Applications of Nonstandard Analysis in Additive Number Theory

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Abstract

This paper reports recent progress in applying nonstandard analysis to additive number theory, especially to problems involving upper Banach density.

1 Prologue

In this paper we use $\mathbf{N}$ for the set of all natural numbers. For any two integers a and b, $[a,b]$ denotes the interval of all integers x between a and b ($a \leqslant x \leqslant b$). If S is an internal set, $|S|$ denotes the internal cardinality of S. The letters A, B, C will be used for sets of natural numbers and the letters i, h, k, m, n will be used for natural numbers. For a set A of natural numbers, the Shnirel'man density $\sigma(A)$, the lower density $\underline{d}(A)$, the upper density $\bar{d}(A)$ and the Banach density $BD(A)$ of A are defined as follows.

$$\sigma(A) = \inf_{n \geqslant 1} \frac{|A \cap [1,n]|}{n},$$

$$\underline{d}(A) = \liminf_{n \to \infty} \frac{|A \cap [1,n]|}{n},$$

$$\bar{d}(A) = \limsup_{n \to \infty} \frac{|A \cap [1,n]|}{n},$$

$$BD(A) = \lim_{n \to \infty} \sup_{m-k=n} \frac{|A \cap [k,m]|}{m-k+1}.$$

In other literature, $BD(A)$ is called the upper Banach density of A. I omit the word "upper" due to the lack of interest in "lower" Banach density. Clearly, the following inequalities are true.

$$0 \leqslant \sigma(A) \leqslant \underline{d}(A) \leqslant \bar{d}(A) \leqslant BD(A) \leqslant 1$$

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for every set A.

This paper is organized as follows. First two recent results are stated, for the purpose of showing the reader the general nature of the applications that are being obtained. The rest of the paper is divided into two parts. One part emphasizes topological aspects of this work and the other emphasizes the measure theoretic aspects. The first two results are easy to state and easy to understand. The reader is encouraged to try to find a relatively short (say, within one page) proof of each result before going further, without using nonstandard analysis, of course. It seems to be difficult to find such proofs, and this fact is part of what lends interest to the methods that are reported on in this paper. In my opinion, the nonstandard methods do offer powerful, convenient, and very efficient tools when dealing with Banach density.

Before stating the first two results in Theorem 1 and Theorem 2, some definitions and notations are needed. Given sets A, B and numbers h, k, let $A \pm B = \{a \pm b : a \in A, b \in B\}$, let $A \pm k = A \pm \{k\}$ and let

$$hA = \{a_1 + a_2 + \cdots + a_h : a_1, a_2, \ldots, a_h \in A\}.$$

A set A is called *thick* if A contains k consecutive integers for every $k \in \mathbf{N}$. A set A is called *piecewise syndetic* if $A + [0, k]$ is thick for some $k \in \mathbf{N}$.

Theorem 1 *For any sets A, B, if $BD(A) > 0$ and $BD(B) > 0$, then $A + B$ is piecewise syndetic.*

Note that it is not hard to construct a set $A_r \subseteq \mathbf{N}$ for every real number $r < 1$ such that $\sigma(A_r) > r$ and A_r is not piecewise syndetic. The idea of the construction, see [8], is similar to the construction of a Cantor set.

Theorem 2 *For any sets A, B, $BD(A + B + \{0, 1\}) \geqslant \min\{BD(A) + BD(B), 1\}$.*

In Theorem 2, $\{0, 1\}$ can be replaced by $\{c, c+1\}$ for any $c \in \mathbf{N}$. Without $\{0, 1\}$, the inequality in Theorem 2 will not be true because one can find a counter-example by letting $A = B$ be the set of all even integers.

Theorem 1 and Theorem 2 each imply a Banach version of Shnirel'man's theorem. A set B is called a *basis of order h* if $hB = \mathbf{N}$. B is called a *basis* if B is a basis of order h for some $h \in \mathbf{N}$. Shnirel'man's theorem (cf. [4] or [13]) says that for every set A, if $\sigma(A) > 0$ and $0 \in A$, then A is a basis. A set B is called a *Banach basis of* *order* h if hB is thick. B is called a *Banach basis* if B is a Banach basis of order h for some $h \in \mathbf{N}$.

Theorem 3 *Let A be a set. If $BD(A) > 0$ and A contains two consecutive numbers, then A is a Banach basis.*

Note that the condition "A contains two consecutive numbers" can be replaced by "the greatest common divisor of all positive integers in $A - a_0$, where a_0 is the least element in A, is 1" because the latter is equivalent to "there exists an $h \in \mathbf{N}$ such that hA contains two consecutive numbers".

The proof of Theorem 3 from Theorem 1 is easy. Since $A+A$ is piecewise syndetic, there is a $k \in \mathbf{N}$ such that $A+A+[0,k]$ is thick. Obviously, $A+A+[n,n+k]$ is thick for any $n \in \mathbf{N}$. Since A contains two consecutive numbers, kA contains an interval $[n, n+k]$ for some n.

Using Theorem 2, one can give a better proof of Theorem 3. Since $A \supseteq \{c, c+1\}$ for some integer c, one has

$$BD(3A) \geqslant BD(A + A + \{c, c+1\}) \geqslant \min\{2BD(A), 1\}.$$

Applying Theorem 2 repeatedly and using the fact that A is thick iff $BD(A) = 1$, one can show that if $BD(A) > 0$, then A is a Banach basis of order $2\lceil \frac{1}{BD(A)} \rceil - 1$, where $\lceil \alpha \rceil$, for every real number α, is the least integer greater than or equal to α. This result is also optimal. Let $A = \{0, 1, 10, 11, 20, 21, \ldots\}$. Then $BD(A) = \frac{1}{5}$. A is a Banach basis of order 9, but not order 8.

I consider the proof of Theorem 3 via Theorem 1 to be topological and via Theorem 2 to be measure-theoretic. What I mean by that should be clear in the next two sections.

2 Topological Aspects

The proofs of all theorems in this section can be found in [6].

First, I would like to mention how Theorem 1 is motivated and what is the connection between Theorem 1 and other research. Thick sets, syndetic sets and piecewise syndetic sets are popular objects in combinatorial number theory. Motivation for Theorem 1 comes in part from the following result proved in [3]: if $BD(A) > 0$, then

$(A - A) \cap \mathbf{N}$ is syndetic. (A set C is called *syndetic* if there exists a $k \in \mathbf{N}$ such that $(C \cup \{0\}) + [0, k] = \mathbf{N}$.) There arises the question whether one can prove a similar result for $A + A$ in place of $A - A$. One quickly sees that there is a set A with $BD(A) > 0$ yet $A + A$ is not syndetic. Hence one sees that Theorem 1 is a reasonable counterpart for $A + A$ to the result from [3] that $(A - A) \cap \mathbf{N}$ is syndetic. However, our proof of Theorem 1 is completely different from the argument in [3]; it seems that this difference stems in part from the fact that piecewise syndeticity is essentially a topological property while syndeticity is not.

Let's fix a countably saturated nonstandard universe. For each set $A \subseteq \mathbf{N}$, let ${}^*\!A$ be the nonstandard version of A in the nonstandard universe. For example, ${}^*\mathbf{N}$ is the set of all natural numbers in the nonstandard universe. All integers in ${}^*\mathbf{N} \setminus \mathbf{N}$ will be called *hyperfinite integers*. Using nonstandard analysis, one can formulate the following equivalent forms of piecewise syndeticity and Banach density. A set A is piecewise syndetic iff there exists an interval $[H, K]$ of hyperfinite length such that the set ${}^*\!A \cap [H, K]$ has only gaps of finite length in $[H, K]$. Also a set $A \subseteq \mathbf{N}$ has Banach density greater than or equal to α iff there exists an interval $[H, K]$ of hyperfinite length such that the set ${}^*\!A \cap [H, K]$ has Loeb measure greater than or equal to α (Loeb measure will be described in the next paragraph). Hence, one can state Theorem 1 in a nonstandard way by roughly saying: if A and B are two subsets of an interval $[0, H - 1]$ of hyperfinite length with positive Loeb measure, then $A + B$ is not "nowhere dense" in a topology similar to an "order–topology". Let me make it rigorous in the next paragraph.

The Loeb measure on a hyperfinite set (cf. [12] or [16]) can be defined as the following. Given an interval $[0, H - 1]$ of hyperfinite length, let $\mu(A) = |A|/H$ for every internal subset A of $[0, H - 1]$. Then μ can be seen as a normalized uniform counting measure defined on the algebra of all internal subsets of $[0, H - 1]$. Let st be the standard part map, i.e. $st(r) = \alpha$ iff r is a real number, α is a standard real number and r is infinitesimally close to α. From the standard point of view, $st \circ \mu$ is an atomless, finitely additive probability measure on the algebra of all internal subsets of $[0, H - 1]$. Now one can define a standard, atomless, countably additive, complete probability measure μ_L, called Loeb measure, on the complete σ-algebra on $[0, H - 1]$ generated by all internal subsets of $[0, H - 1]$ such that $\mu_L(A) = st(\mu(A))$ for every internal $A \subseteq [0, H - 1]$. The verification of the countable additivity of the measure needs countable saturation of the nonstandard universe. Note that the base set $[0, H-1]$ can be replaced by any hyperfinite set such as $[H, K]$ when $K - H$ is hyperfinite.

The U–topologies on an interval of hyperfinite length (cf. [8]) are defined as the following. An initial segment U of ${}^*\mathbf{N}$ is called an (additive) *cut* if $U \supseteq \mathbf{N}$ and $U + U = U$. For example, $\mathbf{N}$ is a cut. A cut is an external set (except $U = {}^*\mathbf{N}$). Let $U \subseteq [0, H-1]$ be a cut. A set $S \subseteq [0, H-1]$ is called U–*open* if for every $x \in [0, H-1]$, there exists a $y > U$, i.e. $y \in {}^*\mathbf{N} \smallsetminus U$, such that $[x-y, x+y] \cap [0, H-1] \subseteq S$. All U–open sets form the U–topology. Note that a U–topology can be defined on any interval of hyperfinite length. Note also that a set $A \subseteq \mathbf{N}$ is piecewise syndetic iff there exists an interval $[H, K]$ of hyperfinite length such that ${}^*A \cap [H, K]$ is not $\mathbf{N}$–nowhere dense (nowhere dense in terms of U–topology for $U = \mathbf{N}$).

In [6], I answered a question in [8] by proving the following.

Theorem 4 *Let H be a hyperfinite integer and $U \subseteq [0, H-1]$ be a cut. For any $A, B \subseteq [0, H-1]$, if both A and B have positive Loeb measure, then $A \oplus B$ is not U–nowhere dense, where $\oplus$ is addition modulo H.*

It is shown in [8] that for any standard real number $\alpha < 1$, there exists a U–nowhere dense set $A \subseteq [0, H-1]$ with $\mu_L(A) > \alpha$. Note that with some adjustments, the interval $[0, H-1]$ can be replaced by any interval of hyperfinite length. Theorem 4 reveals a very interesting phenomenon, I call it the sumset phenomenon, in the standard world. It says that if two sets are large in terms of "measure", then $A + B$ is not small in terms of "order–topology" (usually, there is a meager set in terms of "order–topology" with positive "measure"). When $U = \mathbf{N}$, Theorem 4 implies Theorem 1 which is an example of the sumset phenomenon. When U is the cut $\bigcap_{n \in \mathbf{N}} [0, \lceil H/n \rceil]$, then Theorem 4 implies a well-known example of the sumset phenomenon in real analysis, which says that if A and B are two subsets of real numbers with positive Lebesgue measure, then $A + B$ covers a non-empty open interval of real numbers. Note that there exists a meager subset X of the real line between 0 and 1 such that the Lebesgue measure of X is 1.

The nonstandard universe ${}^*\mathbf{V}$ can be constructed by an ultrapower construction modulo a nonprincipal ultrafilter $\mathcal{F}$ on the countable set $\mathbf{N}$. If ${}^*\mathbf{V}$ is constructed in this way, then every integer in ${}^*\mathbf{N}$ is an equivalence class $[f]_{\mathcal{F}}$ for some $f \in \mathbf{N}^{\mathbf{N}}$ (f is equivalent to g iff $\{n \in \mathbf{N} : f(n) = g(n)\} \in \mathcal{F}$). Let

$$U_{\mathcal{F}} = \mathbf{N} \smallsetminus \{[f]_{\mathcal{F}} : f \in \mathbf{N}^{\mathbf{N}} \text{ and } \lim_{n \to \infty} f(n) = \infty\}.$$

Then, $U_{\mathcal{F}}$ is a cut. Using this kind of cut, one can prove the following example of the sumset phenomenon using Theorem 4.

Theorem 5 *For each number n, let $A_n, B_n \subseteq [0, n-1]$ be such that $|A_n|/n \geqslant \epsilon$ and $|B_n|/n \geqslant \epsilon$ for a fixed standard real number $\epsilon > 0$. Then, for every nonprincipal ultrafilter $\mathcal{F}$ on $\mathbf{N}$, there exists a sequence $\langle [a_n, b_n] : n \in \mathbf{N}\rangle$ of intervals with $[a_n, b_n] \subseteq [0, n-1]$ and $\lim_{n\to\infty}(b_n - a_n) = \infty$ such that for every $X \in \mathcal{F}$, there exists an infinite $Y \subseteq X$ and $k \in \mathbf{N}$ such that the largest gap of $A_n \oplus_n B_n$ in $[a_n, b_n]$ has length k for every $n \in Y$, where $\oplus_n$ is addition modulo n.*

Note that for every standard real number $\alpha < 1$, there exists a sequence $\langle A_n : n \in \mathbf{N}\rangle$ with $A_n \subseteq [0, n-1]$ and $|A_n|/n > \alpha$ such that for every sequence $\langle [a_n, b_n] \subseteq [0, n-1] : n \in \mathbf{N}\rangle$ of intervals with $\lim_{n\to\infty}(b_n - a_n) = \infty$, the length of the largest gap of A_n in $[a_n, b_n]$ approaches infinity.

There is one more example of the sumset phenomenon. A set of the form

$$FS(x_n)_{n=0}^{\infty} = \{\sum_{n\in S} x_n : S \text{ is a non-empty finite subset of } \mathbf{N}\}$$

for some sequence $\langle x_n : n \in \mathbf{N}\rangle$ of natural numbers is called an *IP*-set. It is mentioned in [2] that for every standard real number $\alpha < 1$, there exists a set A with $\bar{d}(A) > \alpha$ (hence $BD(A) > \alpha$) such that A does not contain the set $k + FS(x_n)_{n=0}^{\infty}$ for any $k \in \mathbf{N}$ and any sequence $\langle x_n : n \in \mathbf{N}\rangle$ of natural numbers.

Using Theorem 1 and Hindman's theorem, one can obtain the following result.

Theorem 6 *If $A, B \subseteq \mathbf{N}$ both have positive Banach density, then $A + B$ contains a set of the form $k + FS(x_n)_{n=0}^{\infty}$ for some $k \in \mathbf{N}$ and some sequence $\langle x_n : n \in \mathbf{N}\rangle$ of natural numbers.*

Although Theorem 1, Theorem 5, and Theorem 6 are all standard results, finding a standard proof for each of them may not be easy. On the other hand, the proof of Theorem 4 in the nonstandard world is relatively easy (see the next paragraph). Hence using Theorem 4 to prove the standard results given above is a witness of the power of nonstandard analysis.

To finish this section I give a sketch of the proof of Theorem 4. Let U be a cut, H a hyperfinite integer, and A, B internal subsets of $[0, H-1]$ such that the conclusion of Theorem 4 is false for U, H, A, B. The number H and the sets A and B can be chosen so that the numbers $|A|/H$ and $|B|/H$ almost reach their maxima α and β. By that I mean the following. (1) If $A' \subseteq [0, H'-1]$ and $|A'|/H' > \alpha$, then H' and A' can't be part of any counter-example. (2) There is a small enough standard real number $\epsilon > 0$ such that if $A', B' \subseteq [0, H'-1]$ are such that $|A'|/H' > \alpha - \epsilon$ and $|B'|/H' > \beta$, then H', A' and B' can't be part of any counter-example. (3) $|A|/H > \alpha - \epsilon$ and $|B|/H > \beta - \epsilon$. Note that if $(|A| + |B|)/H > 1$, then $A \oplus B = [0, H-1]$. Hence we have $\beta \leqslant \alpha$ and $\beta \leqslant \frac{1}{2}$. Since $A \oplus B$ is U–nowhere dense, then $A \oplus B \oplus [0, k]$ is also U–nowhere dense for every $k \in U$. By the maximality of β, one has $|B \oplus [0, k]|/H \leqslant \beta$ for every $k \in U$. By the overspill principle, one can find $K > U$ such that $|B \oplus [0, 2K]|/H \leqslant \beta$. Next we divide the interval $[0, H-1]$ into subintervals of length K. Case 1: At least one-third of these subintervals are disjoint from B. Then there is a subinterval $[a, a+K-1]$ among the rest such that $|B \cap [a, a+K-1]|/K \geqslant \beta + \epsilon$. The number K and the set $B \cap [a, a+K-1]$ will lead to a counter-example. Case 2: At least two-thirds of the subintervals contains elements from B. Then $|B \oplus [0, 2K]|/H \geqslant \frac{2}{3}$ which contradicts $\beta \leqslant \frac{1}{2}$.

3 Measure-Theoretic Aspects

There are many interesting theorems about lower density and Shnirel'man density in additive number theory (cf. [4]). There are also a few interesting results about Banach density in combinatorial number theory (cf. [3]). But I can rarely find any results about Banach density in additive number theory. In this section, I will present a general method, using nonstandard analysis and Birkhoff ergodic theorem, of formulating and proving a theorem about Banach density corresponding to each theorem about lower density or Shnirel'man density. All proofs of the results in this section can be found in [7].

In the last section, I showed how one can use U–topologies on an interval of hyperfinite length to prove results in the standard world. I have referred to that kind of proof as topological. In this section, the methods I use seem more measure-theoretic. Let's look at the way of proving Theorem 3 using Theorem 2. By adding more and more copies of the set A, one can increase the Banach density of the sum until it reaches 1. This proof involves no topology. If one views Banach density as a sort of measure, then the proof merely shows that the measure of the sum increases to 1 as more and more copies of A are added.

There is another reason why I consider the proof of Theorem 3 via Theorem 2 measure-theoretic. In the last section, I mentioned that a set A has Banach density greater than or equal to α iff there is an interval $[H, K]$ of hyperfinite length such that the Loeb measure of the set ${}^*A \cap [H, K]$ on $[H, K]$ is greater than or equal to α. Therefore, one can apply measure-theoretic techniques to the Loeb space on $[H, K]$ to obtain results about ${}^*A$. Given a probability space (Ω, Σ, μ), a bijection T from Ω to Ω is called a *measure-preserving transformation* if $T[E] \in \Sigma$, $T^{-1}[E] \in \Sigma$, and $\mu(E) = \mu(T[E])$ for every $E \in \Sigma$. The measure-theoretic result needed here is the following

Birkhoff Ergodic Theorem (cf. [15] or [3]) *Let (Ω, Σ, μ) be a probability space and T be a measure-preserving transformation from Ω to Ω. For every $f \in L_1(\Omega)$, there exists a $\bar{f} \in L_1(\Omega)$ such that for μ-almost all $x \in \Omega$,*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = \bar{f}(x),$$

where T^0 is the identity map and $T^{k+1}(x) = T(T^k(x))$ for every $k \in \mathbf{N}$.

Given an interval $[H, K]$ of hyperfinite length, let T be the map from $[H, K]$ to $[H, K]$ such that $T(K) = H$ and $T(x) = x + 1$ for every $x \in [H, K - 1]$. Clearly, T is a Loeb measure-preserving transformation. If $x \in \cap_{n \in \mathbf{N}} [H, K - n]$ and f is the characteristic function of a set $C \subseteq [H, K]$, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = \alpha$$

implies $\underline{d}((C - x) \cap \mathbf{N}) = \alpha$. Now one can apply the Birkhoff ergodic theorem to get the first half of the following lemma about the relationship between Banach density and lower density. The second half of the lemma is just a consequence of the overspill principle and the transfer principle in nonstandard analysis.

Lemma 1 *Let A be a set such that $BD(A) = \alpha$. Then there is an interval $[H, K]$ of hyperfinite length such that for almost all $x \in [H, K]$ in terms of the Loeb measure, $\underline{d}(({}^*A - x) \cap \mathbf{N}) = \alpha$. On the other hand, if there is an $x \in {}^*\mathbf{N}$ such that $\underline{d}(({}^*A - x) \cap \mathbf{N}) = \alpha$, then $BD(A) \geqslant \alpha$.*

Next two lemmas are about the relationship between Banach density and Shnirel'man density. They can be proven using Lemma 1 and a little extra work in nonstandard analysis. For a set A, let

$$BSD(A) = \lim_{n\to\infty} \sup_{m-k=n} \inf_{k\leqslant i\leqslant m} \frac{|A\cap[k,i]|}{i-k+1}$$

be called the *Banach–Shnirel'man density* of A.

Lemma 2 *Let $A \subseteq \mathbf{N}$. Then $BSD(A) \geqslant \alpha$ iff there exists an $x \in {}^*\mathbf{N}$ such that $\sigma(({}^*A - x)\cap \mathbf{N}) \geqslant \alpha$.*

Lemma 3 *For every set A, $BD(A) = BSD(A)$.*

Lemma 1 tells us that if $BD(A) = \alpha$, then the lower density of ${}^*A$ in a remote copy of $\mathbf{N}$ is also α. Hence one can apply an existing theorem about lower density to the remote copy of $\mathbf{N}$ and obtain a result. Pulling the result down to the standard world then gives a parallel theorem about Banach density. By Lemma 2 and Lemma 3, one can do exactly the same for Shnirel'man density. For example, Theorem 3 is parallel to Shnirel'man's theorem and Theorem 2 is parallel to Mann's theorem and to Besicovitch's theorem (cf. [4]). Mann's theorem says that if $0 \in A\cap B$, then $\sigma(A+B) \geqslant \min\{\sigma(A)+\sigma(B), 1\}$, and Besicovitch's theorem says that if $1 \in A$, $0 \in B$ and $\inf_{n\geqslant 1} \frac{|B\cap[1,n]|}{n+1} \geqslant \beta$, then $\sigma(A + B) \geqslant \min\{\sigma(A) + \beta, 1\}$. In fact, the idea just described can be used to derive a parallel result to each existing theorem about lower density and Shnirel'man density. Next, I will give two examples applying this idea to problems involving essential components. A set B is called an *essential component* if for every set A with $0 < \sigma(A) < 1$, $\sigma(A + B) > \sigma(A)$.

The first example is a theorem parallel to Plünnecke's theorem (cf. [14]) which says that if B is a basis of order h, then for every set A, $\sigma(A+B) \geqslant \sigma(A)^{1-\frac{1}{h}}$. Consequently, a basis of finite order is an essential component. For formulating the parallel theorem, we define a piecewise basis. A set B is called a *piecewise basis of order h* if there exists a sequence $\langle [a_n, b_n] : n \in \mathbf{N}\rangle$ of intervals in $\mathbf{N}$ with $\lim_{n\to\infty}(b_n - a_n) = \infty$ such that

$$h((B - a_n)\cap \mathbf{N}) + a_n \supseteq [a_n, b_n]$$

for every $n \in \mathbf{N}$. It is easy to see that a basis of order h is a piecewise basis of order h. A piecewise basis of order h is a Banach basis of order h.

Theorem 7 *If $B \subseteq \mathbf{N}$ is a piecewise basis of order h, then for every set A,*

$$BD(A+B) \geqslant BD(A)^{1-\frac{1}{h}}.$$

The second example is a theorem parallel to Erdös–Landau's theorem and to Rohrbach's theorem (cf. [4]). Let B be a basis of order h. For every $m \in \mathbf{N}$, let $h(m) = \min\{h' : m \in h'B\}$. Then, the number

$$h^* = \sup \frac{1}{n} \sum_{m=1}^{n} h(m)$$

is called the *average order* of B. Since $h(m) \leqslant h$ for every $m \in \mathbf{N}$, one has $h^* \leqslant h$. Erdös–Landau's theorem says that if B is a basis of average order h^*, then for every set A,

$$\sigma(A+B) \geqslant \sigma(A) + \frac{1}{2h^*}\sigma(A)(1-\sigma(A)).$$

A set $B \subseteq \mathbf{N}$ is called an *asymptotic basis of order* h if $\mathbf{N} \smallsetminus hB$ is a finite set, where $\smallsetminus$ means set subtraction. Suppose B is an asymptotic basis of order h such that $\mathbf{N} \smallsetminus hB \subseteq [0, b-1]$ for some $b \in \mathbf{N}$. Then the number

$$h_a^* = \limsup_{n\to\infty} \frac{1}{n} \sum_{m=b}^{n} h(m)$$

is called the *average asymptotic order of* B. Rohrbach's theorem says that if B is an asymptotic basis of average asymptotic order h_a^*, then for every set A

$$\underline{d}(A+B) \geqslant \underline{d}(A) + \frac{1}{2h_a^*}\underline{d}(A)(1-\underline{d}(A)).$$

A set B is called a *piecewise asymptotic basis of piecewise asymptotic order* h_{pa} if there exists a $k \in \mathbf{N}$ and a sequence $\langle [a_n, b_n] : n \in \mathbf{N}\rangle$ of intervals in $\mathbf{N}$ with $\lim_{n\to\infty}(b_n - a_n) = \infty$ such that

$$h_{pa}((B-a_n)\cap \mathbf{N}) + a_n \supseteq [a_n + k, b_n]$$

for every $n \in \mathbf{N}$. Suppose B is a piecewise asymptotic basis of piecewise asymptotic order h_{pa} with the number k and the sequence $\langle [a_n, b_n] : n \in \mathbf{N}\rangle$ given as above. For each $n \in \mathbf{N}$ and $m \in [a_n + k, b_n]$, let

$$h_n(m) = \min\{h' : m \in h'((B-a_n)\cap \mathbf{N}) + a_n\}.$$

Then, the number

$$h^*_{pa} = \limsup_{n\to\infty} \sup_{a_n+k\leqslant m\leqslant b_n} \frac{1}{m-a_n-k+1}\sum_{i=a_n+k}^{m} h_n(i)$$

is called a *piecewise asymptotic average order of* B. Now we are ready to state the parallel theorem.

Theorem 8 *If* B *is a piecewise asymptotic basis of piecewise asymptotic average order* h^*_{pa}*, then for every set* A*,*

$$BD(A+B) \geqslant BD(A) + \frac{1}{2h^*_{pa}} BD(A)(1-BD(A)).$$

4 Epilogue

Since its discovery by A. Robinson about forty years ago, nonstandard analysis has been reaching its maturity. Besides their great philosophical importance, the ideas and techniques of nonstandard analysis are being applied to many other mathematical fields [1]. During my study of the subjects discussed in this report, I have noticed many advantages of nonstandard methods in dealing with Banach density problems. There are two main advantages I would like to point out here.

(1) Nonstandard methods are used here to reduce the complexity of the mathematical objects that one needs in a proof. Very often a sequence of standard elements can be treated as one nonstandard element, and an asymptotic argument in the standard world can be translated into a direct argument in the nonstandard world. For example, the statement $\sigma(A) > 0$ involves a sequence of intervals while the equivalent statement in the nonstandard world involves only one interval of hyperfinite length. This complexity reduction from second order to first order enables us to see the path towards solutions more clearly with a better understanding, hence produce a shorter proof with greater efficiency.

(2) Nonstandard methods offer a better intuition. In additive number theory, people are usually first interested in problems involving Shnirel'man density. Then, results about Shnirel'man density are generalized to results about lower density because lower density and Shnirel'man density have very similar behavior. From the inequalities among those densities, the next step seems to be generalization to results

about upper density. Unfortunately, the behavior of upper density is quite different. For example, there are sets A and B in $\mathbf{N}$ such that $\bar{d}(A) = \frac{1}{2}$, $\bar{d}(B) = \frac{1}{2}$ and $\bar{d}(A+B) = \bar{d}(A+B+\{0,1\}) = \frac{1}{2}$ (let $A = \cup_{n=1}^{\infty}[2^{f(2n)}, 2^{f(2n)+1}-1]$ and $B = \cup_{n=1}^{\infty}[2^{f(2n+1)}, 2^{f(2n+1)+1}-1]$, where $f(n) = 2^{2^n}$ for every $n \in \mathbf{N}$). Therefore, we can not find a nice parallel theorem about upper density to Mann's theorem. Since Banach density is even greater than or equal to upper density, one might think that the behavior of Banach density is even more different from the behavior of Shnirel'man density or lower density. But the situation is completely changed when nonstandard methods are used. By Lemma 1, Lemma 2, and Lemma 3, one can see clear connections between Banach density and Shnirel'man density, and between Banach density and lower density. The connections offer a good understanding of Banach density and make it easy to derive a parallel theorem about Banach density to each theorem about Shnirel'man density or lower density.

There may be other sources of advantages using nonstandard analysis. For example, use of saturation (Loeb space construction needs countable saturation) may increase proof-theoretic strength [5]. But I am not sure if the use of saturation here is essential in increasing proof-theoretic strength.

I started working on this subject when I found an answer to Problem 9.13 of [8]. S. Leth pointed out to me that Theorem 1 is a consequence of Theorem 4. After trying to produce an elementary proof of Theorem 1, I realized that the use of nonstandard methods there is essential. I found that the elementary proof of the theorem, if it is produced, would be much longer, less general and unnatural, in contrast to the simplicity, generality and elegance of the proof of Theorem 4.

There are also other papers, such as [9], [10] and [11], on the study of sequences of natural numbers using nonstandard methods.

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