On Some Questions of Hrbacek and Di Nasso

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Abstract. We answer the first two of the five questions posed by Mauro Di Nasso and Karel Hrbacek in their recent paper about combinatorial principles in nonstandard analysis.

Introduction

In [**D**] and [**DH**] several interesting combinatorial principles for nonstandard analysis are introduced. These principles are closely related to the model theoretic properties for nonstandard universes introduced and studied in [**H**], [**R**], [**J1**], [**J2**], etc. They can also be viewed as the analogues of forcing axioms in set theory.

At the end of [**DH**], five questions concerning these principles are posed. The primary purpose of this article is to give solutions to the first two of these question.

Although the reader can find all necessary notations and definitions from [**D**] and [**DH**] for this article we would like to repeat them here for the reader's convenience. All the definitions involving the combinatorial principles are due to Di Nasso and Hrbacek.

Let X be a infinite set of atoms, *i.e.*, we assume each element in X has no members[1]. A superstructure based on X is the structure $(V(X);\in)$, where

$$V_0(X) = X,$$

$$V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X)),$$

and

$$V(X) = \bigcup_{n\in\omega} V_n(X).$$

For convenience we often call $V(X)$ a superstructure without mentioning $\in$. A formula $\varphi(\bar{x})$ in the language $\{\in\}$ is called a bounded formula if it belongs to the smallest set of $\{\in\}$–formulas containing all atomic formulas and closed under logical connectives and bounded quantifiers $\forall x \in y$ and $\exists x \in y$. By a nonstandard universe we mean a triple

$$\mathcal{U} = (V(X), V(Y), *),$$

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[1] In fact, what we need is that for any distinct $x, y \in X$, $x \cap y = \emptyset$.

where $* : V(X) \mapsto V(Y)$ is a bounded elementary embedding with ${}^*X = Y$, *i.e.*, for any bounded formula $\varphi(\bar{x})$ and for any $\bar{a}$ in $V(X)$,

$$V(X) \models \varphi(\bar{a}) \text{ iff } V(Y) \models \varphi({}^*\bar{a}),$$

where ${}^*s$ represents the image of s under the map $*$.[2] Every element in $\bigcup_{n\in\omega} {}^*V_n(X)$ is called a $\mathcal{U}$–internal set. We often call a $\mathcal{U}$–internal set just an internal set when the nonstandard universe $\mathcal{U}$ we are working within is clear. For a model theorist, a nonstandard universe can be viewed as a proper elementery extension of the superstructure $(V(X), \in)$ trancated at rank ω. Hence we write $|\mathcal{U}|$ for the cardinality of the collection of all internal sets in $\mathcal{U}$ and write $a \in \mathcal{U}$ for a being an internal set in $\mathcal{U}$.

Let $\mathbf{V}$ be the set theoretic universe. In set theory forcing methods are used to provide the proofs of the independence of some mathematical statements. The main idea is to expand $\mathbf{V}$ to a larger set theoretic universe $\mathbf{V}[G]$ by including the generic filter G of some partial order $\mathbb{P} \in \mathbf{V}$. If $\mathbb{P} \in \mathbf{V}$ is separable and $\mathcal{D} \in \mathbf{V}$ is the collection of all dense subsets of $\mathbb{P}$ in $\mathbf{V}$, then one can assume that there is a $\mathcal{D}$–generic filter $G \subseteq \mathbb{P}$, *i.e.*, G is a filter and has a non-empty intersection with every $D \in \mathcal{D}$. It is easy to show that the $\mathcal{D}$–generic filter G is not in $\mathbf{V}$.

Suppose we construct a nonstandard universe $\mathcal{U}$ in $\mathbf{V}$. Let $\mathbb{P}$ be an internal separative partial order and let $\mathcal{D}_{\mathbb{P}}$ be the internal collection of all internal dense subsets of $\mathbb{P}$. Then there is a $\mathcal{D}_{\mathbb{P}}$–generic filter $G \subseteq \mathbb{P}$. By the transfer principle, G cannot be internal. In general, G is not in $\mathbf{V}$. However, if the nonstandard universe $\mathcal{U}$ is specifically constructed, then G can exist in $\mathbf{V}$.

Let Δ_0 represent the following statement: For any internal partial order $\mathbb{P}$, there is a $\mathcal{D}_{\mathbb{P}}$–generic filter $G \subseteq \mathbb{P}$ (in $\mathbf{V}$).

A nonstandard universe satisfying Δ_0 exists. In fact, one can require the nonstandard universe to be κ–saturated for any given cardinal κ. A nonstandard universe is κ–saturated if for any internal set A and any collection $\mathcal{C}$ of less than κ internal subsets of A with the finite intersection property, the intersection of $\mathcal{C}$ is not empty. Let $\Delta_0(\kappa)$ be Δ_0 plus κ–saturation. In [**D**], it is shown that $\Delta_0(\kappa)$ is a consequence of the κ–isomorphism property. The existence of a nonstandard universe satisfying the κ–isomorphism property has been known for a long time [**H**].

In set theory, the forcing axioms are often used for constructing a generic function. For example, a Cohen real is just a generic function from ω to 2. Similarly, $\Delta_0(\kappa)$ also has an equivalent form in terms of functions.

Let $\mathcal{F}$ be an internal collection of internal functions. Let $D_{\mathcal{F}} = dom(\mathcal{F}) = \bigcup\{dom(f) : f \in \mathcal{F}\}$ and $R_{\mathcal{F}} = range(\mathcal{F}) = \bigcup\{range(f) : f \in \mathcal{F}\}$. The collection $\mathcal{F}$ is called extendible if for any $f \in \mathcal{F}$ and any $a \in D_{\mathcal{F}}$, there exists a $g \in \mathcal{F}$ such that $a \in dom(g)$ and $f \subseteq g$.

For an extendible $\mathcal{F}$ and a regular cardinal κ, a partial function F_0 from $D_{\mathcal{F}}$ to $R_{\mathcal{F}}$ is called a κ–partial path if for any $D_0 \subseteq F_0$ with $|D_0| < \kappa$, there is an $f \in \mathcal{F}$

[2] There are many different approaches for constructing a nonstandard universe. In order to have the transfer principle, a nonstandard universe should be an "elementary" extension of a standard structure containing enough standard mathematical objects such as the real field $\mathbb{R}$, the real-valued functions on $\mathbb{R}$, the functionals on the set of functions on $\mathbb{R}$, etc. The advantages for choosing the superstructure approach are (1) $V(X)$ is simple but big enough to contain most of familiar standard mathematical objects, (2) the interpretation of the symbol $\in$ in $V(Y)$ is the real membership relation in set theory.

such that $F_0 \upharpoonright D_0 \subseteq f$. A κ–partial path F from $D_{\mathcal{F}}$ to $R_{\mathcal{F}}$ is called a κ–path if $dom(F) = D_{\mathcal{F}}$. A κ–path $F : D_{\mathcal{F}} \mapsto R_{\mathcal{F}}$ is called a strong κ–path if for any $D_0 \subseteq D_{\mathcal{F}}$ with $|D_0| < \kappa$, there is an $f \in \mathcal{F}$ such that $F \upharpoonright D_0 \subseteq f \subseteq F$. By a partial path, a path, or a strong path we mean an $\aleph_0$–partial path, an $\aleph_0$–path, or a strong $\aleph_0$–path, respectively. Note that partial path, path, or strong path are usually not internal.

The following theorem is [**DH**, Theorem 1.2], which gives the equivalence between $\Delta_0(\kappa)$ and a statement involving functions, path, and strong path.

THEOREM 0.1 (M. Di Nasso and K. Hrbacek). *The following are equivalent:*
(i) $\Delta_0(\kappa)$.
(ii) For any internal extendible $\mathcal{F}$ *and any partial path* F_0 *from* $D_{\mathcal{F}}$ *to* $R_{\mathcal{F}}$ *with* $|F_0| < \kappa$*, there exists a* κ*–path* $F : D_{\mathcal{F}} \mapsto R_{\mathcal{F}}$ *extending* F_0*.*
(iii) For any internal extendible $\mathcal{F}$ *and any partial path* F_0 *from* $D_{\mathcal{F}}$ *to* $R_{\mathcal{F}}$ *with* $|F_0| < \kappa$*, there exists a strong* κ*–path* $F : D_{\mathcal{F}} \mapsto R_{\mathcal{F}}$ *extending* F_0*.*

In the theorem above, the cardinal κ occurs twice in each of *(ii)* and *(iii)*. The first occurrence is for $|F_0| < \kappa$ and the second is for F being a (strong) κ–path. It is natural to ask if F_0 can be omitted in the first occurrence or "F being a (strong) κ–path" can be replaced simply by "F being a (strong) path".

1. Question 1

The first of the five questions in [**DH**] is about possibility of eliminating the second occurrence of κ in (ii) and (iii) of Theorem 0.1. We split the question into two questions. One for (ii) and one for (iii). The reason for the splitting is obvious: They have different answers.

Let $\Delta_0'(\kappa)$ be the statement: For any internal extendible $\mathcal{F}$ and any partial path F_0 from $D_{\mathcal{F}}$ to $R_{\mathcal{F}}$ with $|F_0| < \kappa$, there exists a path $F : D_{\mathcal{F}} \mapsto R_{\mathcal{F}}$ extending F_0. Let $\Delta_0'(\kappa)_s$ be same as $\Delta_0'(\kappa)$ plus requiring F being a strong path. Note that $\Delta_0'(\kappa)$ and $\Delta_0'(\kappa)_s$ are same as (ii) and (iii), respectively, except requiring F being a κ–path.

QUESTION 1.1. *Does* $\Delta_0'(\kappa)$ *imply* κ*–saturation?*

QUESTION 1.2. *Does* $\Delta_0'(\kappa)_s$ *imply* κ*–saturation?*

The answer for Question 1.1 is No and the answer to Question 1.2 is Yes. The following are the proofs.

THEOREM 1.3. $\Delta_0'(\kappa)$ *does not imply* κ*–saturation.*

PROOF. Let $\langle \mathcal{U}_n : n \in \omega \rangle$ be an elementary chain of nonstandard universes such that $|\mathcal{U}_0| \geqslant \kappa$ and $\mathcal{U}_{n+1}$ is $|\mathcal{U}_n|^+$–saturated. Let $\mathcal{U}$ be the union of the chain. Clearly, $\mathcal{U}$ is not ω_1–saturated. Next we need to show that $\mathcal{U}$ satisfies $\Delta_0'(\kappa)$.

Let $\mathcal{F}$ be an internal and extendible family of functions and let D be the domain of $\mathcal{F}$. Without loss of generality, we can assume that $\mathcal{F} \in \mathcal{U}_0$. Let $D_n = D \cap \mathcal{U}_n$. Hence D_n for $n = 0, 1, 2, \ldots$ is an increasing sequence of sets. Suppose F_0 is a partial path of $\mathcal{F}$ with $|F_0| < \kappa$. Let $F_0^n = F_0 \cap \mathcal{U}_n$. By saturation, one can construct a sequence of functions $f_n \in \mathcal{F} \cap \mathcal{U}_{n+1}$ inductively on n such that $f_0 = \emptyset$, $F_0^n \subseteq f_n$, $D_n \subseteq dom(f_n)$, and $f_{n-1} \cap U_{n-1} \subseteq f_n$. Note that if $(a, b) \in F_0$ and $(a, b) \in \mathcal{U}_{n-1}$, then $(a, b) \in f_{n-1}$. Let

$$F = \bigcup_{n \in \omega} (f_n \cap U_n).$$

It is easy to see that F is a function and extends F_0.

Claim 1.3.1: F is a path.

Proof of Claim 1.3.1: Clearly, $D = \bigcup_{n\in\omega} D_n \subseteq dom(F)$. Let a be a finite subset of D. Then there is an n such that $a \subseteq D_n$. Hence

$$F \restriction a = f_n \restriction a \subseteq f_n.$$

This ends the proof of the claim and the theorem. In fact, we proved that $\Delta_0'(\kappa)$ does not imply $\aleph_1$–saturation for any κ.

$\square$

THEOREM 1.4. *$(\Delta_0'(\kappa))_s$ implies κ–saturation. Hence $(\Delta_0'(\kappa))_s$ is equivalent to $\Delta_0(\kappa)$.*

PROOF. Assume $(\Delta_0'(\kappa))_s$ is true. Let $\{A_i : i \in I\}$ with $|I| < \kappa$ be a family of internal subsets of an internal set A with the finite intersection property. We need to show that

$$\bigcap_{i\in I} A_i \neq \emptyset.$$

Let

$$\mathcal{F} = \{f \in Fun({}^*I, {}^*\mathcal{P}(A)) : dom(f) = {}^*I \text{ and } \bigcap range(f) \neq \emptyset\},$$

where $Fun(A, B)$ is the set of all internal partial functions from A to B.[3] Then $\mathcal{F}$ is internal and trivially extendible. Let $F_0 = \{(i, A_i) : i \in I\}$. Then $|F_0| < \kappa$.

Claim 1.4.1 F_0 is a partial path.

Proof of Claim 1.4.1: Let a be a finite subset of I. Let f_a be an internal function defined as $f_a(i) = A_i$ when $i \in a$ and $f_a(i) = A$ when $i \in {}^*I \smallsetminus a$. Then

$$\bigcap range(f_a) = \bigcap_{i\in a} A_i \neq \emptyset.$$

So $f_a \in \mathcal{F}$ and $F_0 \restriction a \subseteq f_a$. $\square$ (Claim 1.4.1)

By $(\Delta_0'(\kappa))_s$, the family $\mathcal{F}$ has a strong path $F \supseteq F_0$. Let i be a single element in I. Then there is an $f \in \mathcal{F}$ such that

$$F \restriction \{i\} \subseteq f \subseteq F.$$

Since $dom(f) = {}^*I$ and $f \subseteq F$, then $f \restriction I = F_0$. Hence

$$\bigcap_{i\in I} A_i \supseteq \bigcap range(f) \neq \emptyset.$$

This ends the proof of the theorem.

$\square$

[3] In fact, ${}^*I$ can be replaced by any internal set X with $I \subseteq X$.

2. Question 2

The second question in [**DH**] is about possibility of eliminating F_0 in (ii) and (iii) of Theorem 0.1.

Let $\Delta_0''(\kappa)$ be the statement: For any internal extendible $\mathcal{F}$, there exists a strong κ–path $F : D_{\mathcal{F}} \mapsto R_{\mathcal{F}}$.

QUESTION 2.1. *Does $\Delta_0''(\kappa)$ imply κ–saturation?*

In [**DH**] the definition of $\Delta_0''(\kappa)$ contains two cases. One of the cases requires that F be a strong κ–path and another case does not. We state the definition of $\Delta_0''(\kappa)$ by requiring that F be a strong κ–path because even for that case, Question 2.1 has a negative answer.

THEOREM 2.2. *Let $\kappa > 2^\omega$. Then $\Delta_0''(\kappa)$ does not imply κ–saturation.*

PROOF. Let $\kappa > \beth_\omega$ be a regular cardinal. We prove the theorem in two steps. First we construct a nonstandard universe $\mathcal{U}$ such that in $\mathcal{U}$ every internal extendible $\mathcal{F}$ has a strong κ–path. Then we extend $\mathcal{U}$ to a nonstandard universe $\mathcal{W}$ such that on one hand, every internal extendible $\mathcal{F}$ in $\mathcal{W}$ still has a strong κ–path and, on the other hand, $\mathcal{W}$ is not $(2^\omega)^+$–saturated. The nonstandard universe $\mathcal{W}$ is in fact an ultrapower of $\mathcal{U}$ modulo an ultrafilter over a countable set.

Claim 2.2.1 Let $\mathcal{U}$ be a nonstandard universe. Suppose $\mathcal{F} \in \mathcal{U}$ is an internal and extendible family of functions and $\mathcal{F}_0$ is a linearly ordered (maybe external) subset of $\mathcal{F}$, i.e. $f \subseteq g$ or $g \subseteq f$ for any $f, g \in \mathcal{F}_0$. Then $\mathcal{U}$ has an elementary extension $\mathcal{U}'$ such that $|\mathcal{U}'| = |\mathcal{U}|$ and in $\mathcal{U}'$ there is an $f \in \mathcal{F}$ with $\bigcup \mathcal{F}_0 \subseteq f$ and $dom(\mathcal{F}) \cap \mathcal{U} \subseteq dom(f)$.

We leave the proof of Claim 2.2.1 to the reader. The proof is straightforward. For example, it can be done by a combination of a compactness argument and Löwenheim–Skolem Theorem in model theory.

Claim 2.2.2 Let $\kappa > \beth_\omega$ be a regular cardinal and let $\mathcal{U}$ be a nonstandard universe such that every infinite internal subset has cardinality κ. Then $\mathcal{U}$ has an elementary extension $\mathcal{U}'$ such that $|\mathcal{U}'| = \kappa$ and every internal extendible $\mathcal{F}$ in $\mathcal{U}'$ has a strong κ–path.

Proof of Claim 2.2.2: The proof is a routine bookkeeping argument.

Fix a surjection $i : \kappa \mapsto \kappa \times \kappa$ such that $i(\alpha) = (\beta, \gamma)$ implies $\alpha > \beta$ except $i(0) = (0, 0)$ and for each $(\beta, \gamma) \in \kappa \times \kappa$, there are cofinally many $\alpha \in \kappa$ such that $i(a) = (\beta, \gamma)$. (i can be constructed by the following argument: Let $i_0 : \kappa \mapsto \kappa \times \kappa \times \kappa$ be a bijection with $i_0(0) = (0, 0, 0)$. Let

$$i(\alpha) = \begin{cases} (\beta, \gamma) & \text{if } i_0(\alpha) = (\beta, \gamma, \delta) \text{ for some } \delta \in \kappa \text{ and } \alpha > \beta \\ (0, 0) & \text{otherwise} \end{cases}$$

It is easy to check that the function i defined above is what we need.)

We now construct inductively a continuous elementary chain of the nonstandard universes $\langle \mathcal{U}_\alpha : \alpha < \kappa \rangle$ with $|\mathcal{U}_\alpha| = \kappa$ for every $\alpha < \kappa$. Let $\mathcal{U} = \mathcal{U}_0$. Every time when $\mathcal{U}_{\beta+1}$ is constructed, we always fix an enumeration $\{\mathcal{F}_{\beta,\gamma} : \gamma < \kappa\}$ of all internal and extendible families of functions in $\mathcal{U}_{\beta+1} \smallsetminus \mathcal{U}_\beta$. During the construction, we also find a function $f_{\alpha,\beta,\gamma} \in \mathcal{F}_{\beta,\gamma} \cap U_{\alpha+1}$ for any $\alpha, \beta, \gamma \in \kappa$ with $i(\alpha) = (\beta, \gamma)$ such that $dom(\mathcal{F}_{\beta,\gamma}) \cap U_\alpha \subseteq dom(f_{\alpha,\beta,\gamma})$ and if $\alpha' < \alpha$ and $i(\alpha) = i(\alpha') = (\beta, \gamma)$, then $f_{\alpha',\beta,\gamma} \subseteq f_{\alpha,\beta,\gamma}$. In the end, let $\mathcal{U}'$ be the union of the chain and we will show that $\mathcal{U}'$ satisfies $\Delta_0''(\kappa)$. Note that if α is a limit ordinal and $\mathcal{U}_{\alpha'}$ are constructed for all $\alpha' < \alpha$, then $\mathcal{U}_\alpha$ is just the union of $\{\mathcal{U}_{\alpha'} : \alpha' < \alpha\}$. So assume that $\mathcal{U}_{\alpha'}$ is constructed for every $\alpha' \leqslant \alpha$ and we need to construct $\mathcal{U}_{\alpha+1}$.

Let $i(\alpha) = (\beta, \gamma)$. We now apply Claim 2.2.1 to the case for $\mathcal{U} = \mathcal{U}_\alpha$, $\mathcal{F} = \mathcal{F}_{\beta,\gamma}$, and $\mathcal{F}_0 = \{f_{\alpha',\beta,\gamma} : \alpha' < \alpha \text{ and } i(\alpha') = (\beta,\gamma)\}$. Following the claim, let $\mathcal{U}_{\alpha+1} = \mathcal{U}'$ and $f_{\alpha,\beta,\gamma} = f$. This finishes the construction.

Now we show that the claim follows from the construction. Let $\mathcal{F}$ be an internal extendible family of functions in $\mathcal{U}'$. Let $\beta = \min\{\alpha < \kappa : \mathcal{F} \in \mathcal{U}_{\alpha+1}\}$. Then there is a $\gamma < \kappa$ such that $\mathcal{F} = \mathcal{F}_{\beta,\gamma}$. Let $\Gamma = \{\alpha < \kappa : i(\alpha) = (\beta,\gamma)\}$. Then Γ is unbounded in κ. Let $F = \bigcup\{f_{\alpha,\beta,\gamma} : \alpha \in \Gamma\}$. Then F is a function and $dom(F) = \bigcup\{dom(f_{\alpha,\beta,\gamma}) : \alpha \in \Gamma\} \supseteq \bigcup\{dom(\mathcal{F}_{\beta,\gamma}) \cap \mathcal{U}_\alpha : \alpha \in \Gamma\} = dom(\mathcal{F})$. Let $D \subseteq dom(\mathcal{F})$ be a set with $|D| < \kappa$. Then there is an $\alpha \in \Gamma$ such that $D \subseteq \mathcal{U}_\alpha$. By the construction, $D \subseteq dom(f_{\alpha,\beta,\gamma})$. Hence $F \restriction D \subseteq f_{\alpha,\beta,\gamma} \subseteq F$. Note that $f_{\alpha,\beta,\gamma} \in \mathcal{F}$. So F is a strong κ–path of $\mathcal{F}$. $\square$ (Claim 2.2.2)

Claim 2.2.1 and Claim 2.2.2 finish the first step. Notice that in Claim 2.2.2 the requirement for every infinite internal set in $\mathcal{U}$ having cardinality κ is not necessary. It is only for the notational convenience. Next we do the second step.

Claim 2.2.3 Assume that $\mathcal{U}$ is a nonstandard universe such that $|\mathcal{U}| = \kappa$ and every internal extendible $\mathcal{F}$ in $\mathcal{U}$ has a strong κ–path. Let $\mathcal{D}$ be the ultrafilter on ω and let $\mathcal{W} = \mathcal{U}^\omega/\mathcal{D}$, *i.e.* $\mathcal{W}$ is an ultrapower of U modulo an ultrafilter $\mathcal{D}$ on ω. Then every internal extendible $\mathcal{F}$ in $\mathcal{W}$ has a strong κ–path.

Proof of Claim 2.2.3: Note that a is an internal set in $\mathcal{W}$ iff there is a sequence of $\mathcal{U}$–internal sets $\langle a_n : n \in \omega\rangle$ such that $a = [\langle a_n : n \in \omega\rangle]$, where $[x]$ means the class of all elements equivalent to x modulo $\mathcal{D}$. Let $\mathcal{F}$ be an internal extendible family of functions in $\mathcal{W}$. Then there exists a sequence of $\mathcal{U}$–internal extendible family of functions $\langle \mathcal{F}_n : n \in \omega\rangle$ such that

$$\mathcal{F} = [\langle \mathcal{F}_n : n \in \omega\rangle].$$

For each $n \in \omega$ let F_n be a strong κ–path for $\mathcal{F}_n$ and let

$$F = [\langle F_n : n \in \omega\rangle].$$

Note that F_n may not be internal in $\mathcal{U}$. Hence F may not be internal in $\mathcal{W}$. However, F is well defined in the ultrapower of the set theoretic universe modulo $\mathcal{D}$. We now show that F is a strong κ–path of $\mathcal{F}$.

For each $x = [\langle x_n : n \in \omega\rangle] \in dom(\mathcal{F})$, one can assume $x_n \in dom(\mathcal{F}_n)$. This implies $x \in dom(F)$ and hence $dom(F) = dom(\mathcal{F})$.

Let $X = \{x_\alpha : \alpha < \lambda\} \subseteq dom(F)$ for some cardinal $\lambda < \kappa$. We need to find an $f \in \mathcal{F}$ such that $X \subseteq dom(f)$ and $f \subseteq F$.

For each $\alpha < \lambda$ let $x_\alpha = [\langle x_{\alpha,n} : n \in \omega\rangle]$. For each $n \in \omega$ let $X_n = \{x_{\alpha,n} : \alpha < \lambda\}$. Then $X_n \subseteq dom(F_n)$ and $|X_n| \leqslant \lambda$. Since F_n is a strong κ–path for $\mathcal{F}_n$, there exists an $f_n \in \mathcal{F}_n$ such that $X_n \subseteq dom(f_n)$ and $f_n \subseteq F_n$. Let $f = [\langle f_n : n \in \omega\rangle]$. Since each f_n is an internal element in $\mathcal{F}_n$, f is an internal element in $\mathcal{F}$. From the construction of f, it is clear now that $X \subseteq dom(f)$ and $f \subseteq F$. $\square$ (Claim 2.2.3)

To complete the proof of Theorem 2.2, we need only point out that in any ultrapower of the set $\mathbb{N}$ of all standard natural numbers modulo an ultrafilter on ω, a decreasing sequence of hyperfinite integers has the length at most 2^ω. Hence $\mathcal{W}$ is not $(2^\omega)^+$–saturated. So we actually proved that $\Delta_0''(\kappa)$ does not imply $(2^\omega)^+$–saturation.

□

REMARK 2.3. (1) If *CH* is true, then the nonstandard universe $\mathcal{W}$ constructed in Theorem 2.2 is not $\aleph_2$ saturated.

(2) It is consistent with *ZFC* that $2^\omega > \aleph_1$ and there exists an ultrafilter $\mathcal{D}$ on ω such that the interval $[0, H]$ as an ordered set for a hyperfinite integer H in $\mathbb{N}^\omega/\mathcal{D}$ is not $\aleph_2$–saturated. Hence it is consistent with *ZFC* that $2^\omega > \aleph_1$ and the nonstandard universe $\mathcal{W}$ constructed in Theorem 2.2 is not $\aleph_2$ saturated.

(3) We don't know whether or not $\Delta_0''(\kappa)$ for a regular cardinal $\kappa \geqslant \aleph_1$ implies $\aleph_1$–saturation.

3. Three Remaining Questions

The interested reader should consult [**DH**] for the other three questions not answered here. Karel Hrbacek once commented that Question 3 in [**DH**] is the most interesting one.

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