard equivalence of lower asymptotic density by the transfer principle.

**Lemma 2.1** *Let  $A \subseteq \mathbb{N}$ . Then  $\underline{d}(A) \geq \alpha$  iff for every hyperfinite integer  $H$*

$$\frac{{}^*A(0, H)}{H + 1} \gtrsim \alpha. \quad (4)$$

**Proof** Suppose  $\underline{d}(A) \geq \alpha$  and let  $\epsilon$  be a standard positive real number. Then there is  $m \in \mathbb{N}$  such that  $\frac{A(0, n)}{n+1} > \alpha - \epsilon$  for any  $n > m$ . Hence  $\frac{{}^*A(0, H)}{H+1} > \alpha - \epsilon$  for any hyperfinite integer  $H$  by the transfer principle. Since  $\epsilon$  is arbitrary, we have (4).

Suppose (4) is true for every hyperfinite integer  $H$ . Let  $\epsilon$  be a standard positive real number and  $X_\epsilon$  be defined by

$$X_\epsilon = \left\{ n \in {}^*\mathbb{N} : \frac{{}^*A(0, n)}{n + 1} > \alpha - \epsilon \right\}.$$



Then  $X_\epsilon$  is an internal set containing all hyperfinite integers. Hence there is  $m \in \mathbb{N}$  such that  $X_\epsilon$  contains all integers above  $m$ . This means that  $\frac{A(0,n)}{n+1} > \alpha - \epsilon$  for every standard  $n > m$ . Therefore,  $\underline{d}(A) \geq \alpha$ .  $\square$

Let  $A$  and  $B$  be in Theorem 1.5 such that  $\underline{d}(A) = \alpha$  and  $\underline{d}(hB) = 1$ . Without loss of generality, we can assume  $0 < \alpha < 1$ . Let  $H$  be any hyperfinite integer. We want to show that

$$\frac{*(A+B)(0,H)}{H+1} = \frac{(*A+*B)(0,H)}{H+1} \gtrsim \alpha^{1-\frac{1}{h}},$$

which implies Theorem 1.5 by Lemma 2.1. Choose hyperfinite integers  $N < K < H$  such that  $\frac{H-K}{H} \approx 0$  and  $\frac{K-N}{H-N} \approx 0$  (for example  $K = H - \lfloor \sqrt{H} \rfloor$  and  $N = K - \lfloor \sqrt[4]{H} \rfloor$  satisfy the requirements). Let  $C_0 = *A \cap [0, K]$ . Then  $\frac{C_0(0,H)}{H} \gtrsim \alpha$ . Next we want to trim  $C_0$  so that the density of the trimmed set in an interval  $[x, H]$  for every  $x \leq K$  would not be too large. We define  $C_k$  inductively for  $k = 0, 1, \dots, N$  so that  $C_0 \supseteq C_1 \supseteq \dots \supseteq C_N$ ,  $\frac{C_N(0,H)}{H+1} \approx \alpha$ , and  $\frac{C_N(x,H)}{H-x+1} \lesssim \alpha$  for any  $x \leq K$ . We have had  $C_0$ . For each  $k < N$  let

$$C_{k+1} = \begin{cases} C_k, & \text{if } \frac{C_k(N-k,H)}{H-N+k+1} \leq \alpha \\ C_k \setminus \{N-k\}, & \text{otherwise.} \end{cases}$$

It is easy to see that  $C_0, C_1, \dots, C_N$  has the desired properties. Let  $A_0 = C_N$  and

$$D_{A_0, *B, H, h} = \frac{(A' + h*B)(0, H)}{A'(0, H)}$$

for some non-empty internal set  $A' \subseteq A_0$ . Let  $z = \min A'$ . Then  $z \leq K$ . Hence  $H - z$  is hyperfinite, which implies  $\frac{(h*B)(z, H)}{H-z+1} \approx 1$ . Hence by Theorem 1.2 we have

$$\begin{aligned} & \frac{(A_0 + *B)(0, H)}{A_0(0, H)} \\ & \geq D_{A_0, *B, H, 1} \geq (D_{A_0, *B, H, h})^{1/h} = \left( \frac{(A' + h*B)(0, H)}{A'(0, H)} \right)^{1/h} \\ & \gtrsim \left( \frac{(z + h*B)(0, H)}{A'(0, H)} \right)^{1/h} \gtrsim \left( \frac{(h*B)(0, H-z)/(H-z+1)}{A'(z, H)/(H-z+1)} \right)^{1/h} \end{aligned}$$

$$\approx \left( \frac{1}{A_0(z, H)/(H - z + 1)} \right)^{1/h} \approx \frac{1}{\alpha^{1/h}},$$

which implies

$$\frac{*(A + B)(0, H)}{H + 1} \geq \frac{(A_0 + *B)(0, H)}{H + 1} \approx \frac{A_0(0, H)}{H + 1} \cdot \frac{1}{\alpha^{1/h}} \approx \alpha^{1 - \frac{1}{h}}.$$

Since  $H$  is an arbitrary hyperfinite integer, we have proven Theorem 1.5 by Lemma 2.1.

### 3 Proof of Theorem 1.6

In [6, Theorem 12 on page 39] a thin basis of order 2 was constructed by Cassels. We construct an upper asymptotic basis  $B$  here following Cassels' steps and construct  $A$  at the same time so that  $A$  and  $B$  satisfy the desired properties.

Let  $f_0 = 0, f_1 = 1$ , and  $f_{n+2} = f_{n+1} + f_n$  for  $n = 0, 1, 2, \dots$  be Fibonacci sequence. Let  $g_1 = 0$  and  $g_j = \sum_{i=2}^j f_{i-1} f_{i+2}$  for  $j > 1$ . For each  $j = 1, 2, \dots$  let  $G_j = \{g_j + r f_j : 0 \leq r < f_{j+3}\}$ . For each  $j < k$  let  $G_{j,k} = \bigcup_{i=j-1}^{k+2} G_i$ . As a part of a proof it is shown in [6, page 40–41] that

$$[2g_j, 2g_k] \subseteq G_{j,k} + G_{j,k}. \quad (5)$$

Note that  $\min G_{j,k} = g_{j-1}$  and  $\max G_{j,k} = g_{k+3}$ . Note also that for any two elements  $a < b$  in  $G_{j,k}$  we have  $b - a \geq f_{j-1}$ . We now construct  $A$  as the union of intervals  $[a_n, 2a_n]$  and  $B$  as the union of  $G_{j_n, k_n}$ . We select  $a_n, j_n$ , and  $k_n$  alternately by induction on  $n$ .

Fix any positive integer  $a_0$ . Let  $j_0$  be large enough so that  $f_{j_0-1} > 2a_0$ . Let  $k_0 > j_0$  be large enough so that  $g_{k_0} > g_{j_0}^2$ . Suppose we have constructed  $a_i, j_i, k_i$  for  $i = 0, 1, \dots, n$ . Let  $a_{n+1}$  be large enough so that  $a_{n+1} > g_{k_n+3}^2$ . Let  $j_{n+1}$  be large enough so that  $f_{j_{n+1}-1} > a_{n+1}^2 + g_{k_n+3}$  and let  $k_{n+1}$  be large enough so that  $g_{k_{n+1}} > g_{j_{n+1}}^2 + (2a_n)^2$ . Let

$$A = \bigcup_{n=0}^{\infty} [a_n, 2a_n] \text{ and } B = \{0\} \cup \bigcup_{n=0}^{\infty} G_{j_n, k_n}.$$

Now we show that  $A$  and  $B$  constructed above have the desired properties.

Since  $\lim_{n \rightarrow \infty} \frac{a_{n-1}}{a_n} = 0$ , we clearly have  $\bar{d}(A) = \frac{1}{2}$ .

For each  $n$ , by (5) we have

$$\frac{(2B)(0, 2g_{k_n})}{2g_{k_n} + 1} \geq \frac{2(g_{k_n} - g_{j_n}) + 1}{2g_{k_n} + 1} \geq 1 - \frac{g_{j_n}}{g_{k_n}}.$$

Hence  $\bar{d}(2B) = 1$ , *i.e.*,  $B$  is an upper asymptotic basis of order 2.

For verifying  $\bar{d}(A + B) = \frac{1}{2}$  it suffices to show that  $\frac{(A+B)(0,x)}{x+1} \leq \frac{1}{2} + o(1)$  for every  $x \in [a_n, a_{n+1} - 1]$ . Let  $a_n \leq x < g_{j_n-1}$ . Note that  $g_{k_{n-1}+3}^2 \leq a_n$  and  $a_n + g_{k_{n-1}+3} < g_{j_n-1}$ . Then

$$\begin{aligned} \frac{(A+B)(0,x)}{x+1} &\leq \frac{(A+B)(0, 2a_n + g_{k_{n-1}+3})}{2a_n + 1} \\ &\leq \frac{2a_{n-1} + g_{k_{n-1}+3} + a_n + g_{k_{n-1}+3}}{2a_n} \\ &\leq \frac{a_{n-1}}{a_n} + \frac{g_{k_{n-1}+3}}{a_n} + \frac{1}{2} = \frac{1}{2} + o(1). \end{aligned}$$

Let  $g_{k_{n-1}} \leq x < a_{n+1}$ . Note that

$$(A+B) \cap [2a_n + g_{k_{n-1}+3} + 1, a_{n+1} - 1] \subseteq ((A \cap [0, 2a_n]) + G_{j_n, k_n}).$$

Since  $g_{j_n-1} > f_{j_n-1} > a_n^2 + g_{k_{n-1}+3}$  and any two consecutive elements in  $G_{j_n, k_n}$  have a distance  $\geq f_{j_n-1}$ , we have that

$$\begin{aligned} &\frac{(A+B)(0,x)}{x+1} \\ &= \frac{(A+B)(0, g_{j_n-1} - 1)}{x+1} + \frac{(A+B)(g_{j_n-1}, x)}{x+1} \\ &\leq \left(\frac{1}{2} + o(1)\right) \cdot \frac{2a_n}{x+1} + \frac{A(2a_n) \cdot |G_{j_n, k_n} \cap [g_{j_n-1}, x]|}{x+1} \\ &\leq \left(\frac{1}{2} + o(1)\right) \cdot \frac{2a_n}{x+1} + \frac{A(2a_n)(x - g_{j_n-1})}{f_{j_n-1} \cdot (x+1)} + \frac{A(2a_n)}{x+1} \\ &\leq \left(\frac{1}{2} + o(1)\right) \cdot \frac{2a_n}{x+1} + \left(\frac{1}{2} + o(1)\right) \cdot \frac{(x - g_{j_n-1})}{x+1} + o(1) \\ &\leq \frac{1}{2} + o(1). \end{aligned}$$

Hence  $\bar{d}(A) \leq \bar{d}(A+B) \leq \frac{1}{2}$ . This shows  $\bar{d}(A+B) = \frac{1}{2}$ .

## 4 Proof of Theorem 1.7

First we prove a nonstandard equivalence of upper Banach density.

**Lemma 4.1** *Let  $A \subseteq \mathbb{N}$ . Then  $BD(A) \geq \alpha$  iff there is an integer  $n \in {}^*\mathbb{N}$  and a hyperfinite integer  $H$  such that*

$$\frac{{}^*A(n, n+H)}{H+1} \approx \alpha.$$

**Proof** Suppose  $\frac{{}^*A(n, n+H)}{H+1} \approx \alpha$  for some hyperfinite integer  $H$ . For each standard positive real  $\epsilon$  and standard  $m \in \mathbb{N}$ , the statement “there is an interval  $[a, b] \subseteq {}^*\mathbb{N}$  such that  $b - a > m$  and  $\frac{{}^*A(a, b)}{b-a+1} > \alpha - \epsilon$ ” is true in the nonstandard universe. Hence the statement without  $*$  is true in the standard universe by the transfer principle. This means  $BD(A) \geq \alpha$ .

Suppose  $BD(A) \geq \alpha$ . Let  $\mathcal{I} = \{[a_n, b_n] \subseteq \mathbb{N} : n \in \mathbb{N}\}$  be a sequence of intervals such that  $\lim_{n \rightarrow \infty} (b_n - a_n) = \infty$  and  $\lim_{n \rightarrow \infty} \frac{A(a_n, b_n)}{b_n - a_n + 1} = BD(A)$ . Then  $\mathcal{I}$  can be extended to a nonstandard sequence  ${}^*\mathcal{I} = \{[a_n, b_n] \subseteq {}^*\mathbb{N} : n \in {}^*\mathbb{N}\}$ . It is easy to see that for any hyperfinite integer  $N$  we have

$$\frac{{}^*A(a_N, b_N)}{b_N - a_N + 1} \approx BD(A) \geq \alpha.$$

We can now let  $n = a_N$  and  $H = b_N - a_N$  for any hyperfinite integer  $N$ .  $\square$

From the last line of the proof of Lemma 4.1 we see that  $H$  can be chosen arbitrarily large or arbitrarily small in  ${}^*\mathbb{N} \setminus \mathbb{N}$ .

Let  $A$  and  $B$  be in Theorem 1.7. Theorem 1.7 is trivially true if  $BD(A) = 0$  or  $BD(A) = 1$ . Suppose  $0 < \alpha = BD(A) < 1$  and  $BD(hB) = 1$ . We can find a hyperfinite interval  $[n, n+K]$  such that  $[n, n+K] \subseteq (h^*B)$ . Choose  $H$  large enough so that  $\frac{n+K}{H} \approx 0$  and  $\frac{{}^*A(m, m+H)}{H+1} \approx \alpha$  for some  $m \in {}^*\mathbb{N}$ . It suffices to show that

$$\frac{(({}^*A \cap [m, m+H]) + {}^*B)(m, m+H)}{H+1} \approx \alpha^{1-\frac{1}{h}}$$

by Lemma 4.1. Let  $A_0 = (*A \cap [m, m + H - n - K]) - m$ . By the choice of  $H$  and  $A_0$  we have

$$\frac{A_0(0, H)}{H + 1} \approx \alpha \text{ and } \frac{(A_0 + *B)(0, H)}{H + 1} \approx \frac{(*A + *B)(m, m + H)}{H + 1}.$$

It now suffices to show that  $\frac{(A_0 + *B)(0, H)}{H + 1} \gtrsim \alpha^{1 - \frac{1}{h}}$ .

Let  $A' \subseteq A_0$  be non-empty such that  $D_{A_0, *B, H, h} = \frac{(A' + h*B)(0, H)}{A'(0, H)}$ .

**Lemma 4.2**

$$\frac{(A' + h*B)(0, H)}{A'(0, H)} = D_{A_0, *B, H, h} \gtrsim \frac{1}{\alpha}.$$

**Proof** Let  $N = \lfloor K/2 \rfloor$  and let  $I_i = [iN, (i+1)N - 1]$  for  $i = 0, 1, \dots, \lfloor H/N \rfloor - 1$ , and let  $I_{\lfloor H/N \rfloor} = [\lfloor H/N \rfloor \cdot N, H]$ . Suppose

$$\mathcal{I} = \{I_i : i = 0, 1, \dots, \lfloor H/N \rfloor \text{ and } I_i \cap A' \neq \emptyset\}.$$

Then

$$(A' + h*B)(0, H) \geq |\mathcal{I}| \cdot N$$

because  $N \leq K/2$ , every element in  $A'$  is less than or equal to  $H - n - K$ , and  $N + n + I_i \subseteq (A' + h*B) \cap [0, H]$  if  $A' \cap I_i \neq \emptyset$  for every  $i = 0, 1, \dots, \lfloor H/N \rfloor$ . Given a positive standard real  $\epsilon$ , we have

$$A'(0, H) \leq |\mathcal{I}| \cdot (\alpha + \epsilon)N$$

because  $\frac{|A' \cap I_i|}{|I_i|} \lesssim \alpha$  when  $|I_i|$  is hyperfinite by Lemma 4.1. Because  $\epsilon$  is an arbitrary standard positive real number, we have that

$$\frac{(A' + h*B)(0, H)}{A'(0, H)} \gtrsim \frac{|\mathcal{I}| \cdot N}{|\mathcal{I}| \cdot \alpha N} = \frac{1}{\alpha}.$$

□

Combine the arguments above and Theorem 1.2 we now have

$$\begin{aligned} \frac{(A_0 + *B)(0, H)}{A_0(0, H)} &\gtrsim D_{A_0, *B, H, 1} \geq (D_{A_0, *B, H, h})^{1/h} \\ &= \left( \frac{(A' + h*B)(0, H)}{A'(0, H)} \right)^{1/h} \gtrsim \frac{1}{\alpha^{1/h}}. \end{aligned}$$

Hence

$$\frac{*(A+B)(0, H)}{H+1} \gg \frac{(A_0 + *B)(0, H)}{H+1} \gg \frac{A_0(0, H)}{H+1} \cdot \frac{1}{\alpha^{1/h}} \approx \alpha^{1-\frac{1}{h}},$$

which implies Theorem 1.7 by Lemma 4.1.

## 5 Proof of Theorem 1.8

Let  $f_i, g_i$ , and  $G_{j,k}$  be defined in section 3. We construct  $a_n, b_n, j_n, k_n$  inductively on  $n$ . We will let  $A = \bigcup_{n \in \mathbb{N}} [a_n, b_n]$  and  $B = \{0\} \cup \bigcup_{n \in \mathbb{N}} G_{j_n, k_n}$ .

Let  $a_0 = 0, j_0 = 0, k_0 = 0$ , and  $b_n = 10$  (in fact  $b_0$  can be any fixed positive integer). Suppose we have constructed  $a_n, j_n, k_n$ , and  $b_n$ . Let  $a_{n+1} = 2b_n$  (this insures  $\underline{d}(A) = \frac{1}{2}$ ). Let  $j_{n+1}$  be large enough so that  $g_{j_{n+1}-1} > a_{n+1}$ . Let  $k_{n+1}$  be large enough so that  $g_{k_{n+1}} > g_{j_{n+1}}^2$  (this insures  $\bar{d}(2B) = 1$ ). Let  $b_{n+1} > g_{k_{n+1}+3}^2 + b_n$  (this insures that  $A[0, b_n] + G_{j_{n+1}, k_{n+1}} \subseteq [a_{n+1}, b_{n+1}] \subseteq A$ ).

It is now similar to the proof of Theorem 1.6 to verify that  $\underline{d}(A) = \frac{1}{2} = \underline{d}(A+B)$  and  $B$  is an upper asymptotic basis of order 2.

## 6 Proof of Theorem 1.9

Similar to the proofs of Lemma 2.1 and Lemma 4.1, the following nonstandard equivalence of upper asymptotic density can be proven by the transfer principle. The proof is left to the reader.

**Lemma 6.1** *For each  $A \subseteq \mathbb{N}$ ,  $\bar{d}(A) \geq \alpha$  iff there is a hyperfinite integer  $H$  such that*

$$\frac{*A(0, H)}{H+1} \gg \alpha.$$

Let  $A, B \subseteq \mathbb{N}$  be as in Theorem 1.9 such that  $0 < \bar{d}(A) = \alpha < 1$  and  $\underline{d}(hB) = 1$ . By Lemma 6.1 it suffices to find a hyperfinite integer  $H$  such that  $\frac{(*A+*B)(0, H)}{H+1} \gg \alpha$ . By Lemma 6.1 again we can find a hyperfinite integer  $N$  such that  $\frac{*A(0, N)}{N+1} \approx \alpha$ .

**Case 1** For every  $n \in [0, N]$ ,  $0 \ll \frac{n}{N} \ll 1$  implies  $\frac{{}^*A(n, N)}{N-n+1} \approx \alpha$ .

We need to show that  $H = N + \lfloor \sqrt{N} \rfloor$  is what we want. Note that  $\frac{{}^*A(0, H)}{H+1} \approx \frac{{}^*A(0, N)}{N+1} \cdot \frac{N+1}{H+1} + \frac{{}^*A(N, H)}{H+1} \approx \alpha \cdot 1 + 0 = \alpha$ . Since  $\frac{{}^*A(0, N)}{N+1} \approx \alpha$ , we have that  $\frac{{}^*A(0, n)}{n+1} \approx \alpha$  when  $0 \ll \frac{n}{N} \ll 1$ . Similar to the construction of  $C_i$  in the proof of Theorem 1.5 we can construct  $A_0 \subseteq {}^*A \cap [0, N]$  such that  $\frac{A_0(0, H)}{H+1} \approx \alpha$  and  $\frac{A_0(n, H)}{H-n+1} \lesssim \alpha$  for every  $n \in [0, N]$ . Then by Theorem 1.2 we have

$$\begin{aligned} & \frac{(A_0 + {}^*B)(0, H)}{A_0(0, H)} \\ & \geq D_{A_0, {}^*B, H, 1} \geq (D_{A_0, {}^*B, H, h})^{1/h} = \left( \frac{(A' + h {}^*B)(0, H)}{A'(0, H)} \right)^{1/h} \\ & \gtrsim \left( \frac{(z + h {}^*B)(0, H)}{A'(0, H)} \right)^{1/h} \gtrsim \left( \frac{(h {}^*B)(0, H-z)/(H-z+1)}{A'(z, H)/(H-z+1)} \right)^{1/h} \\ & \gtrsim \left( \frac{1}{A_0(z, H)/(H-z+1)} \right)^{1/h} \gtrsim \frac{1}{\alpha^{1/h}} \end{aligned}$$

where  $A'$  is a non-empty subset of  $A_0$  and  $z = \min A'$ . Hence

$$\frac{{}^*(A+B)(0, H)}{H+1} \gtrsim \frac{(A_0 + {}^*B)(0, H)}{H+1} \gtrsim \frac{A_0(0, H)}{H+1} \cdot \frac{1}{\alpha^{1/h}} \approx \alpha^{1-\frac{1}{h}} > \alpha.$$

**Case 2** There is  $n \in [0, N]$  such that  $0 \ll \frac{n}{N} \ll 1$  and  $\frac{{}^*A(n, N)}{N-n+1} \gg \alpha$ .

Let

$$\beta = \inf \left( \{1\} \cup \left\{ st \left( \frac{m}{N} \right) : m \in [0, N], \frac{n}{N} \leq \frac{m}{N} \ll 1, \text{ and } \frac{{}^*A(m, N)}{N-m+1} \lesssim \alpha \right\} \right)$$

where  $st$  is called the standard part map, i.e.,  $st(r)$  is the unique standard real  $\alpha$  such that  $r \approx \alpha$  for each  $r \in {}^*\mathbb{R}$  with  $|r| \leq \gamma$  for some positive standard real number  $\gamma$ . Then  $\beta \gg \frac{n}{N}$ . Let  $H' \in [n, N]$  be such that  $\frac{H'}{N} \approx \beta$ . Now we have that for every  $x \in [n, H']$ ,

$$\frac{n}{H'} \leq \frac{x}{H'} \ll 1 \text{ implies } \frac{{}^*A(x, H')}{H'-x+1} \gg \alpha \quad (6)$$

by the definition of  $\beta$ . Since  $\frac{(h {}^*B)(0, H')}{H'+1} \approx 1$  we can find  $m$  such that  $0 \ll \frac{m}{H'} \leq \frac{H'-n}{H'}$  and  $m \in h {}^*B$ . Let  $m = b_1 + b_2 + \dots + b_h$  where  $b_i \in {}^*B$ . Then there

is  $i \in [1, h]$  such that  $b_i \geq \frac{m}{h}$ . Hence  $\frac{H'-n}{H'} \geq \frac{b_i}{H'} \gg 0$  and  $\frac{n}{H'} \leq \frac{H'-b_i+1}{H'} \ll 1$ . Let  $H = H' + b_i$ . Then by (6) we have

$$\begin{aligned}
& \frac{*(A+B)(0, H)}{H+1} \\
& \approx \frac{*A(0, H') + (b_i + *A)(H'+1, H'+b_i)}{H'+b_i+1} \\
& \approx \frac{*A(0, H')}{H'+1} \cdot \frac{H'+1}{H'+b_i+1} + \frac{*A(H'-b_i+1, H')}{b_i} \cdot \frac{b_i}{H'+b_i+1} \\
& \gg \alpha \frac{H'+1}{H'+b_i+1} + \alpha \frac{b_i}{H'+b_i+1} = \alpha.
\end{aligned}$$

Combining two cases above we have  $\bar{d}(A+B) > \alpha$  by Lemma 6.1.

## 7 Questions

In Theorem 1.9 we proved that a lower asymptotic basis is an essential component in terms of upper asymptotic density. Can we improve that?

**Question 7.1** *Suppose  $B$  is a lower asymptotic basis of order  $h$ . Is it true that  $\bar{d}(A+B) \geq \bar{d}(A)^{1-\frac{1}{h}}$  for every  $A \subseteq \mathbb{N}$ ? If not, what should be the optimal lower bound  $b$  of  $\bar{d}(A+B)$  as a function of  $\bar{d}(A)$ ?*

As a corollary of Theorem 1.5, we have  $\underline{d}(A+P) \geq \underline{d}(A)^{2/3}$ . Can we improve this? More specifically we ask the following question.

**Question 7.2** *Can we find a better lower bound of  $\underline{d}(A+P)$ , i.e., can we find a continuous function  $f(x) \geq x^{2/3}$  for all  $x \in [0, 1]$  and  $f(x) > x^{2/3}$  for some  $x \in [0, 1]$  such that  $\underline{d}(A+P) \geq f(\underline{d}(A))$  for any  $A \subseteq \mathbb{N}$ ?*

## References

- [1] V. Bergelson, *Ergodic Ramsey Theory—an Update, Ergodic Theory of  $\mathbb{Z}^d$  Actions* (edited by M. Pollicott and K. Schmidt), London Math. Soc. Lecture Note Series 228 (1996), 1–61.



- [2] P. Bihani and R. Jin, *Kneser's theorem for upper Banach density*, Journal de theorie des nombres de Bordeaux, **18** (2006), No. 2, 323–343.
- [3] H. Davenport, *On Waring's problem for cubes*, Acta Mathematica, **71** (1939), 123–143.
- [4] T. Estermann, *On Goldbach's problem: proof that almost all even positive integers are sums of two primes*, Proceedings of London Mathematical Society, 44 (1938), 307–314.
- [5] H. Furstenberg, **Recurrence in Ergodic Theory and Combinatorial Number Theory**, Princeton University Press, 1981.
- [6] H. Halberstam and K. F. Roth, **Sequences**, Oxford University Press, 1966
- [7] V. B. Hornfeck and E. Wirsing, *Über die schwache Basisordnung*, Archiv der Mathematik, **7** (1956), 450–452.
- [8] G. Grekos, V. Toma, and J. Tomanová, *A note on uniform or Banach density*, Annales Mathématiques Blaise Pascal, **17** (2010), 153–163.
- [9] C. W. Henson, *Foundations of nonstandard analysis—A gentle introduction to nonstandard extension*, in **Nonstandard Analysis: Theory and Applications**, ed. by N. J. Cutland, C. W. Henson, and L. Arkeryd, Kluwer Academic Publishers 1997.
- [10] R. Jin, *Nonstandard methods for upper Banach density problems*, The Journal of Number Theory, **91** (2001), 20–38.
- [11] R. Jin, *Solution to the inverse problem for upper asymptotic density*, Journal für die reine und angewandte Mathematik, **595** (2006), 121–166.
- [12] R. Jin, *Freiman's inverse problem with small doubling property*, Advances in Mathematics, **216** (2007), No. 2, 711–752.

- [13] T. Lindstrom, *An invitation to nonstandard analysis*, in **Nonstandard Analysis and Its Application**, ed. by N. Cutland, Cambridge University Press 1988.
- [14] M. B. Nathanson, **Additive Number Theory—Inverse Problems and the Geometry of Sumsets**, Springer, 1996.
- [15] H. Plünnecke, *Eine zahlentheoretische Anwendung der Graphentheorie*, *Journal für die reine und angewandte Mathematik*, **234** (1970), 171–183.