Type Two Cuts, Bad Cuts and Very Bad Cuts[1]

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Abstract

Type two cuts, bad cuts and very bad cuts are introduced in [KL] for studying the relationship between Loeb measure and U-topology of a hyperfinite time line in an ω_1-saturated nonstandard universe. The questions concerning the existence of those cuts are asked there. In this paper we answer, fully or partially, some of those questions by showing that: (1) type-two cuts exist, (2) the $\aleph_1$-isomorphism property implies that bad cuts exist, but no bad cuts are very bad.

0 Introduction

All nonstandard universes mentioned in this paper are ω_1-saturated. Given a nonstandard universe V, let ${}^*\mathbb{N}$ denote the set of all positive integers in V and $\mathbb{N}$ denote the set of all standard positive integers. A non-empty initial segment U of ${}^*\mathbb{N}$ (under the natural order of ${}^*\mathbb{N}$) is called a cut if U is closed under addition, *i.e.* $(\forall x, y \in U)\,(x + y \in U)$ is true. For example, $\mathbb{N}$ is the smallest cut and ${}^*\mathbb{N}$ is the largest cut. There are several ways of constructing new cuts from given cuts shown in [KL]. For example, if U is a cut and x is an element in ${}^*\mathbb{N}$, then the set

$$xU = \{y \in {}^*\mathbb{N} : (\exists z \in U)\,(y < xz)\}$$

is a cut. If the element x is in ${}^*\mathbb{N} \smallsetminus U$, then the set

$$x/U = \{y \in {}^*\mathbb{N} : (\forall z \in U)\,(y < x/z)\}$$

is also a cut.

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Cuts are used in [KL] for defining U-topologies on a hyperfinite time line. Given a hyperinteger $H \in {}^*\mathbb{N} \smallsetminus \mathbb{N}$. The set $\mathcal{H} = \{1, 2, \ldots, H\} \subseteq {}^*\mathbb{N}$ is called a hyperfinite time line. Let $U \subseteq \mathcal{H}$ be a cut. A set $O \subseteq \mathcal{H}$ is called U-open if

$$(\forall x \in O)\,(\exists y \in \mathcal{H} \smallsetminus U)\,(\{z \in \mathcal{H} : x - y < z < x + y\} \subseteq O)$$

is true. The U-topology of $\mathcal{H}$ is the topology of all U-open sets in $\mathcal{H}$. A U-topology could also be viewed as an analogue of order topology (note that the natural order topology of $\mathcal{H}$ is discrete). Given $x, y \in \mathcal{H}$, define $a \sim_U y$ iff $|x - y| \in U$. Then it is easy to see that $\sim_U$ is an equivalence relation (here we use the fact that U is closed under addition). Let $x \in \mathcal{H}$. A $\sim_U$ equivalence class containing x is called a U-monad of x. Given $x, y \in \mathcal{H}$. Define $x \ll_U y$ iff $x < y$ and $x \not\sim_U y$. For any $x, y \in \mathcal{H}$ let

$$I(x, y) = \{z \in \mathcal{H} : x \ll_U z \ll_U y\}.$$

Then the U-topology of $\mathcal{H}$ is actually the topology generated by open "intervals" $I(x, y)$ for all $x, y \in \mathcal{H}$. So a U-topology is like an order topology by ordering all U-monads.

Given a hyperfinite time line $\mathcal{H}$. There is a natural way to define a probability measure called Loeb measure on $\mathcal{H}$. For any internal subset A of $\mathcal{H}$ let $\mu(A) = |A|/H$, where H is the largest number in $\mathcal{H}$. Then μ is a finite additive, internal uniform counting measure on the algebra of all internal subsets of $\mathcal{H}$. The Loeb measure $L(\mu)$ is now the extension of $st \circ \mu$ to the completion of the σ-algebra generated by all internal subsets of $\mathcal{H}$, where st is the standard part map. Loeb measure behaves very much like Lebesgue measure on the unit interval $[0, 1]$ of the standard real line.

In [KL] Keisler and Leth probe the similarities between a hyperfinite time line $\mathcal{H}$ equipped with Loeb measure and a U-topology, and the standard unit interval $[0, 1]$ equipped with Lebesgue measure and the natural order topology. They consider a cut

U behaves nicely if it makes $\mathcal{H}$ much like $[0,1]$. For example, considering the fact that $[0,1]$ contains a meager set of Lebesgue measure one, they call a cut $U \subseteq \mathcal{H}$ a good cut if $\mathcal{H}$ contains a U-meager set of Loeb measure one. A cut is called bad if it is not good. Keisler and Leth discovered that most cuts are good and bad cuts are difficult to construct. In fact, they constructed bad cuts in some nonstandard universes under some extra set theoretic assumption beyond *ZFC* such as $2^\omega < 2^{\omega_1}$. They proved also in [KL] that a bad cut must be a type two cut (see §1 for definition) and a type two cut must have both uncountable cofinality and uncountable coinitiality. Given a cut U, the cofinality of U is the cardinal

$$cof(U) = \min\{card(S) : S \subseteq U \wedge (\forall x \in U)\,(\exists y \in S)\,(x < y)\}$$

and the coinitiality of U is the cardinal

$$coin(U) = \min\{card(S) : S \subseteq {}^*\mathbb{N} \smallsetminus U \wedge (\forall x \in {}^*\mathbb{N} \smallsetminus U)\,(\exists y \in S)\,(y < x)\}$$

The questions whether there exists an nonstandard universe in which there are no bad cuts or no type two cuts or no cuts U with $cof(U) > \omega$ and $coin(U) > \omega$ are asked in [KL]. In [J1] the author showed that (1) bad cuts exist in some nonstandard universe (eliminating the need of the assumption $2^\omega < 2^{\omega_1}$), (2) in any ω_2-saturated nonstandard universe there exist cuts U with $cof(U) > \omega$ and $coin(U) > \omega$, (3) assuming $\mathfrak{b} > \omega_1$, *i.e.* every $B \subseteq \omega^\omega$ of cardinality $\leqslant \omega_1$ is eventually dominated by some $f \in \omega^\omega$, then every hyperfinite time line in any nonstandard universe has cuts U with $cof(U) > \omega$ and $coin(U) > \omega$. Later Shelah [Sh] proved a surprising result that every hyperfinite time line in any nonstandard universe has cuts U with $cof(U) = coin(U)$ without using any extra set theoretic assumption. Note that $cof(U) = coin(U)$ implies $cof(U) > \omega$ and $coin(U) > \omega$ by ω_1-saturation. This paper is a sequel to [KL], [J1] and [Sh].

In the first section we prove that every hyperfinite time line in any nonstandard universe has type two cuts. The main idea of the proof is the combination of Shelah's method of constructing cuts U with $cof(U) > \omega$ and $coin(U) > \omega$ in [Sh] and Keisler-Leth's method of constructing type two cuts in [KL]. In the first half of the second section we prove that if the nonstandard universe satisfies the $\aleph_1$-isomorphism property, than there exist bad cuts in every hyperfinite time line. The proof uses a result from [JS]. In the second half of the second section we deal with very bad cuts (see definition below).

Suppose U is a bad cut in some hyperfinite time line $\mathcal{H}$. By [KL, Proposition 4.5] $\mathcal{H}$ contains no U-meager set with positive Loeb measure. So if $S \subseteq \mathcal{H}$ is a U-meager set, then S is either a non-Loeb measurable set or a Loeb measure zero set. A cut U in $\mathcal{H}$ is called very bad if every U-meager set has Loeb measure zero. In the second section we prove that if the nonstandard universe satisfies the $\aleph_1$-isomorphism property, then for any cut U except $U = H/\mathbb{N}$ in a hyperfinite time line $\mathcal{H}$, there exists a U-nowhere dense set $S \subseteq \mathcal{H}$ such that $S \not\subseteq A$ for any internal $A \subseteq \mathcal{H}$ with $\mu(A) \not\approx 1$, and $A \not\subseteq S$ for any internal $A \subseteq \mathcal{H}$ with $\mu(A) \not\approx 0$ (we then call S has outer Loeb measure one and inner Loeb measure zero). So if U is a bad cut, then there is a non-Loeb measurable U-nowhere dense subset of $\mathcal{H}$. Hence U is not very bad.

The reader is recommended to consult [CK] for background in model theory, to consult [CK], [L] or [SB] for background in nonstandard analysis, nonstandard universes and Loeb measure construction. In this paper we shall write $card(S)$ for the external cardinality of the set S and write $|A|$ for the internal cardinality of A when A is an internal set. Let ${}^*\mathbb{R}$ denote the set of all real numbers in a given nonstandard universe V. For each $r \in {}^*\mathbb{R}$ we shall write $[r]$ for the greatest integer less than or equal to r. We call a number $r \in {}^*\mathbb{R}$ bounded if there is an $n \in \mathbb{N}$ such that $-n < r < n$. Otherwise we call r unbounded. We call an $r \in {}^*\mathbb{R}$ infinitesimal if for any $n \in \mathbb{N}$ we have $-\frac{1}{n} < r < \frac{1}{n}$. We write $r \approx s$ if $r - s$ is an infinitesimal. We call a set infinite if it is externally infinite.

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1 Type Two Cuts

Let's fix a nonstandard universe V through out this section. Given a cut U. Let

$$M(U) = \{y \in {}^*\mathbb{N} : (\forall z \in U)\,(yz \in U)\},$$

where M suggests multiplication. Then $M(U)$ is a cut and closed under multiplication. The cut U is called a type one cut if $U = x/M(U)$ for some $x \in U$ or $U = x/M(U)$ for some $x \in {}^*\mathbb{N} \smallsetminus U$. U is called a type two cut if it is not type one. Mentioned in [KL] that type one-type two cuts are defined in [G]. In this section we show that type two cuts exist.

Theorem 1 *There are type two cuts.*

Proof: In order to avoid multiple superscripts we write $\exp(a, b)$ for a^b when $a, b \in {}^*\mathbb{R}$ and $a > 0$. First we construct sequences $\langle a_{n,\alpha} : \alpha < \lambda \rangle$ and $\langle b_{n,\alpha} : \alpha < \lambda \rangle$ for all $n \in \mathbb{N}$ simultaneously by a transfinite induction on ordinal λ such that for any $n \in \mathbb{N}$ and $\alpha, \beta \in \lambda$ the following conditions are satisfied.

(a) $a_{n,\alpha}$ and $b_{n,\alpha}$ are positive and unbounded in ${}^*\mathbb{R}$.

(b) $a_{n,\alpha} < b_{n,\alpha}$.

(c) $\alpha < \beta \longrightarrow a_{n,\alpha} < a_{n,\beta}$.

(d) $\alpha < \beta \longrightarrow b_{n,\alpha} > b_{n,\beta}$.

(e) $a_{n,\alpha} = \exp(b_{n,\alpha}, 1/b^3_{n+1,\alpha})$.

(f) $\alpha + 1 < \lambda \longrightarrow b_{n,\alpha+1} = \exp(b_{n,\alpha}, 1/b_{n+1,\alpha})$.

(g) $\alpha + 1 < \lambda \longrightarrow \exp(a_{n,\alpha}, b_{n+1,\alpha}) \leqslant a_{n,\alpha+1}$.

Suppose the construction is done up to stage λ. It is easy to see that for each $n \in \mathbb{N}$ the sequence $\langle a_{n,\alpha} : \alpha < \lambda \rangle$ is increasing, the sequence $\langle b_{n,\alpha} : \alpha < \lambda \rangle$ is decreasing and all $a_{n,\alpha}$'s are below all $b_{n,\alpha}$'s. For each $n \in \mathbb{N}$ let

$$J_{n,\lambda} = \bigcap_{\alpha<\lambda} \{x \in {}^*\mathbb{N} : a_{n,\alpha} < x < b_{n,\alpha}\}.$$

We shall show that if $J_{n,\lambda} \neq \emptyset$ for all $n \in \mathbb{N}$, then the inductive construction continues. So when the construction can not go further, there must be an $n \in \mathbb{N}$ such that $J_{n,\lambda} = \emptyset$. In this case, we shall use the sequence $\langle a_{n,\alpha} : \alpha < \lambda \rangle$ to define a type two cut.

Given any hyperinteger H, we choose a decreasing sequence $\langle d_n : n \in \mathbb{N} \rangle$ in ${}^*\mathbb{N} \setminus \mathbb{N}$ such that $d_1 \leqslant H$ and $\exp(d_{n+1}, d^3_{n+1}) < d_n$ for each $n \in \mathbb{N}$. The sequence $\langle d_n : n \in \mathbb{N} \rangle$ exists by overspill principle. For the first step of the induction we choose $b_{n,0} = d_n$ and $a_{n,0} = \exp(d_n, 1/d^3_{n+1})$. It is easy to see that for $\lambda = 1$ the conditions (c), (d), (f) and (g) are vacuously true and the conditions (b) and (e) are trivially true. For (a) since $d_{n+1} \leqslant \exp(d_n, 1/d^3_{n+1})$, then $d_{n+1} \leqslant a_{n,0}$.

Suppose now the sequences $\langle a_{n,\alpha} : \alpha < \lambda \rangle$ and $\langle b_{n,\alpha} : \alpha < \lambda \rangle$ have been constructed such that for any $n \in \mathbb{N}$ and $\alpha, \beta < \lambda$ the conditions (a)—(g) are satisfied.

Case 1: $\lambda = \gamma + 1$ for some ordinal γ. For each $n \in \mathbb{N}$ let $b_{n,\lambda} = \exp(b_{n,\gamma}, 1/b_{n+1,\gamma})$ and let $a_{n,\lambda} = \exp(b_{n,\lambda}, 1/b^3_{n+1,\lambda})$. We need to show that the sequences

$$\langle a_{n,\alpha} : \alpha < \lambda + 1 \rangle \text{ and } \langle b_{n,\alpha} : \alpha < \lambda + 1 \rangle$$

satisfy the conditions (a)—(g) with λ replaced by $\lambda + 1$. Note that the conditions (b), (d), (e) and (f) are trivially true.

Claim 1.1 The condition (g) is true.

Proof of Claim 1.1: First we have

$$\begin{aligned} b_{n+1,\gamma} &= \exp(b_{n+1,\lambda}, b_{n+2,\gamma}) \\ &> \exp(b_{n+1,\lambda}, 3) \\ &= b^3_{n+1,\lambda} \end{aligned}$$

since $b_{n+1,\lambda} = \exp(b_{n+1,\gamma}, 1/b_{n+2,\gamma})$ and $b_{n+2,\gamma} > 3$. So we now have

$$\begin{aligned} a_{n,\lambda} &= \exp(b_{n,\lambda}, 1/b^3_{n+1,\lambda}) \\ &= \exp(\exp(b_{n,\gamma}, 1/b_{n+1,\gamma}), 1/b^3_{n+1,\lambda}) \\ &> \exp(\exp(b_{n,\gamma}, 1/b_{n+1,\gamma}), 1/b_{n+1,\gamma}) \\ &= \exp(\exp(b_{n,\gamma}, 1/b^3_{n+1,\gamma}), b_{n+1,\gamma}) \\ &= \exp(a_{n,\gamma}, b_{n+1,\gamma}) \end{aligned}$$

Hence the condition (g) is true.

It is easy to see that (c) follows from (g) and (a) follows from (b) and (c).

Case 2: λ is a limit ordinal. If there exists an $n_0 \in \mathbb{N}$ such that $J_{n_0,\lambda} = \emptyset$, then stop and the construction is finished. Otherwise choose $c_n \in J_{n,\lambda}$ for each $n \in \mathbb{N}$. Let $b_{n,\lambda} = c_n$ and let $a_{n,\lambda} = \exp(b_{n,\lambda}, 1/b^3_{n+1,\lambda})$. We need to check that the sequences

$$\langle a_{n,\alpha} : \alpha < \lambda + 1 \rangle \text{ and } \langle b_{n,\alpha} : \alpha < \lambda + 1 \rangle$$

satisfy the conditions (a)—(g) with λ replaced by $\lambda + 1$. Note that (b), (d), (e), (f) and (g) are trivially true.

Claim 1.2 The condition (c) is true.

Proof of Claim 1.2: Given any $\alpha < \lambda$. Since $c_{n+1} < b_{n+1,\beta}$ for any $\beta < \lambda$, we have

$$\exp(a_{n,\alpha}, c_{n+1}^3) < \exp(a_{n,\alpha}, (b_{n+1,\alpha} b_{n+1,\alpha+1} b_{n+1,\alpha+2})).$$

Now by (g) we have

$$\begin{aligned} & \exp(a_{n,\alpha}, (b_{n+1,\alpha} b_{n+1,\alpha+1} b_{n+1,\alpha+2})) \\ \leqslant\ & \exp(a_{n,\alpha+1}, (b_{n+1,\alpha+1} b_{n+1,\alpha+2})) \\ \leqslant\ & \exp(a_{n,\alpha+2}, b_{n+1,\alpha+2}) \\ \leqslant\ & a_{n,\alpha+3} < c_n. \end{aligned}$$

So $\exp(a_{n,\alpha}, c_{n+1}^3) < c_n$. Hence

$$a_{n,\alpha} < \exp(c_n, 1/c_{n+1}^3) = a_{n,\lambda}.$$

It is now obvious that (a) follows from (c). This ends the construction.

Suppose the construction halts at stage λ for some ordinal λ. Then λ muct be a limit ordinal and there exists an $n \in \mathbb{N}$ such that $J_{n,\lambda} = \emptyset$. We want to construct a type two cut U from the sequences constructed above. Let

$$U = \{y \in {}^*\mathbb{N} : (\exists \alpha < \lambda)\, (y < \log(a_{n,\alpha}))\},$$

where log is the logarithmic function of base 2. Let

$$M = \{y \in {}^*\mathbb{N} : (\forall \alpha < \lambda)\, (y < b_{n+1,\alpha})\}.$$

Claim 1.3 $\{y \in {}^*\mathbb{N} : (\forall \alpha < \lambda)\, (\log(a_{n,\alpha}) < y < \log(b_{n,\alpha}))\} = \emptyset$.

Proof of Claim 1.3: Suppose the claim is not true. Let $y \in {}^*\mathbb{N}$ such that

$$\log(a_{n,\alpha}) < y < \log(b_{n,\alpha})$$

for all $\alpha < \lambda$. Then for any $\alpha < \lambda$ we have

$$a_{n,\alpha} < 2^y < b_{n,\alpha}.$$

This contradicts that $J_{n,\lambda} = \emptyset$.

Claim 1.4 U is a cut.

Proof of Claim 1.4: It is easy to see that $\mathbb{N} \subseteq U$. We want to show that U is closed under addition. For any $x \in U$ it suffices to show that $2x \in U$. Let $x < \log(a_{n,\alpha})$ for some $\alpha < \lambda$. Then

$$\begin{aligned} 2x &< 2\log(a_{n,\alpha}) \\ &= \log(a_{n,\alpha})^2 \\ &< \log(\exp(a_{n,\alpha}, b_{n+1,\alpha})) \\ &\leqslant \log(a_{n,\alpha+1}). \end{aligned}$$

So $2x \in U$.

Claim 1.5 $M(U) = M$.

Proof of Claim 1.5: Let $x \in M$. Given any $y \in U$, we want to show that $xy \in U$. Let $y < \log(a_{n,\alpha})$ for some $\alpha < \lambda$. Then

$$\begin{aligned} xy &< b_{n+1,\alpha} \log(a_{n,\alpha}) \\ &= \log(\exp(a_{n,\alpha}, b_{n+1,\alpha})) \\ &\leqslant \log(a_{n,\alpha+1}). \end{aligned}$$

So $xy \in U$. This shows that $M \subseteq M(U)$.

Let $x \in {}^*\mathbb{N} \smallsetminus M$. We want to find a $y \in U$ such that $xy \notin U$. By the definition of M there is an $\alpha < \lambda$ such that $x > b_{n+1,\alpha}$. Let $y = [\log(a_{n,\alpha+1})] + 1$. Then $y \in U$. We now have

$$\begin{aligned} xy &> b_{n+1,\alpha} \log(a_{n,\alpha+1}) \\ &= (b_{n+1,\alpha}/b^3_{n+1,\alpha+1}) \log(\exp(a_{n,\alpha+1}, b^3_{n+1,\alpha+1})) \\ &= (b_{n+1,\alpha}/b^3_{n+1,\alpha+1}) \log(b_{n,\alpha+1}). \end{aligned}$$

Since

$$b^3_{n+1,\alpha+1} < \exp(b_{n+1,\alpha+1}, b_{n+2,\alpha}) = b_{n+1,\alpha},$$

we have $(b_{n+1,\alpha}/b^3_{n+1,\alpha+1}) > 1$. So $xy > \log(b_{n,\alpha+1})$. So $xy \notin U$. This shows that $M(U) \subseteq M$.

Claim 1.6 $xM \neq U$ for any $x \in U$ and $x/M \neq U$ for any $x \in {}^*\mathbb{N} \smallsetminus U$.

Proof of Claim 1.6: Given any $x \in U$. We want to show that $xM \neq U$. Let $x < \log(a_{n,\alpha})$ for some $\alpha < \lambda$. For any $y \in M$ we have

$$\begin{aligned} xy &< b_{n+1,\alpha} \log(a_{n,\alpha}) \\ &= \log(\exp(a_{n,\alpha}, b_{n+1,\alpha})) \\ &\leqslant \log(a_{n,\alpha+1}) \end{aligned}$$

by the condition (g). So $xM \subseteq \{1, 2, \ldots, [\log(a_{n,\alpha+1})]\}$. Hence $xM \neq U$ because $[\log(a_{n,\alpha+1})] + 1 \in U \smallsetminus xM$.

Given any $x \in {}^*\mathbb{N} \smallsetminus U$. We want to show that $x/M \neq U$. By Claim 1.3 there is an $\alpha < \lambda$ such that $x > \log(b_{n,\alpha})$. For any $y \in M$ we have

$$\begin{aligned} x/y &> (\log(b_{n,\alpha}))/b_{n+1,\alpha} \\ &= \log(\exp(b_{n,\alpha}, 1/b_{n+1,\alpha})) \\ &= \log(b_{n,\alpha+1}). \end{aligned}$$

So $\{1, 2, \ldots, [\log(b_{n,\alpha+1})]\} \subseteq x/M$. Hence $x/M \neq U$ because $[\log(b_{n,\alpha+1})] \in x/M \smallsetminus U$.

Combining all those claims we have that U is a type two cut. $\square$

Remarks: (1) In the definition of type one–type two cuts and in the proof of Theorem 1 we never use ω_1-saturation. So type two cuts also exist in any non-ω_1-saturated nonstandard universe or any nonstandard model of Peano Arithmetic.

(2) The use of ${}^*\mathbb{R}$ when we construct sequences $\langle a_{n,\alpha} : \alpha < \lambda \rangle$ and $\langle b_{n,\alpha} : \alpha < \lambda \rangle$ is not necessary because we can replace $a_{n,\alpha}$ and $b_{n\alpha}$ by $[a_{n,\alpha}]$ and $[b_{n,\alpha}]$, respectively.

(3) Since a cut U with $cof(U) = coin(U)$ in a nonstandard universe may not be a type two cut, Theorem 1 is stronger than the result of Shelah in [Sh] mentioned in the introduction.

(4) Since the hyperinteger $H \in {}^*\mathbb{N} \smallsetminus \mathbb{N}$ is chosen arbitrarily at the beginning of the proof, we conclude that there are type two cuts U's in any hyperfinite time line $\{1, 2, \ldots, H\}$.

2 Bad Cuts and Very Bad Cuts

Let's recall the definitions. A cut U in a hyperfinite time line $\mathcal{H} = \{1, 2, \ldots, H\}$ is called a good cut if $\mathcal{H}$ contains a U-meager set of Loeb measure one, where a set $X \subseteq \mathcal{H}$ is called U-meager if X is the union of countably many nowhere dense sets under U-topology. U is called bad if it is not good. A bad cut U is called very bad if every U-meager set has Loeb measure zero. We show in this section that the $\aleph_1$-isomorphism property implies that there exist bad cuts and there are no very bad cuts. This means that for any nonstandard universe V if V satisfies the $\aleph_1$-isomorphism property, then there exist bad cuts and there are no very bad cuts in any hyperfinite time line $\mathcal{H}$ in V.

Let's introduce the κ-isomorphism property for any infinite regular cardinal κ. Given a nonstandard universe V. Let $\mathcal{L}$ be a first-order language. An $\mathcal{L}$-structure $\mathfrak{A} = (A; \ldots)$ is called internally presented (in V) if the base set A is internal (in V) and the interpretation in $\mathfrak{A}$ of each predicate symbol or function symbol of $\mathcal{L}$ is internal (in V). Let's fix a nonstandard universe V. V is said to satisfy the κ-isomorphism property if the following is true.

> For any first-order language $\mathcal{L}$ with $card(\mathcal{L}) < \kappa$ and for any two internally presented $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$, if $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent, then $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic.

The κ-isomorphism property was suggested by Henson [H1]. It is shown in [H1] that the κ-isomorphism property implies κ-saturation. The κ-isomorphism property implies also that any two infinite internal sets have same external cardinality because they are elementarily equivalent as structures of empty language. See [H1], [H2], [J2] and [JK] for the existence of nonstandard universes satisfying the κ-isomorphism property. In [JS] there is an equivalent form of the κ-isomorphism property in terms of the satisfiability of some second-order types. This equivalent form makes the use of the κ-isomorphism property very easy. Let's state this result [JS, Main Theorem] below.

Lemma 2 *Let κ be any infinite regular cardinal. Then the κ-isomorphism property is equivalent to the following:*

For any first-order language $\mathcal{L}$ with $card(\mathcal{L}) < \kappa$, for any internally presented $\mathcal{L}$-structure $\mathfrak{A}$ and for any set of $\mathcal{L} \cup \{X\}$-sentences , (X), where X is a new n-ary predicate symbol not in $\mathcal{L}$, if , $(X) \cup Th(\mathfrak{A})$ is consistent, then , (X) is satisfiable in $\mathfrak{A}$, i.e. there exists an n-ary relation $R \subseteq A^n$ where A is the base set of $\mathfrak{A}$ such that $(\mathfrak{A}, R) \models \varphi(R)$ for every $\varphi(X) \in$, (X).

Remark: The original proof of [JS, Main Theorem] has a minor restriction on κ, *e.g.* $\kappa < \beth_\omega$. But this restriction can been easily removed by using κ-saturation. See [Sch].

We need also an equivalent form of the bad-ness of a cut from [KL]. An internal function f with $dom(f) = \{1, 2, \ldots, L_f\}$ for some $L_f \in {}^*\mathbb{N}$ is called an internal sequence. Given a cut U, a strictly increasing internal sequence f of positive integers is called a crossing sequence of U if for any $x \in U$ there exists a $y \in range(f) \cap U$ such that $x < y$. The following lemma is a part of [KL, Proposition 4.5].

Lemma 3 *A cut U is bad iff for any crossing sequence f of U the internal sum*

$$\sum_{m=1}^{L_f - 1} (f(m)/f(m+1))$$

is unbounded.

Theorem 4 *The $\aleph_1$-isomorphism property implies that there exist bad cuts in every hyperfinite time line.*

Proof: Fix a nonstandard universe V satisfying the $\aleph_1$-isomorphism property. Given any hyperfinite time line $\mathcal{H} = \{1, 2, \ldots, H\}$ in V, we want to show that there exist bad cuts in $\mathcal{H}$. First we define an internally presented structure $\mathfrak{A}$. Let

$$\mathcal{F} = \begin{array}{l}\{f : f \text{ is an increasing internal sequence from} \\ \{1, 2, \ldots, L_f\} \text{ for some } L_f \leqslant H \text{ to } \mathcal{H}\}.\end{array}$$

Then $\mathcal{F}$ is internal. Define an internally presented structure

$$\mathfrak{A} = (\mathcal{H} \cup \mathcal{F} \cup {}^*\mathbb{R}; \mathcal{H}, \mathcal{F}, R, S, \leqslant, +, \cdot, n)_{n \in \mathbb{N}},$$

where $A = \mathcal{H} \cup \mathcal{F} \cup {}^*\mathbb{R}$ is the base set of $\mathfrak{A}$, $\mathcal{H}$ and $\mathcal{F}$ are unary relation, R is a ternary relation sucht that $\langle a, b, f\rangle \in R$ iff $f \in \mathcal{F}$, $a \in dom(f)$ and $f(a) = b$, S is a function from $\mathcal{F}$ to ${}^*\mathbb{R}$ such that for any $f \in \mathcal{F}$

$$S(f) = \sum_{m=1}^{L_f - 1} (f(m)/f(m+1)),$$

$\langle {}^*\mathbb{R}; +, \cdot, \leqslant\rangle$ is the real field in V, and n is a constant of the structure for each $n \in \mathbb{N}$. Let $\mathcal{L}$ be the language of $\mathfrak{A}$. Note that the following $\mathcal{L}$-sentences are true in $\mathfrak{A}$.

$$\theta_n = \exists x(\mathcal{H}(x) \wedge x \geqslant n \wedge \forall y(\mathcal{H}(y) \longrightarrow y \leqslant x))$$

for each $n \in \mathbb{N}$, and

$$\eta = \begin{array}{l}\forall f \forall x \forall y(\mathcal{F}(f) \wedge \mathcal{H}(x) \wedge \mathcal{H}(y) \wedge x < y \longrightarrow \\ \exists g(\mathcal{F}(g) \wedge range(g) = range(f) \cap [x, y])).\end{array}$$

Let $X \notin \mathcal{L}$ be a unary predicate symbol. We define , (X) to be the set of $\mathcal{L} \cup \{X\}$-sentences which contains exactly the following:

$$\varphi_1(X) = \forall x(X(x) \longrightarrow \mathcal{H}(x))$$

$$\varphi_2(X) = \forall x \forall y (x \leqslant y \wedge \mathcal{H}(x) \wedge X(y) \longrightarrow X(x))$$

$$\varphi_3(X) = \forall x \forall y (X(x) \wedge X(y) \longrightarrow X(x+y))$$

$$\psi_n = \forall f (\mathcal{F}(f) \wedge \forall x (X(x) \longrightarrow \exists y \exists z (R(y,z,f) \wedge X(z) \wedge x \leqslant z)) \longrightarrow S(f) \geqslant n)$$

for each $n \in \mathbb{N}$.

Note that the sentences $\varphi_1(X)$, $\varphi_2(X)$ and $\varphi_3(X)$ say that X is a cut in $\mathcal{H}$. The sentences $\psi_n(X)$ for $n \in \mathbb{N}$ say that if f is a crossing sequence of X, then the internal sum $S(f)$ is unbounded. So , (X) describes that X is a bad cut by Lemma 3. So if , (X) is satisfiable in $\mathfrak{A}$, then $\mathcal{H}$ must contain a bad cut. By Lemma 2 it suffices to show that , $(X) \cup Th(\mathfrak{A})$ is consistent.

Let $\mathfrak{A}'$ be a countable elementary submodel of $\mathfrak{A}$. Then $Th(\mathfrak{A}') = Th(\mathfrak{A})$. If we can show that , (X) is satisfiable in $\mathfrak{A}'$, then it is clear that $Th(\mathfrak{A}) \cup$, (X) is consistent.

Claim 4.1 , (X) is satisfiable in $\mathfrak{A}'$.

Proof of Claim 4.1: Let $A' = \mathcal{H}' \cup \mathcal{F}' \cup \mathbb{R}'$ be the base set of $\mathfrak{A}'$ and let $\mathcal{F}' = \{f_i : i \in \mathbb{N}\}$. We now inductively construct an increasing sequence $\langle a_i : i \in \mathbb{N} \rangle$ and a decreasing sequence $\langle b_i : i \in \mathbb{N} \rangle$ in $\mathcal{H}'$ such that for each $i \in \mathbb{N}$

(a) $a_i < b_i$,

(b) $2a_i < a_{i+1}$,

(c) b_i/a_i is unbounded in $\mathbb{R}'$,

(d) If $f \in \mathcal{F}'$ such that

$$range(f) = range(f_i) \cap \{x \in \mathcal{H}' : a_i \leqslant x \leqslant b_i\},$$

if $S(f)$ is bounded in $\mathbb{R}'$ and if L_f is unbounded, then there is a $k \in \{1, 2, \ldots, L_f\} \cap \mathcal{H}'$ such that $f(k) \leqslant a_{i+1}$ and $f(k+1) \geqslant b_{i+1}$ (or f has a jump across the interval (a_{i+1}, b_{i+1})).

We show first that the claim follows from the construction. Let

$$U = \{x \in \mathcal{H}' : (\exists i \in \mathbb{N})\,(x \leqslant a_i)\}.$$

Then $\varphi_1(U)$ and $\varphi_2(U)$ are trivially true in $(\mathfrak{A}', U)$. The sentence $\varphi_3(U)$ is true in $(\mathfrak{A}', U)$ by the condition (b). Given any $f_i \in \mathcal{F}'$ such that f_i is a crossing sequence of U. To show that $\psi_n(U)$ is true in $(\mathfrak{A}', U)$ for any $n \in \mathbb{N}$ we need only to show that $S(f_i)$ is unbounded. Suppose $S(f_i)$ is bounded. By the fact that η is true in $\mathfrak{A}'$ there exists a $g \in \mathcal{F}'$ such that

$$range(g) = range(f_i) \cap \{x \in \mathcal{H}' : a_i \leqslant x \leqslant b_i\}.$$

Then $S(g)$ is also bounded because $S(g) \leqslant S(f_i)$. Since f_i is a crossing sequence of U, $a_i \in U$ and $b_i \notin U$, then g is also a crossing sequence of U. Hence L_g is unbounded (since no finite sequence could be a crossing sequence of any cut). By the condition (d) we know that g has a jump from a_{i+1} to b_{i+1}, *i.e.* $g(k) \leqslant a_{i+1}$ and $g(k+1) \geqslant b_{i+1}$ for some $k \in dom(g)$. So g can't be a crossing sequence of U, a contradiction.

We now do the inductive construction. Choose any a_1 and b_1 in $\mathcal{H}'$ such that b_1/a_1 is unbounded (for example, $a_1 = 1$ and $b_1 = H$). Suppose we have found $\langle a_i : i < k\rangle$ and $\langle b_i : i < k\rangle$ for some $k > 1$ such that they satisfy the conditions (a)—(d). We need to find a_k and b_k. Let $g \in \mathcal{F}'$ be such that

$$range(g) = range(f_{k-1}) \cap \{x \in \mathcal{H}' : a_{k-1} \leqslant x \leqslant b_{k-1}\}.$$

Case 1: $S(g)$ is unbounded or L_g is bounded. Simply let $a'_k = a_{k-1}$ and $b'_k = b_{k-1}$.

Case 2: $S(g)$ is bounded and L_g is unbounded. Let $n \in \mathbb{N}$ be such that $S(g) < n$. Since g is an element in $\mathfrak{A}'$ and $\mathfrak{A}' \preceq \mathfrak{A}$, then there is a t in $\mathfrak{A}'$ such that

$$t = \min\{g(m)/g(m+1) : m \in \mathcal{H}' \wedge 1 \leqslant m < L_g\}.$$

Let $m_0 \in \mathcal{H}'$ and $m_0 < L_g$ be such that $t = g(m_0)/g(m_0+1)$. Then

$$t(L_g - 1) \leqslant \sum_{m=1}^{L_g - 1} (g(m)/g(m+1)) = S(g) \leqslant n.$$

So we have $g(m_0+1)/g(m_0) \geqslant (L_g - 1)/n$. Now let $a'_k = g(m_0)$ and $b'_k = g(m_0+1)$.

Clearly we have b'_k/a'_k is unbounded. Let $a_k = 2a'_k$ and $b_k = b'_k - 1$. Then it is easy to see that b_k/a_k is still unbounded. Now it is obvious that the sequences

$$\langle a_i : i < k+1 \rangle \text{ and } \langle b_i : i < k+1 \rangle$$

satisfy the conditions (a)—(d). $\square$

Remarks: (1) We don't know if it is true that bad cuts exist in any nonstandard universe without the $\aleph_1$-isomorphism property.

(2) The $\aleph_1$-isomorphism property is equivalent to the $\aleph_0$-isomorphism property plus ω_1-saturation (see [J3] and [Sch]). In fact every $n \in \mathbb{N}$ is definable in $\mathfrak{A}$. So it is only for convenience to add constants n into the structure $\mathfrak{A}$.

(3) Given any hyperinteger L and K in $\mathcal{H}$ such that K/L is unbounded. Then we can make the bad cut U sitting between L and K, *i.e.* $L \in U$ and $K \notin U$. To do this, just add L and K as constants of $\mathfrak{A}$, add the sentences $X(L)$ and $\neg X(K)$ to , (X) and let $a_1 = L$, $b_1 = K$ at the beginning of the inductive construction. See [KL, Proposition 7.10] for the motivation of this remark.

Next we show that the $\aleph_1$-isomorphism property implies the non-existence of very bad cuts.

Theorem 5 *The $\aleph_1$-isomorphism property implies that for any hyperfinite time line $\mathcal{H} = \{1, 2, \ldots, H\}$ and for any $c \in \mathcal{H}$ such that $c/H \approx 0$ there exists an $X \subseteq \mathcal{H}$ such that*

(1) X has outer Loeb measure one and inner Loeb measure zero, i.e. $|A|/H \approx 1$ *for any internal $A \subseteq \mathcal{H}$ with $X \subseteq A$, and $|A|/H \approx 0$ for any internal $A \subseteq \mathcal{H}$ with $A \subseteq X$,*

(2) for any $x, y \in X$ if $x \neq y$, then $|x - y| \geqslant c$.

Proof: We use same method as in the proof of Theorem 4. Let's fix a nonstandard universe V satisfying the $\aleph_1$-isomorphism property. Let $\mathcal{P}$ be the set of all internal subsets of $\mathcal{H}$. So $\mathcal{P}$ is internal. Define an internally presented structure

$$\mathfrak{A} = (\mathcal{H} \cup \mathcal{P} \cup {}^*\mathbb{R}; \mathcal{H}, \mathcal{P}, \in, \mu, +, \cdot, \leqslant, c, n)_{n \in \mathbb{N}},$$

where $A = \mathcal{H} \cup \mathcal{P} \cup {}^*\mathbb{R}$ is the base set of $\mathfrak{A}$, $\mathcal{H}$ and $\mathcal{P}$ are unary relations, $\in$ is the natural membership relations between the elements of $\mathcal{H}$ and the elements of $\mathcal{P}$, μ is a function from $\mathcal{P}$ to ${}^*\mathbb{R}$ such that for any $A \in \mathcal{P}$, $\mu(A) = |A|/H$, $\langle {}^*\mathbb{R}, +, \cdot, \leqslant \rangle$ is the real field in V, c and n for each $n \in \mathbb{N}$ are constants. Let $\mathcal{L}$ be the language of $\mathfrak{A}$ and let X be a new unary predicate not in $\mathcal{L}$. Let , (X) be the set of $\mathcal{L} \cup \{X\}$-sentences which contains exactly the following:

$$\theta_1(X) = \forall x (X(x) \longrightarrow \mathcal{H}(x))$$

$$\theta_2(X) = \forall x \forall y (X(x) \wedge X(y) \wedge x \neq y \longrightarrow |x - y| \geqslant c)$$

$$\varphi_n(X) = \forall A (\mathcal{P}(A) \wedge X \subseteq A \longrightarrow \mu(A) > 1 - \frac{1}{n})$$

for each $n \in \mathbb{N}$ and

$$\psi_n(X) = \forall A (\mathcal{P}(A) \wedge A \subseteq X \longrightarrow \mu(A) < \frac{1}{n})$$

for each $n \in \mathbb{N}$. It is easy to see that $\theta_1(X)$ says that X is a subset of $\mathcal{H}$, $\theta_2(X)$ says that any two different elements of X have distance greater or equal to c, $\varphi_n(X)$ for all $n \in \mathbb{N}$ say that X has outer Loeb measure one and $\psi_n(X)$ for all $n \in \mathbb{N}$ say that X

has inner Loeb measure zero. So we are done if we can show that , (X) is satisfiable in $\mathfrak{A}$. By Lemma 2 we need only to show the consistency of , $(X) \cup Th(\mathfrak{A})$. Let $\mathfrak{A}'$ be a countable elementary submodel of $\mathfrak{A}$. It suffices to show that , (X) is satisfiable in $\mathfrak{A}'$. Let $A' = \mathcal{H}' \cup \mathcal{P}' \cup \mathbb{R}'$ be the base set of $\mathfrak{A}'$ and let $\mathcal{P}' = \{A_n : n \in \mathbb{N}\}$. We want to construct sets $\{x_n \in \mathcal{H}' : n \in \mathbb{N}\}$ and $\{y_n \in \mathcal{H}' : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$

(a) $\mu(A_n) \ll 1 \longrightarrow x_n \notin A_n$,

(b) $\mu(A_n) \gg 0 \longrightarrow y_n \in A_n$,

(c) $x_n \notin (\bigcup_{m<n}\{x \in \mathcal{H}' : x_m - c \leqslant x \leqslant x_m + c\}) \cup \{y_m : m < n\}$ and $y_n \notin \bigcup_{m \leqslant n} \{x \in \mathcal{H}' : x_m - c \leqslant x \leqslant x_m + c\}$,

where $a \ll b$ means $a < b$ and $a \not\approx b$. Suppose we have found $\{x_m : m < n\}$ and $\{y_m : m < n\}$ such that (a), (b) and (c) are true up to stage $n - 1$. Let

$$I_n = \bigcup_{m<n} \{x \in \mathcal{H}' : x_m - c \leqslant x \leqslant x_m + c\}.$$

Note that

$$\mu(I_n \cup \{y_m : m < n\}) \leqslant (2c + 1)n/H + n/H \approx 0.$$

Thus

$$\mu(\mathcal{H}' \setminus (I_n \cup \{y_m : m < n\})) \approx 1.$$

If $\mu(A_n) \approx 1$, then choose any $x_n \in \mathcal{H}' \setminus (I_n \cup \{y_m : m < n\})$. If $\mu(A_n) \ll 1$, then

$$\mathcal{H}' \setminus (I_n \cup \{y_m : m < n\} \cup A_n) \neq \emptyset.$$

So we can choose x_n in above set. Now let

$$J_n = I_n \cup \{x \in \mathcal{H}' : x_n - c \leqslant x \leqslant x_n + c\}.$$

Then $\mu(J_n) \approx 0$. Let $y_n \in A_n \setminus J_n$ if $\mu(A_n) \gg 0$ and $y_n \in \mathcal{H}' \setminus J_n$ otherwise. This ends the construction.

Let $W = \{x_n : n \in \mathbb{N}\}$. It is clear that $(\mathfrak{A}', W) \models \theta_1(W)$. By the first part of (c) we have $(\mathfrak{A}', W) \models \theta_2(W)$. By (a) we have $(\mathfrak{A}', W) \models \varphi_n(W)$ for every $n \in \mathbb{N}$ and by (b) and the second part of (c) we have $(\mathfrak{A}', W) \models \psi_n(W)$ for every $n \in \mathbb{N}$. So , (X) is satisfiable in $\mathfrak{A}'$. $\square$

Corollary 6 *The $\aleph_1$-isomorphism property implies that there are no very bad cuts.*

Proof: Given any hyperfinite time line $\mathcal{H} = \{1, 2, \ldots, H\}$ and given a bad cut $U \subseteq \mathcal{H}$. Since the cut $H/\mathbb{N}$ is a good cut, then $U \neq H/\mathbb{N}$. Let $c \in H/\mathbb{N} \smallsetminus U$. It is easy to see that $c/H \approx 0$. Let $X \subseteq \mathcal{H}$ be the set obtained in Theorem 5. Then X is a U-nowhere dense set because any interval of length less than c contains at most one element of X. Obviously X does not have Loeb measure zero because it has outer Loeb measure one. So U is not very bad. $\square$

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