$U-$monad Topologies of Hyperfinite Time Lines

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Abstract

In an ω_1–saturated nonstandard universe a cut is an initial segment of the hyperintegers, which is closed under addition. Keisler and Leth in [KL] introduced, for each given cut U, a corresponding U–topology on the hyperintegers by letting O be $U-$open if for any $x \in O$ there is a y greater than all the elements in U such that the interval $[x-y, x+y] \subseteq O$. Let U be a cut in a hyperfinite time line $\mathcal{H}$, which is a hyperfinite initial segment of the hyperintegers. The $U-$monad topology of $\mathcal{H}$ is the quotient topology of the $U-$topological space $\mathcal{H}$ modulo U. In this paper we answer a question of Keisler and Leth about the $U-$monad topologies by showing that when $\mathcal{H}$ is $\kappa-$saturated and has cardinality κ, (1) if the coinitiality of U_1 is uncountable, then the U_1-monad topology and the U_2-monad topology are homeomorphic iff both U_1 and U_2 have the same coinitiality; (2) $\mathcal{H}$ can produce exactly three different $U-$monad topologies (up to homeomorphism) for those U's with countable coinitiality. As a corollary $\mathcal{H}$ can produce exactly four different $U-$monad topologies if the cardinality of $\mathcal{H}$ is ω_1 .

Throughout this paper we work within ω_1–saturated nonstandard universes. We let $\mathcal{M}$ be a nonstandard universe and ${}^*\mathbb{N}$ be the set of all hyperintegers in $\mathcal{M}$ which contains $\mathbb{N}$, the set of all standard positive integers. Let $H \in {}^*\mathbb{N} - \mathbb{N}$; we call $\mathcal{H} = \{n \in {}^*\mathbb{N} : \ltimes \leq \mathbb{H}\}$ a hyperfinite time line or a hyperline for short. We always let H be the largest element of $\mathcal{H}$. Let $[a, b] = \{x \in \mathcal{H} : a \leq x \leq b\}$ be an interval in $\mathcal{H}$.

Let us recall that a cut in $\mathcal{H}$ is an initial segment of $\mathcal{H}$ which is closed under addition. A cut must be external. Let U be a cut in $\mathcal{H}$. A subset O of $\mathcal{H}$ is called $U-$open if for any $x \in O$ there is a $y \in \mathcal{H} - U$ such that $[x - y, x + y] \subseteq O$. All $U-$open sets form a $U-$topology on $\mathcal{H}$.

Let U be a cut in $\mathcal{H}$ and $x \in \mathcal{H}$. $x/U = \{y \in \mathcal{H} : \forall z \in U \ (y \leq x/z)\}$, which is also a cut.

$cf(U)$, the cofinality of U, $= \min\{card(F) : F \subseteq U$ and F is cofinal in $U\}$.

$ci(U)$, the coinitiality of U, $= \min\{card(F) : F \subseteq \mathcal{H} - U$ and F is coinitial in $\mathcal{H} - U\}$.

Let U be a cut in $\mathcal{H}$. For each $x \in \mathcal{H}$ we let $U-$monad(x), the $U-$monad of x, $= \{y \in \mathcal{H} : |y - x| \in U\}$. For a subset B of $\mathcal{H}$ $U-$monad(B)= $\{U-$monad$(x) : x \in B\}$. By a $U-$monad we mean a $U-$monad(x) for some $x \in \mathcal{H}$.

Since the $U-$topology on $\mathcal{H}$ is not Hausdorff, it is sometimes convenient to consider the quotient space of $U-$topological space $\mathcal{H}$ induced by the map $x \longmapsto U-$monad(x). This is called the $U-$monad topology of $\mathcal{H}$ in [KL]. We denote it by $U-$monad$(\mathcal{H})$.

The question about $U-$monad topologies in [KL]:

For which U and V are $U-$monad$(\mathcal{H})$ and $V-$monad$(\mathcal{H})$ homeomorphic?

In this paper we answer the question when $\mathcal{H}$ is $\kappa-$saturated and has cardinality κ. For background in model theory see Chang and Keisler [CK], for background in nonstandard universes see Stroyan and Bayod [SB], and for hyperfinite sets see [KKLM]. This paper was developed under the supervision of H. J. Keisler, to whom the author is deeply grateful.

Throughout this paper we let $card(A)$ mean external cardinality of A and ${}^*card(A)$ mean internal cardinality of A if A is an internal set. When A is hyperfinite, ${}^*card(A)$ is a hyperinteger.

Theorem 1 *Let $\mathcal{H}$ be $\kappa-$saturated (as an ordered set) and $card(\mathcal{H}) = \kappa$. Let U and V be two cuts in $\mathcal{H}$. Then $U-$monad$(\mathcal{H})$ is homeomorphic to $V-$monad$(\mathcal{H})$ if and only if one of the following is true:*

(1) $U = H/\mathbb{N}$ and $V = H/\mathbb{N}$.

(2) $U \neq H/\mathbb{N}$ and $V \neq H/\mathbb{N}$ but there are $x, y \in \mathcal{H}$ such that $U = x/\mathbb{N}$ and $V = y/\mathbb{N}$.

(3) $U \neq x/\mathbb{N}$ and $V \neq y/\mathbb{N}$ for any $x, y \in \mathcal{H}$ but $ci(U) = ci(V)$.

The proof of the theorem is contained in the next eleven lemmas.

For any $A, B \in U-$monad$(\mathcal{H})$ let $A < B$ mean $\forall a \in A\ \forall b \in B(a < b)$. Then $U-$monad$(\mathcal{H})$ is an ordered topological space with order topology. For convenience we consider an ordermorphism instead of a homeomorphism between two monad topological spaces.

From now on we always use "$\cong$" to denote an ordermorphism between two linear orders. And we always let $i = 1, 2$.

Lemma 1 (Hausdorff 1914) *Let $L^{(i)}$ be two $\kappa-$saturated dense linearly ordered sets of power κ such that one of the following is true:*

(1) $L^{(i)}$ both have two end points.

(2) $L^{(i)}$ both have no end points.

(3) $L^{(i)}$ both have only right end points.

(4) $L^{(i)}$ both have only left end points.

Then $L^{(1)} \cong L^{(2)}$.

Lemma 2 *Let $L^{(i)}$ be two linearly ordered sets as in Lemma 1. Let $F^{(i)}$ be a convex segment of $L^{(i)}$ respectively such that $F^{(i)}$ both have left (right) end points and $cf(F^{(1)}) = cf(F^{(2)})$ ($ci(F^{(1)}) = ci(F^{(2)})$). Then $F^{(1)} \cong F^{(2)}$.*

Proof: We can assume $cf(F^{(i)}) = \lambda < \kappa$ by lemma 1.

Let $\langle x^{(i)}_\alpha : \alpha < \lambda \rangle$ be two strictly increasing sequences in $F^{(i)}$ such that the sequences are cofinal in $F^{(i)}$ respectively. Let $F^{(i)}_\alpha = \{x \in F^{(i)} : x \leq^{(i)} x^{(i)}_\alpha\}$.

Now we build an ordermorphism I from $F^{(1)}$ to $F^{(2)}$.

We can assume that $x^{(i)}_0$ are not left end points. By Lemma 1 $F^{(1)}_0 \cong F^{(2)}_0$. Let $I|F^{(1)}_0$ be just this ordermorphism.

Assume that we have $I|F^{(1)}_\beta : F^{(1)}_\beta \longrightarrow F^{(2)}_\beta$ for every $\beta < \alpha$.

Case 1: $\alpha = \beta + 1$.

Let $\tilde{F}^{(i)}_\alpha = (F^{(i)}_\alpha - F^{(i)}_\beta) \bigcup \{x^{(i)}_\beta\}$. Then there exists an $I' : \tilde{F}^{(1)}_\alpha \cong \tilde{F}^{(2)}_\alpha$ by the fact that $x^{(i)}_\alpha > x^{(i)}_\beta$ and by Lemma 1. Let $I|F^{(1)}_\alpha = I|F^{(1)}_\beta \bigcup I'$.

Case 2: α is a limit ordinal below λ.

Let $\tilde{F}^{(i)}_\alpha = F^{(i)}_\alpha - \bigcup_{\beta<\alpha} F^{(i)}_\beta$. Since $cf(\bigcup_{\beta<\alpha} F^{(i)}_\beta) = cf(\alpha) < \kappa$, $ci(\tilde{F}^{(i)}_\alpha) = \kappa$ by $\kappa-$saturation. Since both $\tilde{F}^{(i)}_\alpha$ have right end points $x^{(i)}_\alpha$, there exists an $I' : \tilde{F}^{(1)}_\alpha \cong \tilde{F}^{(2)}_\alpha$ by Lemma 1. Let $I|F^{(1)}_\alpha = (I|\bigcup_{\beta<\alpha} F^{(1)}_\beta) \bigcup I'$.

Now $I = \bigcup_{\alpha<\lambda} I|F^{(1)}_\alpha$ is the ordermorphism from $F^{(1)}$ to $F^{(2)}$. □

From now on we always assume the hyperlines mentioned below are $\kappa-$saturated and have cardinality κ.

Lemma 3 *Let U be a cut in $\mathcal{H}$ such that $ci(U) = \kappa$. Then $U-$monad($\mathcal{H}$) is $\kappa-$saturated (as an ordered set) and has two end points.*

Proof: Easy. □

Lemma 4 *Let $U^{(i)} \subseteq \mathcal{H}^{(i)}$ be two cuts such that $ci(U^{(1)}) = ci(U^{(2)}) = \kappa$. Then $U^{(1)}-$monad($\mathcal{H}^{(1)}$) $\cong U^{(2)}-$monad($\mathcal{H}^{(2)}$).*

Proof: By Lemma 1 and Lemma 3. □

Lemma 5 *$\mathcal{H}^{(1)} \cong \mathcal{H}^{(2)}$ for any two hyperlines $\mathcal{H}^{(i)}$.*

Proof: Since $ci(\mathbb{N}) = \kappa$, then $\mathbb{N}-$monad$(\mathcal{H}^{(1)}) \cong \mathbb{N}-$monad$(\mathcal{H}^{(2)})$ by Lemma 4. Since the left end points of $\mathbb{N}-$monad$(\mathcal{H}^{(i)})$ are the copy of $\mathbb{N}$, the right end points are the copy of the reverse of $\mathbb{N}$ and every other point is just a copy of the integers Z, then we can easily find an ordermorphism from $\mathcal{H}^{(1)}$ to $\mathcal{H}^{(2)}$. □

Lemma 6 *Let $U^{(i)} = H^{(i)}/\mathbb{N}$. Then $U^{(1)}-$monad$(\mathcal{H}^{(1)}) \cong [0,1] \cong U^{(2)}-$monad$(\mathcal{H}^{(2)})$, where $[0,1]$ is the unit interval of the reals.*

Proof: Easy. □

Lemma 7 *If $U^{(i)} = x^{(i)}/\mathbb{N}$ for some $x^{(i)} \in \mathcal{H}^{(i)}$ and $U^{(i)} \neq H^{(i)}/\mathbb{N}$, then $U^{(1)}-$monad$(\mathcal{H}^{(1)}) \cong U^{(2)}-$monad$(\mathcal{H}^{(2)})$.*

Proof: Let $G^{(i)} = \{a \in \mathcal{H}^{(i)} : ax^{(i)} \in \mathcal{H}^{(i)}\}$. Then both $G^{(i)}$ are hyperlines. Hence there exists an ordermorphism $J : G^{(1)} \cong G^{(2)}$ by Lemma 5.

For every $a \in G^{(i)}$ if $a \neq \max G^{(i)}$, let $K_a^{(i)} = [(a-1)x^{(i)}+1, ax^{(i)}]$; if $a = \max G^{(i)}$, let $K_a^{(i)} = [(a-1)x^{(i)}+1, H^{(i)}]$. Then for any $a \in G^{(i)}$ $U^{(i)}-$monad$(K_a^{(i)}) \cong [0,1]$, the unit interval of the reals. So there exists a $j_a : U^{(1)}-$monad$(K_a^{(1)}) \cong U^{(2)}-$monad$(K_{J(a)}^{(2)})$.

Now $I = \bigcup_{a \in G^{(1)}} j_a$ is the ordermorphism from $U^{(1)}-$monad$(\mathcal{H}^{(1)})$ to $U^{(2)}-$monad$(\mathcal{H}^{(2)})$.
□

Lemma 8 *Let $U^{(i)}$ be a cut in $\mathcal{H}^{(i)}$ respectively such that $U^{(i)} \neq x^{(i)}/\mathbb{N}$ for any $x^{(i)} \in \mathcal{H}^{(i)}$ and $ci(U^{(1)}) = ci(U^{(2)}) = \lambda$. Then $U^{(1)}-$monad$(\mathcal{H}^{(1)}) \cong U^{(2)}-$monad$(\mathcal{H}^{(2)})$.*

Proof: Assume $\lambda < \kappa$ (the case $\lambda = \kappa$ has been solved in Lemma 4).

Let $\langle x_\alpha^{(i)} : \alpha < \lambda \rangle$ be a strictly decreasing sequence in $\mathcal{H}^{(i)}$ respectively such that it is coinitial in $\mathcal{H}^{(i)} - U^{(i)}$, $x_0^{(i)} = H^{(i)}$ and $x_\alpha^{(i)}/x_{\alpha+1}^{(i)} > n$ for any $\alpha < \lambda$ and any $n \in \mathbb{N}$.

For convenience we need a notion of trees. (See [To] for the basic notation.)

Let T be any tree of height λ. For any $\alpha < \lambda$ T_α is the $\alpha-$th level of T. $T|\alpha = \bigcup_{\beta<\alpha} T_\beta$. For any $t \in T$, $T(t) = \{s \in T : s \leq_T t \text{ or } t \leq_T s\}$. B is a branch of T iff B is a maximal linearly ordered subset of T and $\mathcal{B}(T) = \{B : B \text{ is a branch of } T\}$.

Now we construct two partition trees $T^{(i)}$ of $\mathcal{H}^{(i)}$ of height λ such that:

(1) $\forall t \in T^{(i)}$ (t is a convex segment of $\mathcal{H}^{(i)}$).

(2) $\forall s, t \in T^{(i)}_\alpha$ $(s \neq t \to s \bigcap t = \emptyset)$.

(3) $\forall t \in T^{(i)}_\alpha \, \forall \beta \, (\alpha < \beta < \lambda \to t = \bigcup\{s : s \in T^{(i)}_\beta \bigcap T^{(i)}(t)\})$.

(4) if $\alpha = \beta + 1$, then $\forall t \in T^{(i)}_\alpha$ ($t = [a, b]$ such that $x^{(i)}_\alpha \leq b - a + 1 \leq 2x^{(i)}_\alpha$).

Let $s, t \in T^{(i)}_\alpha$. We define $s \prec t$ iff $\forall x \in s \, \forall y \in t \, (x < y)$.

(5) there exists a tree isomorphism J from $T^{(1)}$ to $T^{(2)}$ such that

$$\forall \alpha \, \forall s, t \in T^{(1)}_\alpha \, (s \prec t \leftrightarrow J(s) \prec J(t)).$$

If we have these two trees and an isomorphism J which satisfies (5), then for any branch $B \in \mathcal{B}(T^{(i)})$, $\bigcap B$ can only intersect one $U^{(i)}-$monad because any two points in $\bigcap B$ have distance inside $U^{(i)}$ by (4). With the help of the isomorphism J we can build an ordermorphism between $U^{(i)}-$monad$(\mathcal{H}^{(i)})$ since J satisties (5).

Let $T^{(i)}_0 = \{\mathcal{H}^{(i)}\}$ and $J(\mathcal{H}^{(1)}) = \mathcal{H}^{(2)}$.

Assume we have $T^{(i)}|\alpha$ and $J|\alpha$, a tree isomorphism from $T^{(1)}|\alpha$ to $T^{(2)}|\alpha$ which satisfies (5).

Case 1: α is a limit ordinal below λ.

Let $T^{(i)}_\alpha = \{\bigcap B : B \in \mathcal{B}(T^{(i)}|\alpha)\}$. By $\kappa-$saturation $\bigcap B \neq \emptyset$ for any $B \in \mathcal{B}(T^{(i)}|\alpha)$. Let $\bigcap B \in T^{(1)}_\alpha$, where $B = \{t_\beta : t_\beta \in T^{(1)}_\beta$ and $\beta < \alpha\}$. We let $J(\bigcap B) = \bigcap_{\beta<\alpha} J(t_\beta)$. Then $J|\alpha + 1$ is a tree isomorphism from $T^{(1)}|\alpha + 1$ to $T^{(2)}|\alpha + 1$ which satisfies (5).

Case 2: $\alpha = \beta + 1$ and $\beta = \beta' + 1$.

Let $t^{(i)} \in T^{(i)}_\beta$, $t^{(i)} = [a^{(i)}, b^{(i)}]$ such that $J(t^{(1)}) = t^{(2)}$. Let $G^{(i)} = \{x \in \mathcal{H}^{(i)} : a^{(i)} + x x^{(i)}_\alpha \leq b^{(i)}\}$. Then there exists a $j : G^{(1)} \cong G^{(2)}$ by the fact that $G^{(i)}$ are hyperlines and by Lemma 5. For every $a \in G^{(i)}$, if $a \neq \max G^{(i)}$, let $K^{(i)}_a = [a^{(i)} + (a-1)x^{(i)}_\alpha + 1, a^{(i)} + a x^{(i)}_\alpha]$; if $a = \max G^{(i)}$, let $K^{(i)}_a = [a^{(i)} + (a-1)x^{(i)}_\alpha + 1, b^{(i)}]$.

Let $T^{(i)}_\alpha \bigcap T^{(i)}(t^{(i)}) = \{K^{(i)}_a : a \in G^{(i)}\}$ and $J(K^{(1)}_a) = K^{(2)}_{j(a)}$ for any $a \in G^{(1)}$.

Case 3: $\alpha = \beta + 1$ and β is a limit ordinal.

Let $t^{(i)} \in T^{(i)}_\beta$ such that $t^{(i)} = \bigcap\{t^{(i)}_\gamma : t^{(i)}_\gamma = [a^{(i)}_\gamma, b^{(i)}_\gamma] \in T^{(i)}_\gamma$, γ is a successor ordinal and $\gamma < \beta\}$ and $J(t^{(1)}) = t^{(2)}$. Then one of the followings has to be true:

(1) both $t^{(i)}$ have no end points.

(2) both $t^{(i)}$ have left end points (if and only if $\langle a^{(i)}_{\gamma+1} : \gamma < \beta\rangle$ is eventually constant.)

(3) both $t^{(i)}$ have right end points (if and only if $\langle b^{(i)}_{\gamma+1} : \gamma < \beta\rangle$ is eventually constant.)

Let us assume that $t^{(i)}$ both have no end points. (the proofs of the other two cases are just half of the proof of this case.)

By $\kappa-$saturation $cf(t^{(i)}) = ci(t^{(i)}) = \kappa$.

Pick an $a^{(i)}_0 \in t^{(i)}$. Let $G^{(i)}_L = \{a \in \mathcal{H}^{(i)} : a^{(i)}_0 - ax^{(i)}_\alpha \in t^{(i)}\}$ and $G^{(i)}_R = \{a \in \mathcal{H}^{(i)} : a^{(i)}_0 + ax^{(i)}_\alpha \in t^{(i)}\}$. Then $cf(G^{(i)}_L) = cf(G^{(i)}_R) = \kappa$ because if $a^{(i)}_\gamma \neq a^{(i)}_{\gamma+1}$, then $a^{(i)}_{\gamma+1} - a^{(i)}_\gamma \geq x^{(i)}_{\gamma+1}$ and if $b^{(i)}_\gamma \neq b^{(i)}_{\gamma+1}$, then $b^{(i)}_\gamma - b^{(i)}_{\gamma+1} \geq x^{(i)}_{\gamma+1}$.

By Lemma 1 and the proof of Lemma 5 there exist $j_L : G^{(1)}_L \cong G^{(2)}_L$ and $j_R : G^{(1)}_R \cong G^{(2)}_R$.

For any $a \in G^{(i)}_L$ let $K^{(i)}_{L,a} = [a^{(i)}_0 - ax^{(i)}_\alpha + 1, a^{(i)}_0 - (a-1)x^{(i)}_\alpha]$. For any $a \in G^{(i)}_R$ let $K^{(i)}_{R,a} = [a^{(i)}_0 + (a-1)x^{(i)}_\alpha + 1, a^{(i)}_0 + ax^{(i)}_\alpha]$. Now let $T^{(i)}_\alpha \bigcap T^{(i)}(t^{(i)}) = \{K^{(i)}_{L,a} : a \in G^{(i)}_L\} \bigcup \{K^{(i)}_{R,a} : a \in G^{(i)}_R\}$ and let $J(K^{(1)}_{L,a}) = K^{(2)}_{L,j_L(a)}$ and $J(K^{(1)}_{R,a}) = K^{(2)}_{R,j_R(a)}$.

We have now finished construction of two trees $T^{(i)}$ and a J satisfying (1)—(5).

For any $\bar{x}^{(1)} \in U^{(1)}-$monad$(\mathcal{H}^{(1)})$ let $B^{(1)} = \{t^{(1)}_\alpha : t^{(1)}_\alpha \in T^{(1)}_\alpha,\ \alpha < \lambda\} \in \mathcal{B}(T^{(1)})$ such that $\bar{x}^{(1)} \bigcap(\bigcap B^{(1)}) \neq \emptyset$. Then let $I(\bar{x}^{(1)}) = \bar{x}^{(2)} \in U^{(2)}-$monad$(\mathcal{H}^{2})$ if $\bar{x}^{(2)} \bigcap(\bigcap\{J(t^{(1)}_\alpha) : t^{(1)}_\alpha \in B^{(1)}\}) \neq \emptyset$.

If there are two $B^{(1)}, C^{(1)} \in \mathcal{B}(T^{(1)})$ such that $\bar{x}^{(1)} \bigcap(\bigcap B^{(1)}) \neq \emptyset$ and $\bar{x}^{(1)} \bigcap(\bigcap C^{(1)}) \neq \emptyset$, then $B^{(1)}$ and $C^{(1)}$ are adjacent branches. That means if $B^{(1)} = \{t^{(1)}_\alpha : t^{(1)}_\alpha \in T^{(1)}_\alpha, \alpha < \lambda\}, C^{(1)} = \{s^{(1)}_\alpha : s^{(1)}_\alpha \in T^{(1)}_\alpha, \alpha < \lambda\}$ and $t^{(1)}_{\alpha+1} = [a_{\alpha+1}, b_{\alpha+1}], s^{(1)}_{\alpha+1} = [c_{\alpha+1}, d_{\alpha+1}]$ such that $\exists \alpha_0 < \lambda\ (t_{\alpha_0} \prec s_{\alpha_0})$, then $\langle b_{\alpha+1} : \alpha < \lambda\rangle$ and $\langle c_{\alpha+1} : \alpha < \lambda\rangle$ are both eventually constant and $c_{\alpha+1} = b_{\alpha+1} + 1$ for $\alpha \geq \alpha_0$. Since J satisfies (5), $\{J(t^{(1)}_\alpha) : \alpha < \lambda\}$ and $\{J(s^{(1)}_\alpha) : \alpha < \lambda\}$ are also adjacent branches in $\mathcal{B}(T^{(2)})$. Hence $\bigcap_{\alpha<\lambda} J(t^{(1)}_\alpha)$ and $\bigcap_{\alpha<\lambda} J(s^{(1)}_\alpha)$ both can only intersect the same $U^{(2)}-$monad. That implies I is a one to one map. Obviously I is onto

I is an ordermorphism from $U^{(1)}-$monad$(\mathcal{H}^{(1)})$ to $U^{(2)}-$monad$(\mathcal{H}^{(2)})$ because J is a tree isomorphism and satisfies property (5). □

Lemma 9 *If $U = H/\mathbb{N} \subseteq \mathcal{H}$ and $V \neq H'/\mathbb{N} \subseteq \mathcal{H}'$, then $U-$monad$(\mathcal{H})$ is not homeomorphic to $V-$monad$(\mathcal{H}')$.*

Proof: $U-$monad$(\mathcal{H}) \cong [0,1]$, the unit interval of the reals. But $V-$monad$(\mathcal{H}')$ is not separable. □

Lemma 10 *Let $U \subseteq \mathcal{H}$ and $V \subseteq \mathcal{H}'$ be two cuts. If $ci(U) \neq ci(V)$, then $U-monad(\mathcal{H})$ is not homeomorphic to $V-monad(\mathcal{H}')$.*

Proof: Every $x \in U-$monad$(\mathcal{H})$ has character $\chi(x) = ci(U)$ and every $y \in V-$monad$(\mathcal{H}')$ has character $\chi(y) = ci(V)$. □

Lemma 11 *If $U \subseteq \mathcal{H}$ and $V \subseteq \mathcal{H}'$ are two cuts such that $U = x/\mathbb{N}$ for some $x \in \mathcal{H}$ and $ci(V) = \omega$ but $V \neq y/\mathbb{N}$ for any $y \in \mathcal{H}'$, then $U-monad(\mathcal{H})$ is not homeomorphic to $V-monad(\mathcal{H}')$.*

Proof: $U-$monad$(\mathcal{H})$ is locally separable but $V-$monad$(\mathcal{H}')$ is not. □

Proof of Theorem 1:

"$\Longrightarrow$" By Lemma 9, Lemma 10 and Lemma 11.

"$\Longleftarrow$" By Lemma 6, Lemma 7 and Lemma 8. □

Corollary 1 *If $card(\mathcal{H}) = \omega_1$, then $\mathcal{H}$ can produce exactly four different monad topologies. They are $\mathbb{N}-monad(\mathcal{H})$, $H/\mathbb{N}-monad(\mathcal{H})$, $x/\mathbb{N}-monad(\mathcal{H})$ for some $x \in H/\mathbb{N} - \mathbb{N}$ and $U-monad(\mathcal{H})$ for some cut U in $\mathcal{H}$ such that $ci(U) = \omega$ but $U \neq x/\mathbb{N}$ for any $x \in \mathcal{H}$.*

In order to show that the assumptions of $\kappa-$saturation and cardinality κ about the hyperlines in this section are necessary in some sense we give two examples

Example 1: In [M, Theorem 8] A. W. Miller built an ω_1-saturated nonstandard universe under continuum hypothesis in which there exists a hyperinteger h such that $card([1, h]) = \omega_1$ but if y is another hyperinteger such that $y > h^n$ for every $n \in \mathbb{N}$, then $card([1, y]) = \omega_2$.

Let H be any hyperinteger such that $H > h^n$ for every $n \in \mathbb{N}$. Let $U = \mathbb{N}$ and $V = h^{\mathbb{N}}$, then $ci(U) = ci(V) = \omega_1$.

But the left end point of $U-$monad$(\mathcal{H})$ has a neighborhood of cardinality ω_1 and $V-$monad$(\mathcal{H})$ is locally ω_2 .

So $U-$monad$(\mathcal{H})$ is not homeomorphic to $V-$monad$(\mathcal{H})$.

Example 2: In [Ca, Chapter 4] M. Canjar constructs low-saturated $\omega-$ultrapowers of $\mathbb{N}$ within the model obtained by adding κ many random reals into a model of *GCH*.

(In that model $2^\omega = \kappa > \omega_1$.) In his low-saturated ultrapower ${}^*\mathbb{N}$, U, the union of all the skies below the top one, is $\kappa-$saturated but for any H in the top sky there exists a cut W with both cofinality and coinitiality ω_1 such that $U \subseteq W \subseteq [1, H] = \mathcal{H}$.

Let V be a cut in $\mathcal{H}$ such that $ci(V) = ci(U), V \subseteq U$ and $V \neq U$. Then $V-$monad($\mathcal{H}$) has a closed $\kappa-$saturated initial segment. But every segment of $U-$monad($\mathcal{H}$) is not $\kappa-$saturated.

So $U-$monad($\mathcal{H}$) is not ordermorphic or anti-ordermorphic to $V-$monad($\mathcal{H}$).

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