$U-$Lusin Sets In Hyperfinite Time Lines

Renling Jin

Abstract

In an ω_1–saturated nonstandard universe a cut is an initial segment of the hyperintegers, which is closed under addition. Keisler and Leth in [KL] introduced, for each given cut U, a corresponding U–topology on the hyperintegers by letting O be $U-$open if for any $x \in O$ there is a y greater than all the elements in U such that the interval $[x-y, x+y] \subseteq O$. Let U be a cut in a hyperfinite time line $\mathcal{H}$, which is a hyperfinite initial segment of the hyperintegers. A subset B of $\mathcal{H}$ is called a $U-$Lusin set in $\mathcal{H}$ if B is uncountable and for any Loeb–Borel $U-$meager subset X of $\mathcal{H}$, $B \bigcap X$ is countable. Here a Loeb–Borel set is an element of the σ–algebra generated by all internal subsets of $\mathcal{H}$. In this paper we answer some questions of Keisler and Leth about the existence of $U-$Lusin sets by proving that: (1) If $U = x/\mathbb{N} = \{\curvearrowright \in \mathcal{H} : \forall \ltimes \in \mathbb{N}\ (\curvearrowright < \curvearrowleft/\ltimes)\}$ for some $x \in \mathcal{H}$, then there exists a $U-$Lusin set of power κ if and only if there exists a Lusin set of the reals of power κ; (2) If $U \neq x/\mathbb{N}$ but the coinitiality of U is ω, then there are no $U-$Lusin sets if *CH* fails; (3) Under *ZFC* there exists a nonstandard universe in which $U-$Lusin sets exist for every cut U with uncountable cofinality and coinitiality; (4) In any ω_2-saturated nonstandard universe there are no $U-$Lusin sets for all cuts U except $U = x/\mathbb{N}$.

Throughout this paper we work within ω_1-saturated nonstandard universes. We let $\mathcal{M}$ be a nonstandard universe and ${}^*\mathbb{N}$ be the set of all hyperintegers in $\mathcal{M}$ which contains $\mathbb{N}$, the set of all standard positive integers. Let $H \in {}^*\mathbb{N} - \mathbb{N}$; we call $\mathcal{H} = \{n \in {}^*\mathbb{N} : \ltimes \leq \mathbb{H}\}$ a hyperfinite time line or a hyperline for short. We always let H be the largest element of $\mathcal{H}$. Let $[a, b] = \{x \in \mathcal{H} : a \leq x \leq b\}$ be an interval in $\mathcal{H}$ and $[r] = \max\{n \in {}^*\mathbb{N} : \ltimes \leq \diagdown\}$ for any hyperreal r.

A notion of Loeb measure for $\mathcal{H}$, which is the standard part of the countably additive extension of the counting measure on $\mathcal{H}$, was introduced by P. Loeb (cf.[Lo]) as a counterpart of Lebesgue measure for the reals. Recently H. J. Keisler and S. Leth (cf.[KL]) introduced $U-$topologies on $\mathcal{H}$ for any cuts U as an analogue of the order topology on the reals. They discussed the relationship between $U-$meager sets and Loeb measure zero sets and the existence of Loeb–Sierpiński sets and $U-$Lusin sets. They listed many questions at the end of the paper. In this paper we discuss some of those questions about the existence of $U-$Lusin sets. Most of the questions discussed here were motivated by the results of [KL], [M1] and [M2]. For background in model theory see Chang and Keisler [CK], for background in nonstandard universes and the Loeb measure see Stroyan and Bayod [SB], for standard Sierpiński sets and Lusin sets see [M1] and for Loeb–Borel sets and countably determined sets see [KKLM]. This paper was developed under the supervision of H. J. Keisler, to whom the author is deeply grateful.

Throughout this paper we let $card(A)$ mean external cardinality of A and ${}^*card(A)$ mean internal cardinality of A if A is an internal set. When A is hyperfinite, ${}^*card(A)$ is a hyperinteger.

Let us recall that a cut in $\mathcal{H}$ is an initial segment of $\mathcal{H}$ which is closed under addition. A cut must be external. Let U be a cut in $\mathcal{H}$. A subset O of $\mathcal{H}$ is called $U-$open if for any $x \in O$ there is a $y \in \mathcal{H} - U$ such that $[x-y, x+y] \subseteq O$. All $U-$open sets form a $U-$topology on $\mathcal{H}$. A subset of $\mathcal{H}$ is called $U-$nowhere dense ($U-$meager) if it is nowhere dense (meager) in the $U-$topology.

A subset B of $\mathcal{H}$ is called a Loeb–Sierpiński set if B is uncountable and for each set X of Loeb measure zero, $B \bigcap X$ is countable

Let U be a cut in $\mathcal{H}$. We recall that a subset B of $\mathcal{H}$ is called a $U-$Lusin set if B is uncountable and for every Loeb–Borel $U-$meager set X, $B \bigcap X$ is countable. (A subset of $\mathcal{H}$ is called a Loeb–Borel set if it is an element of the $\sigma-$algebra generated by all internal subsets of $\mathcal{H}$.)

We restrict the X above to be a Loeb–Borel set because otherwise $U-$Lusin sets will not exist for most of the cuts U by Proposition 8.3 in [KL].

Now we list some results from [KL] and [M2], which motivate the questions discussed here.

[KL, Proposition 8.9]. Loeb–Sierpiński sets do not exist in any ω_2-saturated nonstandard universe.

[KL, Proposition 8.10]. (CH). Assume that the family of all internal subsets of $\mathcal{H}$ has cardinality ω_1 . Let $\mathcal{U}$ be a set of cuts which is well ordered by the relation $\supseteq$. Then there is a set $B \subseteq \mathcal{H}$ which is $U-$Lusin for every cut $U \in \mathcal{U}$. In particular, for each cut U there is a $U-$Lusin set.

[M2, Theorem 13]. It is consistent with *ZFC* that there exists a Sierpiński set of the reals, but no Loeb–Sierpiński set in any ω_1-saturated nonstandard universe. This shows that Loeb–Sierpiński sets are harder to get than Sierpiński sets of the reals.

[M2, Theorem 16]. It is relatively consistent with *ZFC* that the continuum hypothesis is false but in some ω_1-saturated universe there is a Loeb–Sierpiński set.

The questions follow:

(1) Are there models of *ZFC* in which Lusin sets of the reals exist but there are no nonstandard universes in which $U-$Lusin sets exist?

(2) Can $U-$Lusin sets exist in an ω_2-saturated nonstandard universe?

(3) Can there ever exist $U-$Lusin sets of cardinality ω_2 ?

Let U be a cut in $\mathcal{H}$ and $x \in \mathcal{H}$. The following are cuts:

$xU = \{y \in \mathcal{H} : \exists z \in U \ (y \leq xz)\}$

$x/U = \{y \in \mathcal{H} : \forall z \in U \ (y \leq x/z)\}$

Let U be a cut in $\mathcal{H}$; we let

$cf(U)$, the cofinality of U, $= \min\{card(F) : F \subseteq U$ and F is cofinal in $U\}$.

$ci(U)$, the coinitiality of U, $= \min\{card(F) : F \subseteq \mathcal{H} - U$ and F is coinitial in $\mathcal{H} - U\}$.

U is called a (κ, λ) cut ($(\geq \kappa, \geq \lambda)$ cut) if $cf(U) = \kappa$ and $ci(U) = \lambda$ ($cf(U) \geq \kappa$ and $ci(U) \geq \lambda$).

Theorem 1 *Let $U = x/\mathbb{N}$ for some $x \in \mathcal{H}$. Then there exists a $U-$Lusin set of cardinality κ if and only if there exists a Lusin set of the reals of cardinality κ.*

We need two lemmas before we prove this theorem.

Lemma 1 (Keisler and Leth) *There exists an $H/\mathbb{N}-$Lusin set of cardinality κ implies that there exists a Lusin set of the reals of cardinality κ.*

Proof: See [KL, Proposition 8.4]. □

Lemma 2 *There exists a Lusin set of the reals of cardinality κ implies that there exists an $H/\mathbb{N}-$Lusin set of cardinality κ in $\mathcal{H}$.*

Proof: Let B' be a Lusin set of the reals and $card(B') = \kappa$. We can assume that $B' \subseteq I = [0, 1]$, the unit interval of the reals. For any $x \in \mathcal{H}$ we let $st(x)$ be the standard part of x/H. Then st is a function from $\mathcal{H}$ to I and $|st(x) - st(y)| = 0$ iff $|x - y| \in H/\mathbb{N}$.

For each $b \in B'$ let $x_b \in \mathcal{H}$ such that $st(x_b) = b$ and $B = \{x_b : b \in B'\}$. We claim that B is an $H/\mathbb{N}-$Lusin set and $card(B) = \kappa$.

Let X be any $H/\mathbb{N}-$nowhere dense set in $\mathcal{H}$ (X is not necessarily a Loeb–Borel set) and $X' = \{st(x) : x \in X\} \subseteq I$.

Claim: X' is nowhere dense in I.

Proof of the claim: Let $a', b' \in I$ and $a' < b'$. Let $a, b \in \mathcal{H}$ such that $st(a) = a'$ and $st(b) = b'$. Since $|b - a| \notin U$ and X is $U-$nowhere dense in $\mathcal{H}$, there exist $\bar{c}, \bar{d} \in \mathcal{H}$ such that $a < \bar{c} < \bar{d} < b$, $|\bar{d} - \bar{c}| \notin U$ and $[\bar{c}, \bar{d}] \bigcap X = \emptyset$. Let $c = \bar{c} + [\frac{\bar{d}-\bar{c}}{3}]$ and $d = \bar{c} + [\frac{2(\bar{d}-\bar{c})}{3}]$. Let $c' = st(c)$ and $d' = st(d)$. Then $a' \leq st(\bar{c}) < c' < d' < st(\bar{d}) \leq b'$. Hence $(c', d') \bigcap X' = \emptyset$. This ends the proof of the claim.

$X' \bigcap B'$ is countable since B' is a Lusin set in I. And $X \bigcap B$ is also countable because st is one to one on B. $card(B) = \kappa$ since $st''B = B'$. $\Box$

Proof of Theorem 1:

"$\Longleftarrow$": There exists a Lusin set of cardinality κ in I $\Rightarrow$ There exists an $x/\mathbb{N}-$Lusin set in $[1, x]$ by Lemma 2 $\Rightarrow$ There exists an $x/\mathbb{N}-$Lusin set in $\mathcal{H}$.

"$\Longrightarrow$": Let $x \in \mathcal{H} - \mathbb{N}$. It is sufficient to prove that there exists an $x/\mathbb{N}-$Lusin set of cardinality κ in $\mathcal{H}$ implies that there exists an $x/\mathbb{N}-$Lusin set of cardinality κ in $[1, x]$, by Lemma 1.

Let $B \subseteq \mathcal{H}$ be an $x/\mathbb{N}-$Lusin set and $card(B) = \kappa$.

For any $y \in B$ there exists a $z \in \mathcal{H} \bigcup \{0\}$ such that $xz < y \leq x(z + 1)$. Let $a_y = y - xz$, then $A = \{a_y : y \in B\} \subseteq [1, x]$.

Claim: A is an $x/\mathbb{N}-$Lusin set in $[1, x]$ and $card(A) = \kappa$.

Proof of Claim: If $card(A) < \kappa$ then $\exists a \in A$ such that $card(\{y \in B : a_y = a\}) > \omega$.

Let $A_a = \{a + xz : z \in \mathcal{H}\} \bigcap \mathcal{H}$, then A_a is internal and $x/\mathbb{N}-$nowhere dense. If $a_y = a$ for some $y \in B$, then $y - xz = a \Longrightarrow y \in A_a$, hence $card(A_a \bigcap B) > \omega$. But B is an $x/\mathbb{N}-$Lusin set, a contradiction.

Let $S \subseteq [1, x]$, we define

$$F(S) = \{s + xz : s \in S \text{ and } z \in \mathcal{H} \bigcup \{0\}\} \bigcap \mathcal{H}.$$

Then for any $S \subseteq [1, x]$ and $z \in \mathcal{H} \bigcup \{0\}$, $F(S) \bigcap [xz + 1, x(z + 1)] = \{s + xz : s \in S\} \bigcap \mathcal{H}$. So $S \subseteq [1, x]$ is $x/\mathbb{N}-$nowhere dense $\Longrightarrow$ $F(S)$ is $x/\mathbb{N}-$nowhere dense in $\mathcal{H}$ and $S \subseteq [1,x]$ is internal $\Longrightarrow F(S)$ is internal. Besides, F preserves union, intersection and complement. So if $S \subseteq [1,x]$ is a Loeb–Borel set, then $F(S)$ is also a Loeb–Borel set and S and $F(S)$ are at the same level in the Loeb–Borel hierarchy.

If A is not an $x/\mathbb{N}-$Lusin set in $[1,x]$, then there exists a Loeb–Borel $x/\mathbb{N}-$meager set S in $[1,x]$ such that $S \bigcap A$ is uncountable. For any $a \in S \bigcap A$ there is a $y \in B$ and $z \in \mathcal{H} \bigcup \{0\}$ such that $a = a_y = y - xz$. Then $y = a + xz \in F(S)$, which implies that $y \in F(S) \bigcap B$. Hence $\{a_y : y \in F(S) \bigcap B\}$ is uncountable, so $F(S) \bigcap B$ is uncountable. But $F(S)$ is a Loeb–Borel $x/\mathbb{N}-$meager set and B is an $x/\mathbb{N}-$Lusin set, a contradiction. $\square$

Remark: Let us call B a strong $U-$Lusin set if B is uncountable and has countable intersection with every $U-$meager (without Loeb–Borel) set. Then in the case of $U = x/\mathbb{N}$ the existence of $U-$Lusin sets and the existence of strong $U-$Lusin sets are equivalent.

Theorem 2 *Assume $2^\omega > \omega_1$. Let U be a cut in $\mathcal{H}$ such that $ci(U) = \omega$ and $U \neq x/\mathbb{N}$ for any $x \in \mathcal{H}$. Then there are no $U-$Lusin sets in $\mathcal{H}$.*

Proof: Let $B \subseteq \mathcal{H}$ be a $U-$Lusin set with cardinality ω_1. Let $\langle x_n : n \in \omega \rangle$ be a decreasing sequence in $\mathcal{H}$ such that it is coinitial in $\mathcal{H} - U$ and for each $n \in \omega$ x_n/x_{n+1} is infinite. For any $n \in \omega$ there is an $a_n \in \mathcal{H} - \mathbb{N}$ such that $a_n x_{n+1} < x_n/2$. For any $y \in [1, a_n]$, let

$$A_y = \bigcup \{\, [\,(za_n + y - 1)x_{n+1} + 1, (za_n + y)x_{n+1}] \bigcap \mathcal{H} : z \in \mathcal{H} \bigcup \{0\}\}.$$

Then A_y is internal and $A_y \bigcap A_{y'} = \emptyset$ if $y \neq y'$.

Since $card(B) = \omega_1$ and $card([1, a_n]) = 2^\omega > \omega_1$, then there exists a $y_n \in [1, a]$ such that $A_{y_n} \bigcap B = \emptyset$.

Let

$$F = \mathcal{H} - \bigcup_{n \in \omega} A_{y_n} = \bigcap_{n \in \omega} (\mathcal{H} - A_{y_n}).$$

Then F is a Π^0_1 $U-$nowhere dense set since every interval of length x_n has a subinterval of length x_{n+1} from A_{y_n} which is disjoint from F. But $B \subseteq F$. This contradicts that B is a $U-$Lusin set. $\square$

Theorem 3 *(ZFC) There exists a nonstandard universe $\mathcal{M}$ with a set B in each hyperline $\mathcal{H} \in \mathcal{M}$ such that B is a $U-$Lusin set for every $(\geq \omega_1, \geq \omega_1)$ cut U in $\mathcal{H}$.*

We need a lemma from [Zi].

Lemma 3 (B. Živaljević) *Let U be a $(\geq \omega_1, \geq \omega_1)$ cut in $\mathcal{H}$. Then every countably determined $U-$nowhere dense set can be covered by countably many internal $U-$nowhere dense sets.*

Proof: See [Zi, Proposition 1.2]. □

Remark: Every Loeb–Borel set is countably determined. (cf. [KKLM])

Proof of Theorem 3: Let $\mathcal{M}_0$ be the standard superstructure. We build an $\omega_1 + 1-$elementary chain of nonstandard universes by taking

$$\mathcal{M}_{\alpha+1} \cong \mathcal{M}_\alpha^{\kappa_\alpha}/\mathcal{F}_\alpha$$

for some $\kappa_\alpha \geq$ the cardinality of the family of all internal subsets of ${}^*\mathbb{N}_\alpha$ in $\mathcal{M}_\alpha$. Here ${}^*\mathbb{N}_\alpha$ is the set of all hyperintegers in $\mathcal{M}_\alpha$ and $\mathcal{F}_\alpha$ is a regular ultrafilter over κ_α and

$$\mathcal{M}_\alpha \cong \bigcup_{\beta<\alpha} \mathcal{M}_\beta$$

if α is a limit ordinal below $\omega_1 + 1$. The ω_1-saturated nonstandard universe we want is $\mathcal{M}_{\omega_1}$. For any $H \in {}^*\mathbb{N}_{\omega\nVdash} - \mathbb{N}$ we want to find a $U-$Lusin set in $\mathcal{H}$. Without loss of generality we assume $H \in {}^*\mathbb{N}_{\nVdash}$. For all $\alpha < \omega_1$, let $,\ _\alpha(x)$ be the type of formulas

$$\{x \notin A : A \subseteq \mathcal{H} \text{ is } \mathcal{M}_\alpha\text{–internal } U-\text{nowhere dense for some cut } U \text{ in } \mathcal{H} \cap {}^*\mathbb{N}_\alpha\} \bigcup \{\curvearrowleft \leq \mathbb{H}\}.$$

$,\ _\alpha(x)$ is finitely satisfiable in $\mathcal{M}_\alpha$ and $card(,\ _\alpha(x)) \leq \kappa_\alpha$. By the saturation of the κ_α–regular ultrapower over $\mathcal{M}_\alpha$ there exists an $x_\alpha \in {}^*\mathbb{N}_{\alpha+\nVdash}$ realizing $,\ _\alpha(x)$ in $\mathcal{M}_{\alpha+1}$.

Claim: $\{x_\alpha : \alpha < \omega_1\}$ is a $U-$Lusin set for any $(\geq \omega_1, \geq \omega_1)$ cut U in $\mathcal{H}$.

Proof of the claim: It suffices to prove that $\{x_\alpha : \alpha < \omega_1\}$ has countable intersection with every internal $U-$nowhere dense set by Lemma 3.

Let A be an internal $U-$nowhere dense subset of $\mathcal{H}$ in $\mathcal{M}_{\omega_1}$. Then $\exists \alpha < \omega_1 (A \in \mathcal{M}_\alpha)$. Let $U_\alpha = \{x \in {}^*\mathbb{N}_\alpha : \curvearrowleft \in \mathbb{U}\}$. Then U_α is a cut in ${}^*\mathbb{N}_\alpha \bigcap \mathcal{H}$. Now A is $U_\alpha-$nowhere dense because for any interval I of length y such that $y \in \mathcal{H} \bigcap \mathcal{M}_\alpha$ and $y \notin U_\alpha$,

$$z_0 = \max\{z \in \mathcal{H} : \text{there is a subinterval } J \text{ of } I \text{ such that } {}^*card(J) = z \text{ and } J \bigcap A = \emptyset\}$$

is in $\mathcal{M}_\alpha$ and is not in U. Hence $x_\beta \notin A$ if $\beta \geq \alpha$. $\square$

Remarks: (1) In $\mathcal{H}$ there are $(\geq \omega_1, \geq \omega_1)$ good and bad cuts. (A cut U in $\mathcal{H}$ is called good if there is a U–meager set in $\mathcal{H}$ with Loeb measure one. Otherwise U is called bad.)

(2) In some universes $U-$Lusin sets are easier to get than Lusin sets of the reals for every $(\geq \omega_1, \geq \omega_1)$ cut U.

Theorem 4 *For each hyperline $\mathcal{H}$ in an ω_2-saturated nonstandard universe, if $U \neq x/\mathbb{N}$ for all $x \in \mathcal{H}$, then there are no $U-$Lusin sets in $\mathcal{H}$.*

Before the proof we need one more definition and a lemma by Keisler and Leth.

Let A be an internal subset of $\mathcal{H}$ and $x \in \mathcal{H}$. Define

> $g_A(x) = \max\{b \in \mathcal{H}$: for every interval I of length $\geq x$, there is a subinterval J of I such that ${}^*card(J) \geq b$ and $J \bigcap A = \emptyset$, or $b = 1\}$.

g_A is an internal function if A is internal.

It is easy to see that if $A \subseteq \mathcal{H}$ be internal, then A is $U-$nowhere dense iff $\forall x \notin U(g_A(x) \notin U)$.

Lemma 4 (Keisler and Leth) *In any ω_2-saturated nonstandard universe, if $cf(U) = \omega$, then there are no $U-$Lusin sets.*

Proof: Let U be any cut in $\mathcal{H}$ and $x \in \mathcal{H}$. We call the set $\{y \in \mathcal{H} : |y - x| \in U\}$, the $U-$monad of x. If $cf(U) = \omega$, then every $U-$monad is a Σ^0_1 set and $U-$nowhere dense.

Assume that there is a $U-$Lusin set B. Then the intersection of B and each $U-$monad should be at most countable. Since B is uncountable, we can assume that B has at most one element in every $U-$monad.

Since $cf(U) = \omega$, and by ω_1-saturation, there is an internal sequence $\langle x_n : n < a\rangle$ for some $a \in \mathcal{H} - \mathbb{N}$ such that $\langle x_n : n \leq \omega\rangle$ is increasing, cofinal in U and $\forall n < a \ (x_{n+1}/x_n > 2^{n+1})$. Let , (X) be the type

$$\{\forall n \in [1, a] \ (g_X(x_{n+1}) \geq x_n)\} \bigcup \{F \subseteq X : F \in [B]^{<\omega}\}.$$

Claim: , (X) is finitely satisfiable.

Proof of the claim: Let F be any finite subset of B. Since B has at most one element in each $U-$monad, then $z_0 = \min\{|x-y| : x, y \in F, x \neq y\} \notin U$. Let I be any interval of $\mathcal{H}$ such that ${}^*card(I) \geq x_{n+1}$. If $x_{n+1} \in U$ (that means n is finite), then there exists a subinterval J of I such that ${}^*card(J) \geq x_n$ and $J \bigcap F = \emptyset$ because $z_0 \notin U$ and $x_{n+1}/x_n \geq 2^{n+1} > 2$. If $x_{n+1} \notin U$ (that means n is infinite), then I can be divided into infinitely many disjoint subintervals of length $\geq x_n$ because x_{n+1}/x_n is infinite. One of those subintervals should be disjoint from F since F is finite.

So $g_F(x_{n+1}) \geq x_n$ for every $n \in [1, a]$. This ends the proof of the claim.

Since $card(, (X)) < \omega_2$, there is an internal set A which is $U-$nowhere dense and $B \subseteq A$, a contradiction. □

Proof of Theorem 4: We prove the theorem case by case.

Assume that $B = \{x_\alpha : \alpha \in \omega_1\}$ is a $U-$Lusin set in $\mathcal{H}$.

Case 1: $U = x\mathbb{N}$ for some $x \in \mathcal{H}$.

This contradicts Lemma 4 since $cf(x\mathbb{N}) = \omega$.

Case 2: $U \neq x\mathbb{N}$ for any $x \in \mathcal{H}$ and $cf(U) = \kappa \leq \omega_1$.

Let $\langle y_\alpha : \alpha < \kappa \rangle$ be increasing and cofinal in U such that $y_{\alpha+1}/y_\alpha > n$ for any $n \in \mathbb{N}$ (there is such a sequence because $U \neq x\mathbb{N}$).

Let

$$, (X) = \{g_X(y_{\alpha+1}) \geq y_\alpha : \alpha < \kappa\} \bigcup \{F \subseteq X : F \in [B]^{<\omega}\}.$$

Claim: $, (X)$ is finitely satisfiable.

Proof of the claim: Let F be any finite subset of B and I be an interval of $\mathcal{H}$ such that ${}^*card(I) \geq y_{\alpha+1}$. Then I can be divided into infinitely many subintervals of length $\geq y_\alpha$ since $y_{\alpha+1}/y_\alpha$ is infinite. One of those subintervals should be disjoint from F since F is finite. So $g_F(y_{\alpha+1}) \geq y_\alpha$. This ends the proof of the claim.

Since $card(, (X)) < \omega_2$, there exists an A which realizes $, (X)$ by ω_2-saturation. Then A is internal $U-$nowhere dense and $B \subseteq A$, a contradiction.

Case 3: $U \neq x/\mathbb{N}$ for any $x \in \mathcal{H}$ and $ci(U) = \kappa \leq \omega_1$.

Let $\langle y_\alpha : \alpha < \kappa \rangle$ be decreasing and coinitial in $\mathcal{H} - U$ such that $y_\alpha/y_{\alpha+1} > n$ for all $n \in \mathbb{N}$ (there is such a sequence because $U \neq x/\mathbb{N}$).

Let

$$, (X) = \{g_X(y_\alpha) \geq y_{\alpha+1} : \alpha < \kappa\} \bigcup \{F \subseteq X : F \in [B]^{<\omega}\}.$$

Then , (X) is finitely satisfiable for the same reason as in Case 2. By ω_2-saturation there exists an internal $U-$nowhere dense set A which covers B, a contradiction.

Case 4: U is a $(\geq \omega_2, \geq \omega_2)$ cut.

Let $D(B) = \{|x-y| : x, y \in B\}$. Then $card(D(B)) \leq \omega_1$. Hence

$$\exists x, y \in \mathcal{H}\ (x \notin U \text{ and } y \in U \text{ such that } [y,x] \bigcap D(B) = \emptyset).$$

Let

$$, (X) = \{\forall z \in \mathcal{H}\ (3y \leq z \leq x \rightarrow g_X(z) \geq [z/3])\} \bigcup \{F \subseteq X : F \in [B]^{<\omega}\}.$$

Claim: , (X) is finitely satisfiable.

Proof of the claim: Let F be any finite subset of B, $z \in [3y, x]$ and $[c, d]$ be an interval of $\mathcal{H}$ such that $|d-c| \geq z$. Let $J_1 = [c, c+[z/3]], J_2 = [c+[z/3]+1, c+2[z/3]]$ and $J_3 = [c+2[z/3]+1, c+3[z/3]]$. Then $J_1 \bigcap F \neq \emptyset$ implies $J_3 \bigcap F = \emptyset$ because $D(B) \bigcap [y,x] = \emptyset$. So $g_F(z) \geq [z/3]$ and this ends the proof of the claim. By ω_2-saturation there exists an internal $U-$nowhere dense subset of $\mathcal{H}$ which covers B, a contradiction. $\square$

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Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA.