# Compactness of Loeb Spaces[1]

Renling Jin[2] & Saharon Shelah[3]

Abstract

In this paper we show that the compactness of a Loeb space depends on its cardinality, the nonstandard universe it belongs to and the underlying model of set theory we live in. In §1 we prove that Loeb spaces are compact under various assumptions, and in §2 we prove that Loeb spaces are not compact under various other assumptions. The results in §1 and §2 give a quite complete answer to a question of D. Ross in [R1], [R2] and [R3].

## 0 Introduction

In [R1] and [R2] D. Ross asked: *Are (bounded) Loeb measure spaces compact?* J. Aldaz then, in [A], constructed a counterexample. But Aldaz's example is atomic, while most of Loeb measure spaces people are interested are atomless. So Ross re-asked his question in [R3]: *Are atomless Loeb measure spaces compact?* In this paper we answer the question. Let's assume that all measure spaces mentioned throughout this paper are atomless probability spaces.

Given a probability space $(\Omega, \Sigma, P)$. A subfamily $\mathcal{C} \subseteq \Sigma$ is called compact if for any $\mathcal{D} \subseteq \mathcal{C}$, $\mathcal{D}$ has f.i.p. *i.e.* finite intersection property, implies $\bigcap \mathcal{D} \neq \emptyset$. We call a compact family $\mathcal{C}$ inner-regular on $\Omega$ if for any $A \in \Sigma$

$$P(A) = \sup\{P(C) : C \subseteq A \wedge C \in \mathcal{C}\}.$$

A probability space $(\Omega, \Sigma, P)$ is called compact if $\Sigma$ contains an inner-regular compact subfamily. Clearly, the definition of compactness is a generalization of Radon spaces with no topology involved. In fact, Ross proved in [R2] that a compact probability space is essentially Radon, *i.e.* one can topologize the space so that every

[1] *Mathematics Subject Classification* Primary 28E05, 03H05, 03E35.

[2]The research of the first author was supported by NSF postdoctoral fellowship #DMS-9508887. This research was started when the first author spent a wonderful year as a visiting assistant professor in University of Illinois-Urbana Champaign during 94-95. He is grateful to the logicians there.

[3]The research of the second author was supported by The Israel Science Foundation administered by The Israel Academy of Sciences and Humanities. This paper is number 613 on the second author's publication list.

measurable set $A$ contains a compact subset of measure at least half of the measure of $A$.

Loeb measure spaces are important tools in nonstandard analysis (see, for example, [AFHL] and [SB]). Ross proved in [R2] that every compact probability space is the image, under a measure preserving transformation, of a Loeb measure space. This shows, by a word of Ross, some evidence that Loeb spaces themselves may be compact.

In this paper we show that the compactness of a Loeb space depends on its cardinality, the nonstandard universe it belongs to, and even the underlying world of set theory we live in (suppose we live in a transitive model of ZFC).

Throughout this paper we always denote $\mathcal{M}$ for our underlying transitive model of set theory ZFC. We sometimes use $\mathcal{N}$ for another transitive model of ZFC. If we make a statement without mentioning a particular model, this statement is always assumed to be relative to $\mathcal{M}$. Let $\mathbb{N}$ be the set of all standard natural numbers. Using $\mathbb{N}$ as a set of urelements, we construct the standard universe $(V, \in)$ by letting

$$V_0 = \mathbb{N},\ V_{n+1} = V_n \cup \mathcal{P}(V_n) \text{ and } V = \bigcup_{n\in\omega} V_n.$$

A nonstandard universe $({}^*V,\ {}^*\in)$ is the truncation, at ${}^*\in$-rank $\omega$, of an elementary extension of the standard universe such that ${}^*\mathbb{N} \smallsetminus \mathbb{N} \neq \emptyset$. We always assume the nonstandard universe ${}^*V$ we work within is at least $\omega_1$-saturated. In fact, $\omega_1$-saturation is needed in Loeb measure construction. For any set $S$ we use $|S|$ for its set theoretic cardinality. If $S$ is an internal set (in ${}^*V$), then ${}^*|S|$ means the internal cardinality of $S$. For any object $S$ in the standard universe we always denote ${}^*S$ for its nonstandard version in ${}^*V$. For example, if $\Omega$ is an internal set, then ${}^*\mathcal{P}(\Omega)$ denote the set of all internal subsets of $\Omega$. Let $\Sigma_0 \subseteq {}^*\mathcal{P}(\Omega)$ be an internal algebra and let $P : \Sigma_0 \mapsto {}^*[0,1]$ be an internal finitely additive probability measure. We call $(\Omega, \Sigma_0, P)$ an internal probability space. Let $st : {}^*[0,1] \mapsto [0,1]$ be the standard part map. Then $(\Omega, \Sigma_0, st\circ P)$ is a standard finitely additive probability space. Then one can use $\Sigma_0$ to generate uniquely an $st\circ P$-complete $\sigma$-algebra $\Sigma$ and extend $st\circ P$ uniquely to a standard complete countably additive probability measure $L_P$. The space $(\Omega, \Sigma, L_P)$ is called a Loeb space generated by $(\Omega, \Sigma_0, P)$. Let $H$ be a hyperfinite integer, *i.e.* $H \in {}^*\mathbb{N} \smallsetminus \mathbb{N}$. Let $\Omega = \{1, 2, \ldots, H\}$, let $\Sigma_0 = {}^*\mathcal{P}(\Omega)$ and let $P(A) = {}^*|A|/H$ for each $A \in \Sigma_0$. We call $(\Omega, \Sigma_0, P)$ a hyperfinite internal space. The space $(\Omega, \Sigma, L_P)$ generated by $(\Omega, \Sigma_0, P)$ as above is called a hyperfinite Loeb space. Hyperfinite Loeb spaces are most useful among other Loeb spaces in nonstandard analysis. For notational simplicity we prove the results only for hyperfinite Loeb spaces in this paper. From now on we denote the symbol $(\Omega, \Sigma, L_P)$ or just $\Omega$ without confusion, exclusively for a hyperfinite Loeb space. Most of the results for hyperfinite Loeb spaces in this paper can be easily generalized to Loeb spaces in the general sense (see Fact 3 at the beginning of §1 and the comments after that).

In §1 we show when a hyperfinite Loeb space is compact. We prove the following results.

*Corollary 3*: Suppose CH (Continuum Hypothesis) holds. Suppose $|{}^*\mathbb{N}| = \omega_1$. Then every hyperfinite Loeb space in ${}^*V$ is compact.

*Corollary 4*: Suppose MA (Martin's Axiom) holds. Suppose ${}^*V$ is $2^\omega$-saturated and $|{}^*\mathbb{N}| = 2^\omega$. Then every hyperfinite Loeb space in ${}^*V$ is compact.

*Corollary 5*: Suppose $\mathcal{M}$ is obtained by adding $\kappa$ Cohen reals to a ZFC model $\mathcal{N}$ for some $\kappa \geqslant (2^\omega)^{\mathcal{N}}$ with $\kappa^\omega = \kappa$ in $\mathcal{N}$. Suppose $|{}^*\mathbb{N}| = 2^\omega$ (in $\mathcal{M}$ now). Then every hyperfinite Loeb space in ${}^*V$ is compact.

*Corollary 7*: Suppose $\mathcal{M}$ is same as in Corollary 5. In $\mathcal{M}$ suppose $\lambda$ is a strong limit cardinal with $cf(\lambda) \leqslant \kappa$. Suppose $|{}^*\mathbb{N}| = \lambda$ and ${}^*V$ satisfies the $\aleph_0$-special model axiom (see [R5] or [J] for the definition). Then every hyperfinite Loeb space in ${}^*V$ is compact.

*Theorem 8*: Suppose $\kappa$ is a strong limit cardinal with $cf(\kappa) = \omega$ and $\lambda = \kappa^+ = 2^\kappa$. Suppose ${}^*V$ is $\lambda$-saturated and has cardinality $\lambda$. Then every hyperfinite Loeb space in ${}^*V$ is compact.

In §2 we show when a hyperfinite Loeb space is not compact. We prove the following results.

*Theorem 9*: Suppose $\lambda$ is a regular cardinal such that $\kappa^\omega < \lambda$ for any $\kappa < \lambda$. Suppose $|\Omega| = \lambda$. Then $\Omega$ is not compact.

*Theorem 10*: Suppose $\lambda$ is a strong limit cardinal, $\kappa = cf(\lambda)$ and $\mu^\omega < \kappa$ for any $\mu < \kappa$. Suppose $|\Omega| = \lambda$. Then $\Omega$ is not compact.

*Theorem 11*: Suppose $\mathcal{M}$ is obtained by adding $\kappa$ random reals to a ZFC model $\mathcal{N}$ for some regular $\kappa > (2^\omega)^{\mathcal{N}}$ with $\kappa^\omega = \kappa$. Suppose $|\Omega| = 2^\omega$ (note $2^\omega = \kappa$ in $\mathcal{M}$). Then $\Omega$ is not compact.

*Theorem 12*: Suppose $\mathcal{M}$ is obtained by adding $\kappa$ random reals to a ZFC model $\mathcal{N}$ for some regular $\kappa > \omega$. Suppose $\lambda$ is a strong limit cardinal such that $cf(\lambda) \leqslant \kappa$. Suppose $|\Omega| = \lambda$ (hence $cf(\lambda) > \omega$). Then $\Omega$ is not compact.

*Theorem 13*: Let $\lambda > |V|$ and $\lambda^\omega = \lambda$. Then there exists a ${}^*V$ such that $|\Omega| = \lambda$ and $\Omega$ is not compact for every $\Omega$ in ${}^*V$.

In this paper we write $\lambda, \kappa, \mu, \ldots$ for cardinals, $\alpha, \beta, \gamma, \ldots$ for ordinals and $k, m, n, \ldots$ for natural numbers. We write $\lambda^\kappa$ ($\lambda^{<\kappa}$) for cardinal exponents and ${}^\kappa\lambda$ (${}^{<\kappa}\lambda$) for sets of functions. For any set $S$ we write $[S]^\lambda$ for the set of all subsets of $S$ with cardinality $\lambda$. For any set $S$ we write $({}^S2, \Sigma({}^S2), \nu_S)$ for the complete probability space generated by all Baire sets of ${}^S2$ such that for any finite $S_0 \subseteq S$ and any $\tau \in {}^{S_0}2$, $\nu_S([\tau]) = 2^{-|S_0|}$, where $[\tau] = \{f \in {}^S 2 : f \restriction S = \tau\}$.

The reader is assumed to know basics of nonstandard analysis and be familiar with nonstandard universes and Loeb space construction. We suggest the reader consult [L] and [SB] for information on those subjects. The reader is also assumed to have basic knowledge on set theory and forcing. The reader is recommended to consult [K1] for that.

# 1 Towards Compactness

We would like to list three facts about hyperfinite Loeb spaces $(\Omega, \Sigma, L_P)$, which will be used frequently throughout this paper.

**Fact 1**: For any $S \in \Sigma$ and any $\epsilon > 0$ there exists an $A \in {}^*\mathcal{P}(\Omega)$ such that $A \subseteq S$ and $L_P(S \smallsetminus A) < \epsilon$.

**Fact 2**: For any internal sets $A_n \subseteq \Omega$ there exists an internal set $B \subseteq \bigcap_{n\in\omega} A_n$ such that $L_P(\bigcap_{n\in\omega} A_n) = L_P(B)$.

A $\lambda$-sequence $\langle A_\alpha : \alpha \in \lambda \rangle$ of measurable subsets of $\Omega$ is called independent if for any finite $I_0 \subseteq \lambda$

$$L_P(\bigcap_{\alpha\in I_0} A_\alpha) = \prod_{\alpha\in I_0} L_P(A_\alpha).$$

**Fact 3**: Suppose $|\Omega| = \lambda$. Then there exists an independent $\lambda$-sequence of internal sets of measure $\frac{1}{2}$ on $\Omega$.

Fact 1 and Fact 2 are direct consequences of $\omega_1$-saturation and Loeb measure construction. Fact 3 can be proved by finite combinatorics and the overspill principle in nonstandard analysis. Call a sequence $\langle A_i : i < m \rangle$ of subsets of a finite probability space of size $n$ with normalized counting measure $\mu$ a $(k, c)$-independent sequence for some $k \in \mathbb{N}$ and $c \in (0,1)$ iff for any $i_1 < i_2 < \ldots < i_{k'}$ with $k' < k$ and any $h \in {}^{k'}2$ one has

$$\mu(\bigcap_{j=1}^{k'} A_{i_j}^{h(j)}) \geqslant \frac{c}{2^{k'}}.$$

Note that given any $k \in \mathbb{N}$ and any $c \in (0,1)$, there is a $(k,c)$-independent sequence of length $n$ in an $n$-elements probability space for large enough $n \in \mathbb{N}$. Let $H$ be an infinite integer. Applying overspill principle one can find an infinite integer $K$ and a nonstandard real $C$ with $st(C) = 1$ such that there is a $(K,C)$-independent sequence of internal sets of length $H$ in $\Omega = \{1, 2, \ldots, H\}$. It is easy to check that the sequence obtained is, in standard sense, an independent sequence of the hyperfinite Loeb probability space $\Omega$.

For a Loeb space in the general sense Fact 1 and Fact 2 are also true. But Fact 3 may not hold. So whether or not a result about hyperfinite Loeb space can be generalized to a general Loeb space may depend on the truth of Fact 3.

A set $t \subseteq {}^{<\omega}2$ is called a tree if for any $s, s' \in {}^{<\omega}2$, $s \subseteq s'$ and $s' \in t$ imply $s \in t$. We use capital letter $T \subseteq {}^{<\omega}2$ exclusively for a tree with no maximal node. So every branch of $T$ is infinite. For a tree $T$ we write $[T]$ for the set of all its branches. In fact, every closed subset of ${}^{\omega}2$ could be written as $[T]$ for some tree $T$.

**Definition 1** *A sequence of trees $\langle T_{\alpha,n} : \alpha \in \lambda \wedge n \in \omega \rangle$ is called a $(\kappa, \lambda)$-witness if*

*(1)* $\nu_\omega([T_{\alpha,n}]) > \frac{n}{n+1}$,

*(2)* $(\forall f \in {}^{\omega}2)(|\{\alpha \in \lambda : \exists n(f \in [T_{\alpha,n}])\}| < \kappa)$.

**Theorem 2** *Suppose there exists a $(\kappa, \lambda)$-witness $\langle T_{\alpha,n} : \alpha \in \lambda \wedge n \in \omega \rangle$ for some uncountable cardinals $\kappa$ and $\lambda$. Suppose ${}^*V$ is $\kappa$-saturated and $|{}^*\mathcal{P}(\Omega)| = \lambda$. Then $\Omega$ is compact.*

**Proof:** Choose $\langle A_n : n \in \omega \rangle$, an independent $\omega$-sequence of internal subsets with measure $\frac{1}{2}$ on $\Omega$. We write $A_n^0 = A_n$ and $A_n^1 = \Omega \setminus A_n$. Then for any finite $s \subseteq \omega$ and any $h \in {}^{s}2$ we have

$$L_P(\bigcap_{n \in s} A_n^{h(n)}) = 2^{-|s|}.$$

For any tree $T$ define

$$A_T = \bigcap_{n \in \omega} \bigcup_{\eta \in {}^{n}2 \cap T} \bigcap_{i=0}^{n-1} A_i^{\eta(i)}.$$

It is easy to see that $L_P(A_T) = \nu_\omega([T])$. Note that $A_T$ is a countable intersection of internal sets. We now want to construct an inner-regular compact family $\mathcal{C}$ of internal subsets on $\Omega$. Let ${}^*\mathcal{P}(\Omega) = \{a_\alpha : \alpha \in \lambda\}$. For each $\alpha \in \lambda$ and $n \in \omega$ let $b_{\alpha,n} \subseteq a_\alpha \cap A_{T_{\alpha,n}}$ be internal such that

$$L_P(b_{\alpha,n}) = L_P(a_\alpha \cap A_{T_{\alpha,n}}).$$

Then let

$$\mathcal{C} = \{A_n^l : n \in \omega \wedge l = 0, 1\} \cup \{b_{\alpha,n} : \alpha \in \lambda \wedge n \in \omega\}.$$

**Claim 2.1**: $\mathcal{C}$ is an inner-regular compact family on $\Omega$.

Proof of Claim 2.1: The inner-regularity is clear. We need to prove the compactness. Let $\mathcal{D} \subseteq \mathcal{C}$ be such that $\mathcal{D}$ has f.i.p. We want to show that $\bigcap \mathcal{D} \neq \emptyset$. Without loss of generality we assume that $\mathcal{D}$ is maximal. So for each $n \in \omega$ either $A_n^0 \in \mathcal{D}$ or $A_n^1 \in \mathcal{D}$ but not both. Let $h \in {}^\omega 2$ be such that for each $n \in \omega$ we have $A_n^{h(n)} \in \mathcal{D}$. Given any $b_{\alpha,n} \in \mathcal{D}$, we want to show that $h \in [T_{\alpha,n}]$. Let $k \in \omega$. Then

$$(\bigcap_{i=0}^{k-1} A_i^{h(i)}) \cap b_{\alpha,n} \neq \emptyset.$$

So we have

$$(\bigcap_{i=0}^{k-1} A_i^{h(i)}) \cap (\bigcap_{m \in \omega} \bigcup_{\eta \in {}^m 2 \cap T_{\alpha,n}} \bigcap_{i=0}^{m-1} A_i^{\eta(i)}) \neq \emptyset.$$

This implies that there exists an $\eta \in T_{\alpha,n} \cap {}^k 2$ such that

$$(\bigcap_{i=0}^{k-1} A_i^{h(i)}) \cap (\bigcap_{i=0}^{k-1} A_i^{\eta(i)}) \neq \emptyset.$$

Hence we have $h \restriction k = \eta \restriction k \in T_{\alpha,n}$. This is true for any $k \in \omega$. So $h \in [T_{\alpha,n}]$. But we assumed that

$$|\{\alpha : \exists n (h \in [T_{\alpha,n}])\}| < \kappa.$$

So $|\mathcal{D}| < \kappa$. Now using $\kappa$-saturation, we get $\bigcap \mathcal{D} \neq \emptyset$. □

**Remark:** From the definition of the compactness we do not have to choose $\mathcal{C}$ as a family of internal sets. We do that because internal sets are more interesting. In this paper if we construct a compact family we always construct a family of internal sets.

**Corollary 3** *Suppose CH holds and $|{}^*\mathbb{N}| = \omega_1$. Then every hyperfinite Loeb space in ${}^*V$ is compact.*

**Proof:** It suffices to construct an $(\omega_1, \omega_1)$-witness. Let ${}^{\omega}2 = \{f_\alpha : \alpha \in \omega_1\}$. For each $n \in \omega$ and $\alpha \in \omega_1$, choose $T_{\alpha,n}$ such that

$$\nu_\omega([T_{\alpha,n}]) > \frac{n}{n+1}$$

and

$$[T_{\alpha,n}] \cap \{f_\beta : \beta \in \alpha\} = \emptyset.$$

It is clear that $\langle T_{\alpha,n} : \alpha \in \omega_1 \wedge n \in \omega \rangle$ is an $(\omega_1, \omega_1)$-witness. $\square$

**Remarks:** (1) The condition $|{}^*\mathbb{N}| = \omega_1$ implies CH by $\omega_1$-saturation of ${}^*V$.

(2) If ${}^*V$ is an $\omega$-ultrapower of the standard universe, then $|{}^*\mathbb{N}| = \omega_1$, provided CH holds.

**Corollary 4** *Suppose for any $S \subseteq {}^{\omega}2$, $|S| < 2^\omega$ implies $\nu_\omega(S) = 0$. Suppose ${}^*V$ is $2^\omega$-saturated and $|{}^*\mathbb{N}| = 2^\omega$. Then every hyperfinite Loeb space in ${}^*V$ is compact.*

**Proof:** By same construction as in Corollary 3 with length $2^\omega$ we can find a $(2^\omega, 2^\omega)$-witness. Now the corollary follows from $2^\omega$-saturation of ${}^*V$ and Theorem 2. $\square$

**Remark:** Obviously, Corollary 3 is a special case of Corollary 4. In the case of $\neg$CH, one has that MA implies $\nu_\omega(S) = 0$ for any set $S \subseteq {}^{\omega}2$ with $|S| < 2^\omega$ and MA implies also $2^\kappa = 2^\omega$ for any $\kappa < 2^\omega$, which guarantees the existence of $2^\omega$-saturated nonstandard universes.

**Corollary 5** *Suppose $\mathcal{M}$ is obtained by adding $\lambda$ Cohen reals to a ZFC model $\mathcal{N}$ for some $\lambda \geqslant (2^\omega)^{\mathcal{N}}$ with $\lambda^\omega = \lambda$ in $\mathcal{N}$. Suppose $|{}^*\mathbb{N}| = \lambda$ ($2^\omega = \lambda$ in $\mathcal{M}$). Then every hyperfinite Loeb space in ${}^*V$ is compact.*

**Proof:** It suffices to construct an $(\omega_1, \lambda)$-witness. Work in $\mathcal{N}$. For each $n \in \omega$ let

$$\mathcal{T}_n = \{t \subseteq {}^{<\omega}2 : (\exists T \subseteq {}^{<\omega}2)(\nu_\omega([T]) > \frac{n+1}{n+2} \wedge \exists m(t = T \restriction m))\}$$

be a forcing notion ordered by the reverse of end-extension of trees (we assume smaller conditions are stronger). It is clear that $\mathcal{T}_n$ is countable and separative. So forcing with $\mathcal{T}_n$ is same as adding a Cohen real. Let $\mathcal{T}_n^\alpha = \mathcal{T}_n$, let $\mathbb{P}_\alpha = \prod_{n\in\omega} \mathcal{T}_n^\alpha$ with finite supports for each $\alpha \in \lambda$ and let $\mathbb{P} = \prod_{\alpha\in\lambda} \mathbb{P}_\alpha$ with finite supports. Without loss of generality we assume that $\mathcal{M} = \mathcal{N}[G]$, where $G \subseteq \mathbb{P}$ is an $\mathcal{N}$-generic filter. For each $\alpha \in \lambda$ and $n \in \omega$ let

$$T_{\alpha,n} = \bigcup (G \cap \mathcal{T}_n^\alpha).$$

We want to show that the sequence $\langle T_{\alpha,n} : \alpha \in \lambda \wedge n \in \omega \rangle$ is an $(\omega_1, \lambda)$-witness.

Given any $f \in {}^\omega 2$ in $\mathcal{M}$, there exists a countable set $S \subseteq \lambda$ in $\mathcal{N}$ such that $f \in \mathcal{N}[G_S]$, where $G_S = G \cap (\prod_{\alpha \in S} \mathbb{P}_\alpha)$. For any $\alpha \in \lambda \setminus S$ and any $n \in \omega$, $G \cap \mathcal{T}_n^\alpha \subseteq \mathcal{T}_n^\alpha$ is a $\mathcal{N}[G_S]$-generic filter. Define

$$D_f = \{t \in \mathcal{T}_n^\alpha : \exists m \, (t \text{ has height } m \wedge f \restriction m \notin t)\}.$$

**Claim 5.1** $\mathcal{D}_f$ is dense in $\mathcal{T}_n^\alpha$.

Proof of Claim 5.1: Let $t \in \mathcal{T}_n^\alpha$. We want to find a $t' \in \mathcal{T}_n^\alpha \cap D_f$ such that $t'$ is an end-extension of $t$. Let $m'$ be the height of $t$. Without loss of generality we assume that $f \restriction m' \in t$. Let

$$T = \{\eta \in {}^{<\omega}2 : \eta \in t \vee \eta \restriction m' \in t\}.$$

It is clear that $\nu_\omega([T]) > \frac{n+1}{n+2}$. Let

$$\epsilon = \nu_\omega([T]) - \frac{n+1}{n+2}$$

and let $n' > m'$ be large enough so that $2^{-n'} < \epsilon$. Let

$$T' = \{\eta \in T : |\eta| \leqslant n' \vee \eta \restriction (n'+1) \neq f \restriction (n'+1)\}.$$

Now

$$\nu_\omega([T']) > \nu_\omega([T]) - \epsilon = \frac{n+1}{n+2}.$$

Let $m = n' + 1$. Then we have

$$t' = T' \restriction m \in \mathcal{T}_n^\alpha \cap D_f$$

and that $t'$ is an end-extension of $t$. □ (Claim 5.1)

Since $D_f$ is dense in $\mathcal{T}_n^\alpha$, then $G \cap \mathcal{T}_n^\alpha \cap D_f \neq \emptyset$. This implies $f \notin [T_{\alpha,n}]$. So

$$\{\alpha \in \lambda : \exists n (f \in [T_{\alpha,n}])\} \subseteq S.$$

This shows that $\langle T_{\alpha,n} : \alpha \in \lambda \wedge n \in \omega \rangle$ is an $(\omega_1, \lambda)$-witness. □

**Remarks:** (1) We don't have requirements for ${}^*V$ because ${}^*V$ is always $\omega_1$-saturated.

(2) Above three corollaries can be easily generalized to general Loeb spaces as long as their cardinalities are $2^\omega$. For example, above three corollaries are also true if we replace a hyperfinite Loeb space by a Loeb space generated by a nonstandard version of Lebesgue measure on unit interval. From now on we will not make similar remarks like this. The reader should be able to do so by himself.

Next we will mention a property of nonstandard universes called the $\aleph_0$-special model axiom (see [R5] or [J] for details). The special model axiom is in fact an axiomatization of special nonstandard universes as models (see [CK] for the definition of a special model). In the proof we need only some simple consequences of the property. Let's list all the consequences we need. If ${}^*V$ satisfies the $\aleph_0$-special model axiom, then

(1) all infinite internal sets have same cardinality, say $\lambda$,

(2) for every hyperfinite internal space $(\Omega,\ {}^*\mathcal{P}(\Omega), P)$ there exists a sequence $\langle(\Omega_\alpha, \Sigma_\alpha) : \alpha \in cf(\lambda)\rangle$, called a specializing sequence, such that

(a) $\Omega = \bigcup_{\alpha\in cf(\lambda)} \Omega_\alpha$,

(b) ${}^*\mathcal{P}(\Omega) = \bigcup_{\alpha\in cf(\lambda)} \Sigma_\alpha$,

(c) if $\{A_n : n \in \omega\} \subseteq \Sigma_\alpha$, then there exists a $B \in \Sigma_{\alpha+1}$ such that $B \subseteq \bigcap_{n\in\omega} A_n$ and $L_P(B) = L_P(\bigcap_{n\in\omega} A_n)$,

(d) if $\mathcal{D} \subseteq \Sigma_\alpha$ and $\mathcal{D}$ has f.i.p., then $(\bigcap \mathcal{D}) \cap \Omega_{\alpha+1} \neq \emptyset$.

**Theorem 6** *Suppose there exists an increasing sequence $\langle Z_\alpha \subseteq {}^\omega 2 : \alpha \in \kappa\rangle$ for some regular cardinal $\kappa > \omega$ such that $\nu_\omega(Z_\alpha) = 0$ for every $\alpha \in \kappa$ and ${}^\omega 2 = \bigcup_{\alpha\in\kappa} Z_\alpha$ (so $\kappa \leqslant 2^\omega$). Suppose $\lambda$ is a strong limit cardinal with $cf(\lambda) = \kappa$. Suppose ${}^*V$ satisfies the $\aleph_0$-special model axiom and $|{}^*\mathbb{N}| = \lambda$. Then every hyperfinite Loeb space in ${}^*V$ is compact.*

**Proof:** Given a hyperfinite Loeb space $\Omega$, let $\langle(\Omega_\beta, \Sigma_\beta) : \beta \in \kappa\rangle$ be a specializing sequence of $\Omega$. For each $\beta \in \kappa$ let $T_{\beta,n} \subseteq {}^{<\omega}2$ be such that

$$\nu_\omega([T_{\beta,n}]) > \frac{n}{n+1} \text{ and } [T_{\beta,n}] \cap Z_\beta = \emptyset.$$

Without loss of generality we can pick an independent sequence $\langle A_n : n \in \omega\rangle$ of internal sets with measure $\frac{1}{2}$ in $\Sigma_0$. Let ${}^*\mathcal{P}(\Omega) = \{a_\alpha : \alpha \in \lambda\}$. For each $\alpha \in \lambda$ let

$$g(\alpha) = \min\{\beta \in \kappa : a_\alpha \in \Sigma_\beta\}.$$

We now construct an inner-regular compact family $\mathcal{C}$ on $\Omega$. For each $\alpha \in \lambda$ and each $n \in \omega$ let $b_{\alpha,n} \in \Sigma_{g(\alpha)+1}$ be such that

$$b_{\alpha,n} \subseteq a_\alpha \cap A_{T_{g(\alpha),n}}$$

and

$$L_P(b_{\alpha,n}) = L_P(a_\alpha \cap A_{T_{g(\alpha),n}})$$

(check Theorem 2 for the definition of $A_T$). Define now

$$\mathcal{C} = \{b_{\alpha,n} : \alpha \in \lambda \wedge n \in \omega\} \cup \{A_n^l : n \in \omega \wedge l = 0, 1\}$$

(recall $A^0 = A$ and $A^1 = \Omega \smallsetminus A$).

**Claim 6.1** $\mathcal{C}$ is an inner-regular compact family on $\Omega$.

Proof of Claim 6.1: Again the inner-regularity is clear. Let $\mathcal{D} \subseteq \mathcal{C}$ be such that $\mathcal{D}$ has f.i.p. We want to show $\bigcap \mathcal{D} \neq \emptyset$. Again we assume $\mathcal{D}$ is maximal. Let

$$\delta = \bigcup \{g(\alpha) : \exists n (b_{\alpha,n} \in \mathcal{D})\}.$$

Case 1: $\delta < \kappa$. Then $\mathcal{D} \subseteq \Sigma_{\delta+1}$. By the special model axiom we have $\bigcap \mathcal{D} \neq \emptyset$.

Case 2: $\delta = \kappa$. Let $h \in {}^\omega 2$ be such that $A_n^{h(n)} \in \mathcal{D}$. Same as the proof of Claim 2.1 we have that $h \in [T_{g(\alpha),n}]$ if $b_{\alpha,n} \in \mathcal{D}$. But there is a $\beta_0 \in \kappa$ such that $h \in Z_{\beta_0}$. So we have $h \notin [T_{g(\alpha),n}]$ for any $g(\alpha) > \beta_0$. This contradicts $\delta = \kappa$. $\square$

**Remarks:** (1) The nonstandard universes satisfying the $\aleph_0$-special model axiom and having cardinality $\lambda$ exist. In fact, those universes are frequently used by Ross (see [R4] and [R5]).

(2) This theorem guarantees the existence of arbitrarily large compact hyperfinite Loeb spaces.

(3) The set theoretical assumption besides ZFC for $\mathcal{M}$ in this theorem is rather weak. The model $\mathcal{M}$ satisfies the assumption if $\mathcal{M}$ is a model of *e.g.* CH or MA, or is obtained by adding enough Cohen reals.

**Corollary 7** *Suppose $\mathcal{M}$ is same as in Corollary 5. In $\mathcal{M}$ suppose $\lambda$ is a strong limit cardinal with $cf(\lambda) \leqslant \kappa$. Suppose $|{}^*\mathbb{N}| = \lambda$ and ${}^*V$ satisfies the $\aleph_0$-special model axiom. Then every hyperfinite Loeb space in ${}^*V$ is compact.*

**Proof:** First we arrange the Cohen forcing such that $\mathcal{M}$ is a forcing extension of some model of ZFC by adding $cf(\lambda)$ Cohen reals. Then by [K2, Theorem 3.20] we know that ${}^{\omega}2$ is a union of an increasing $cf(\lambda)$-sequence of measure zero sets. $\square$

**Remark:** There is another proof by a method similar to the proof of Corollary 5.

**Theorem 8** *Suppose $\kappa$ is a strong limit cardinal with $cf(\kappa) = \omega$ and suppose $\lambda = \kappa^+ = 2^\kappa$. Suppose ${}^*V$ is $\lambda$-saturated and $|{}^*\mathbb{N}| = \lambda$. Then every hyperfinite Loeb space in ${}^*V$ is compact.*

**Proof:** Given a hyperfinite Loeb space $\Omega$ in ${}^*V$, we want to show that $\Omega$ is compact. Let $\kappa = \bigcup_{n\in\omega} \kappa_n$ be such that $\kappa_0 > \omega$ and $\kappa_{n+1} > 2^{\kappa_n}$ for each $n \in \omega$. Choose an independent $\kappa$-sequence

$$\langle A_\alpha \in {}^*\mathcal{P}(\Omega) : \alpha \in \kappa\rangle$$

of internal sets with measure $\frac{1}{2}$. For any $n, m$ let

$$\mathcal{B}_n = \mathcal{B}(\{A_\alpha : \alpha \in \kappa_{n+1}\})$$

be the Boolean algebra generated by $A_\alpha$'s for all $\alpha \in \kappa_{n+1}$ and let

$$Pos_{n,m} = \{X \in \mathcal{B}_n : L_P(X) > \frac{m}{m+1}\}.$$

Note that every $X \in Pos_{n,m}$ is internal with measure $> \frac{m}{m+1}$ because it is a finite Boolean combination of internal sets. For each $n, m \in \omega$ let

$$I_{n,m} = \{E \subseteq Pos_{n,m} : E \text{ has f.i.p.}\}.$$

For each $m \in \omega$ let

$$Pos_{<\omega,m} = \{\bar{X} : \exists n(\bar{X} = \langle X_0, X_1, \ldots, X_{n-1}\rangle \wedge (\forall i < n)(X_i \in Pos_{n,m}))\}.$$

and let

$$\mathcal{F}_m = \{F : F \text{ is a function from } Pos_{<\omega,m} \text{ to } \bigcup_{n\in\omega} I_{n,m} \text{ such that}$$

$$(\forall \bar{X} = \langle X_0, \ldots, X_{n-1}\rangle \in Pos_{<\omega,m})(F(\bar{X}) \in I_{n,m})\}.$$

It is clear that $|\mathcal{F}_m| \leqslant \kappa^\kappa = \lambda$. Let $\mathcal{F}_m = \{F_{\alpha,m} : \alpha \in \lambda\}$ be a fixed enumeration. For each $\alpha \in \lambda$ let's fix an increasing sequence $\langle B_{\alpha,n} \subseteq \alpha : n \in \omega\rangle$ such that $|B_{\alpha,n}| \leqslant \kappa_n$

for each $n \in \omega$ and $\alpha = \bigcup_{n\in\omega} B_{\alpha,n}$. We define a function $f_{\alpha,m}$ from $\omega$ to $\bigcup_{n\in\omega} Pos_{n,m}$ for each $\alpha \in \lambda$ by induction on $n$ such that for each $n \in \omega$

(1) $f_{\alpha,m}(n) \in Pos_{n,m}$,

(2) $f_{\alpha,m}(n+1) \subseteq f_{\alpha,m}(n)$,

(3) $f_{\alpha,m}(n) \notin \bigcup\{F_{\beta,m}(f_{\alpha,m}\restriction n) : \beta \in B_{\alpha,n}\}$.

Suppose we have defined $f_{\alpha,m}\restriction n$.

**Claim 8.1** There is an $X = f_{\alpha,m}(n)$ such that (1), (2) and (3) hold.

Proof of Claim 8.1: For each $\beta \in B_{\alpha,n}$ let $\mathcal{D}_\beta$ be an ultrafilter on $\mathcal{B}_n$ such that

$$\mathcal{D}_\beta \supseteq F_{\beta,m}(f_{\alpha,m}\restriction n)$$

(note that $F_{\beta,m}(f_{\alpha,m}\restriction n)$ has f.i.p.). Let $f_{\alpha,m}(n-1) = C$ at stage $n > 0$ (replace $C$ by $\Omega$ at stage $n = 0$). For each $\gamma \in [\kappa_n, \kappa_{n+1})$ let

$$J_\gamma = \{\beta \in B_{\alpha,n} : C \cap A_\gamma \in \mathcal{D}_\beta\}.$$

Since $|B_{\alpha,n}| \leqslant \kappa_n$ and $2^{\kappa_n} < \kappa_{n+1} = |[\kappa_n, \kappa_{n+1})|$, then there exists an $E \subseteq [\kappa_n, \kappa_{n+1})$ with $|E| = \kappa_{n+1}$ such that for any two different $\gamma, \gamma' \in E$ we have $J_\gamma = J_{\gamma'}$. Let $\gamma_0 < \gamma'_0 < \gamma_1 < \gamma'_1 < \ldots$ be in $E$ and let

$$C_n = (A_{\gamma_n} \cap C)\Delta(A_{\gamma'_n} \cap C) = (A_{\gamma_n}\Delta A_{\gamma'_n}) \cap C,$$

where $\Delta$ means symmetric difference. It is easy to see that for any $n \in \omega$ and any $\beta \in B_{\alpha,n}$ we have $C_n \notin \mathcal{D}_\beta$. It is also clear that

$$L_P(C_n) = L_P(C)L_P(A_{\gamma_n}\Delta A_{\gamma'_n}) = \frac{1}{2}L_P(C).$$

Since $L_P(C) > \frac{m}{m+1}$, there exists a big enough $N \in \omega$ such that

$$(1 - (\frac{1}{2})^N)L_P(C) > \frac{m}{m+1}.$$

Now let $f_{\alpha,m}(n) = \bigcup_{i=0}^{N-1} C_i$. It is easy to see that (2) and (3) hold. For (1) we have

$$L_P(f_{\alpha,m}(n)) = L_P(C \cap (\bigcup_{i=0}^{N-1}(A_{\gamma_i}\Delta A_{\gamma'_i})))$$

$$= L_P(C)L_P(\bigcup_{i=0}^{N-1}(A_{\gamma_i}\Delta A_{\gamma'_i}))$$

$$= (1 - (\frac{1}{2})^N) L_P(C) > \frac{m}{m+1}.$$

$\square$ (Claim 8.1)

We now define an inner-regular compact family $\mathcal{C}$ on $\Omega$. Let ${}^*\mathcal{P}(\Omega) = \{a_\alpha : \alpha \in \lambda\}$ be an enumeration. For each $\alpha \in \lambda$ and $m \in \omega$, let

$$b_{\alpha,m} \subseteq a_\alpha \cap (\bigcap_{n \in \omega} f_{\alpha,m}(n))$$

be internal such that

$$L_P(b_{\alpha,m}) = L_P(a_\alpha \cap (\bigcap_{n \in \omega} f_{\alpha,m}(n))).$$

Now let

$$\mathcal{C} = \{b_{\alpha,m} : \alpha \in \lambda \wedge m \in \omega\}.$$

**Claim 8.2** $\mathcal{C}$ is an inner-regular compact family on $\Omega$.

Proof of Claim 8.2: The inner-regularity of $\mathcal{C}$ is obvious. Let $\mathcal{D} \subseteq \mathcal{C}$ be such that $\mathcal{D}$ has f.i.p. If $|\mathcal{D}| < \lambda$, then $\bigcap \mathcal{D} \neq \emptyset$ by $\lambda$-saturation. So let's assume $|\mathcal{D}| = \lambda$. Hence there exists an $m_0 \in \omega$ such that

$$Z = \{\alpha : b_{\alpha,m_0} \in \mathcal{D}\}$$

has cardinality $\lambda$. Let's prove next claim first.

**Claim 8.3** There exists an $\bar{X} = \langle X_0, X_1, \ldots, X_{n-1}\rangle$ for some $n \in \omega$ such that

$$\{f_{\alpha,m_0}(n) : \alpha \in Z \wedge f_{\alpha,m_0} \restriction n = \bar{X}\} \notin I_{n,m_0}.$$

Proof of Claim 8.3: Suppose not. Then we can define a function

$$F : Pos_{<\omega,m_0} \mapsto \bigcup_{n \in \omega} I_{n,m_0}$$

such that for each $\bar{X} = \langle X_0, \ldots, X_{n-1}\rangle$

$$F(\bar{X}) = \{f_{\alpha,m_0}(n) : \alpha \in Z \wedge f_{\alpha,m_0} \restriction n = \bar{X}\}.$$

It is clear that $F \in \mathcal{F}_{m_0}$. So there is an $\beta \in \lambda$ such that $F = F_{\beta,m_0}$. Since $|Z| = \lambda$ there is an $\alpha \in Z$ such that $\alpha > \beta$. Now choose large enough $n \in \omega$ such that $\beta \in B_{\alpha,n}$. Then

$$f_{\alpha,m_0}(n) \notin F_{\beta,m_0}(f_{\alpha,m_0} \restriction n)$$

by the construction of $f_{\alpha,m_0}$. But this contradicts the definition of $F = F_{\beta,m_0}$. $\square$ (Claim 8.3)

We continue the proof of Claim 8.2. By Claim 8.3 there exists an $\bar{X} = \langle X_0, \ldots, X_{n-1} \rangle$ such that

$$\{f_{\alpha,m_0}(n) : \alpha \in Z \wedge f_{\alpha,m_0} \restriction n = \bar{X}\}$$

does not have f.i.p. Hence $\{b_{\alpha,m_0} : \alpha \in Z\}$ does not have f.i.p. because $b_{\alpha,m_0} \subseteq f_{\alpha,m_0}(n)$. This contradicts that $\mathcal{D}$ has f.i.p. $\square$

**Remarks:** (1) $\lambda$-saturated nonstandard universes of cardinality $\lambda$ exist because $\lambda^{<\lambda} = \lambda$.

(2) Under certain assumptions for $\mathcal{M}$, *e.g.* Singular Cardinal Hypothesis, this theorem guarantees the existence of arbitrarily large compact hyperfinite Loeb spaces with regular cardinality.

(3) The proof of this theorem is implicitly included in [Sh575].

(4) D. Fremlin recently found an easier proof of the theorem based on a generalization of the proof of Corollary 3, and the fact that any subset $S$ of ${}^{\kappa}2$ with $|S| \leqslant \kappa$ has $\nu_\kappa(S) = 0$. Now we have an independent sequence $\langle A_\alpha : \alpha < \kappa \rangle$ of length $\kappa$ instead of length $\omega$. For any closed subset $[T] \subseteq {}^{\kappa}2$ one has a correspondent set $A_T$ with $L_P(A_T) = \nu_\kappa([T])$. Let ${}^{\kappa}2 = \{f_\alpha : \alpha < \lambda\}$ and let $\Omega = \{a_\alpha : \alpha < \lambda\}$. Choose $[T_{\alpha,n}]$ so that $[T_{\alpha,n}] \cap \{f_\beta : \beta < \alpha\} = \emptyset$ and $\nu_\kappa([T_{\alpha,n}]) \geqslant \frac{n}{n+1}$. It is not hard to see that the set $\mathcal{C} = \{a_\alpha \cap A_{T_{\alpha,n}} : \alpha < \lambda,\ n \in \omega\} \cup \{A^j_\alpha : \alpha < \kappa,\ j = 0, 1\}$ is an inner-regular compact family.

## 2 Towards non-compactness

Remember that our nonstandard universes are at least $\omega_1$-saturated. For a measure space $(X, \Sigma, P)$ we write $\bar{\mathcal{B}}(X)$ for the measure algebra of $X$, *i.e.* the Boolean algebra of measurable sets modulo the ideal of measure zero sets. If $\mathcal{D} \subseteq \Sigma$, we write $\bar{\mathcal{B}}(\mathcal{D})$ for the complete subalgebra of $\bar{\mathcal{B}}(X)$ generated by $\mathcal{D}$.

**Theorem 9** *Suppose $\lambda$ is a regular cardinal such that $\kappa^\omega < \lambda$ for every $\kappa < \lambda$. Suppose $|\Omega| = \lambda$. Then $\Omega$ is not compact.*

**Proof:** Suppose not. Let $\mathcal{C}$ be an inner-regular compact family on $\Omega$. Let $\langle A_\alpha : \alpha \in \lambda \rangle$ be an independent $\lambda$-sequence of internal subsets of $\Omega$ of Loeb measure $\frac{1}{2}$. Pick $X^l_{\alpha,n} \in \mathcal{C}$ for every $\alpha \in \lambda$, $n \in \omega$ and $l = 0, 1$ such that $X^l_{\alpha,n} \subseteq A^l_\alpha$ and

$$L_P(\bigcup_{n\in\omega} X^l_{\alpha,n}) = \frac{1}{2}.$$

Note that $A^0_\alpha = A_\alpha$ and $A^1_\alpha = \Omega \setminus A_\alpha$, but generally $X^1_{\alpha,n}$ will not be $\Omega \setminus X^0_{\alpha,n}$.

**Claim 9.1** There exists an $E \in [\lambda]^\lambda$ and there exists an $n(\alpha, l)$ for each $\alpha \in E$ and $l = 0, 1$ such that for any $m \in \omega$, any distinct $\{\alpha_0, \ldots, \alpha_{m-1}\} \subseteq E$ and any $h \in {}^m2$ we have

$$L_P(\bigcap_{i=0}^{m-1} X^{h(i)}_{\alpha_i, n(\alpha_i, h(i))}) > 0.$$

The theorem follows from the claim. Since $|\Omega| = \lambda < 2^\lambda$, we can find $f \in {}^E2$ such that

$$\bigcap_{\alpha\in E} X^{f(\alpha)}_{\alpha, n(\alpha, f(\alpha))} = \emptyset.$$

But $\{X^{f(\alpha)}_{\alpha, n(\alpha, f(\alpha))} : \alpha \in E\} \subseteq \mathcal{C}$ has f.i.p.

Proof of Claim 9.1: For each measurable set $A \subseteq \Omega$ let $\bar{A}$ denote the element in the measure algebra, represented by $A$. For each $\alpha \in \lambda$ let

$$\bar{\mathcal{B}}_\alpha = \bar{\mathcal{B}}(\{A_\beta : \beta < \alpha\} \cup \{X^l_{\beta,n} : \beta < \alpha \wedge n \in \omega \wedge l = 0, 1\}).$$

Recall that $\bar{\mathcal{B}}(X)$ for some family $X$ of measurable sets is a complete subalgebra of measure algebra on $\Omega$ generated by $X$. By c.c.c. of measure algebra it is easy to see that $|\bar{\mathcal{B}}_\alpha| \leqslant |\alpha|^\omega$. Notice that the sequence $\langle \bar{\mathcal{B}}_\alpha : \alpha \in \lambda \rangle$ is increasing and $\bar{\mathcal{B}}_\alpha = \bigcup_{\beta<\alpha} \bar{\mathcal{B}}_\beta$ when $cf(\alpha) > \omega$. Let $\mathcal{B}$ be a complete Boolean algebra. Given $a \in \mathcal{B}$ and a subalgebra $\mathcal{B}' \subseteq \mathcal{B}$, define a function

$$g(a, \mathcal{B}') = \inf\{b \in \mathcal{B}' : b \geqslant a\}.$$

$g(a, \mathcal{B}')$ exists since $\mathcal{B}$ is complete. Let

$$D = \{\alpha \in \lambda : cf(\alpha) > \omega\}.$$

Then $D$ is stationary in $\lambda$. For each $\alpha \in D$ let

$$d(\alpha) = \min\{\beta : g(\bar{A}^0_\alpha, \bar{\mathcal{B}}_\alpha) \in \bar{\mathcal{B}}_\beta \wedge g(\bar{A}^1_\alpha, \bar{\mathcal{B}}_\alpha) \in \bar{\mathcal{B}}_\beta\}.$$

Then $d(\alpha) < \alpha$ for every $\alpha \in D$. By Pressing-Down Lemma we can find a stationary subset $E \subseteq D$ and an $\alpha_0 \in \lambda$ such that $d(\alpha) = \alpha_0$ for every $\alpha \in E$. Since

$$|\bar{\mathcal{B}}_{\alpha_0}| \leqslant |\alpha_0|^\omega < \lambda,$$

we can assume that there are $b_0, b_1 \in \bar{\mathcal{B}}_{\alpha_0}$ such that for all $\alpha \in E$ we have

$$g(\bar{A}^0_\alpha, \bar{\mathcal{B}}_\alpha) = b_0 \text{ and } g(\bar{A}^1_\alpha, \bar{\mathcal{B}}_\alpha) = b_1.$$

By thinning $E$ further we can assume that $\bar{A}_\alpha \notin \bar{\mathcal{B}}_{\alpha_0}$ for each $\alpha \in E$. Hence $\bar{A}_\alpha \notin \bar{\mathcal{B}}_\alpha$ for each $\alpha \in E$. It is easy to see that $b_0 \wedge b_1 \neq 0$ because otherwise we have, for any $\alpha \in E$,

$$\bar{A}^0_\alpha \leqslant b_0 \leqslant -b_1 \leqslant -\bar{A}^1_\alpha = \bar{A}^0_\alpha$$

and this implies $\bar{A}^0_\alpha = b_0 \in \bar{\mathcal{B}}_{\alpha_0}$.

**Claim 9.2** For any $\alpha \in E$ there exist $n(\alpha, 0)$ and $n(\alpha, 1)$ such that

$$g(\bar{X}^0_{\alpha,n(\alpha,0)}, \bar{\mathcal{B}}_\alpha) \wedge g(\bar{X}^1_{\alpha,n(\alpha,1)}, \bar{\mathcal{B}}_\alpha) \neq 0.$$

Proof of Claim 9.2: Suppose not. Then for any $n, m \in \omega$ we have

$$\bar{X}^0_{\alpha,n} \leqslant g(\bar{X}^0_{\alpha,n}, \bar{\mathcal{B}}_\alpha) \leqslant -g(\bar{X}^1_{\alpha,m}, \bar{\mathcal{B}}_\alpha) \leqslant -\bar{X}^1_{\alpha,m}.$$

So then

$$\bar{A}^0_\alpha = \bigvee_{n\in\omega} \bar{X}^0_{\alpha,n} \leqslant \bigvee_{n\in\omega} g(\bar{X}^0_{\alpha,n}, \bar{\mathcal{B}}_\alpha) \leqslant \bigwedge_{m\in\omega} (-g(\bar{X}^1_{\alpha,m}, \bar{\mathcal{B}}_\alpha))$$
$$\leqslant \bigwedge_{m\in\omega} (-\bar{X}^1_{\alpha,m}) = -\bigvee_{m\in\omega} \bar{X}^1_{\alpha,m} = -\bar{A}^1_\alpha = \bar{A}^0_\alpha.$$

This implies

$$\bar{A}_\alpha = \bar{A}^0_\alpha = \bigvee_{n\in\omega} g(\bar{X}^0_{\alpha,n}, \bar{\mathcal{B}}_\alpha) \in \bar{\mathcal{B}}_\alpha,$$

a contradiction. $\square$ (Claim 9.2)

We continue to prove Claim 9.1. By thinning $E$ even further we can assume that there exist $d_0, d_1 \in \bar{\mathcal{B}}_{\alpha_0}$ such that for every $\alpha \in E$,

$$g(\bar{X}^0_{\alpha,n(\alpha,0)}, \bar{\mathcal{B}}_\alpha) = d_0 \text{ and } g(\bar{X}^1_{\alpha,n(\alpha,1)}, \bar{\mathcal{B}}_\alpha) = d_1.$$

It follows from Claim 9.2 that $d = d_0 \wedge d_1 \neq 0$.

We now prove, by induction on $m$, that for any $m \in \omega$, for any distinct $\{\alpha_0, \ldots, \alpha_{m-1}\} \subseteq E$ and for any $h \in {}^m 2$,

$$d \wedge (\bigwedge_{i=0}^{m-1} \bar{X}^{h(i)}_{\alpha_i, n(\alpha_i, h(i))}) \neq 0.$$

Suppose the above is true for $m$, but not true for $m+1$. Pick some distinct $\{\alpha_0, \ldots, \alpha_m\} \subseteq E$ and $h \in {}^{m+1}2$ such that $\alpha_i < \alpha_m$ for each $i < m$ and

$$d \wedge (\bigwedge_{i=0}^{m} \bar{X}^{h(i)}_{\alpha_i, n(\alpha_i, h(i))}) = 0.$$

Without loss of generality let $h(m) = 0$. Then

$$(d \wedge (\bigwedge_{i=0}^{m-1} \bar{X}^{h(i)}_{\alpha_i, n(\alpha_i, h(i))})) \wedge \bar{X}^0_{\alpha_m, n(\alpha_m, 0)} = 0.$$

Then

$$\bar{X}^0_{\alpha_m, n(\alpha_m, 0)} \leqslant -(d \wedge (\bigwedge_{i=0}^{m-1} \bar{X}^{h(i)}_{\alpha_i, n(\alpha_i, h(i))})).$$

So

$$g(\bar{X}^0_{\alpha_m, n(\alpha_m, 0)}, \bar{\mathcal{B}}_{\alpha_m}) = d_0 \leqslant -(d \wedge (\bigwedge_{i=0}^{m-1} \bar{X}^{h(i)}_{\alpha_i, n(\alpha_i, h(i))})).$$

This implies

$$d_0 \wedge (d \wedge (\bigwedge_{i=0}^{m-1} \bar{X}^{h(i)}_{\alpha_i, n(\alpha_i, h(i))})) = d \wedge (\bigwedge_{i=0}^{m-1} \bar{X}^{h(i)}_{\alpha_i, n(\alpha_i, h(i))}) = 0.$$

This contradicts the inductive hypothesis. □

**Remarks:** (1) Not like other results so far, Theorem 9 is a consequence of ZFC.

(2) When $\lambda = (\eta^\theta)^+$ for some infinite cardinals $\eta$ and $\theta$, we have $\kappa^\omega < \lambda$ for any $\kappa < \lambda$. So ZFC implies the existence of arbitrarily large non-compact hyperfinite Loeb spaces in some nonstandard universes.

(3) The proof of this theorem is implicitly included in [Sh92].

(4) The proof works also for general Loeb spaces if they have an independent $\lambda$-sequence of measure $\frac{1}{2}$.

(5) The reader could find a shorter proof by using Maharam theorem and a similar idea in the proof of Theorem 11.

**Theorem 10** *Suppose $\lambda$ is a strong limit cardinal, $\kappa = cf(\lambda)$ and $\mu^\omega < \kappa$ for any $\mu < \kappa$. Suppose $|\Omega| = \lambda$. Then $\Omega$ is not compact.*

**Proof:** Suppose $\Omega$ is compact and let $\mathcal{C}$ be an inner-regular compact family on $\Omega$. Let $\langle \lambda_\alpha : \alpha \in \kappa \rangle$ be an increasing sequence such that $2^{\lambda_\alpha} < \lambda_{\alpha+1}$ for each $\alpha \in \kappa$ and $\lambda = \bigcup_{\alpha\in\kappa} \lambda_\alpha$. Let $\Omega = \{a_\beta : \beta \in \lambda\}$ be an enumeration and let $\Omega_\alpha = \{a_\beta : \beta < \lambda_\alpha\}$ for each $\alpha \in \kappa$. Choose an independent $\lambda$-sequence of internal sets of measure $\frac{1}{2}$, say $\langle A_\beta : \beta \in \lambda \rangle$, on $\Omega$.

**Claim 10.1** There exist two different ordinals $\gamma_\alpha, \gamma'_\alpha \in [\lambda_\alpha, \lambda_{\alpha+1})$ such that $A_{\gamma_\alpha} \Delta A_{\gamma'_\alpha} \cap \Omega_\alpha = \emptyset$ for each $\alpha \in \kappa$.

Proof of Claim 10.1: Since $2^{\lambda_\alpha} < \lambda_{\alpha+1}$, then there exist two different $\gamma_\alpha$ and $\gamma'_\alpha$ in $[\lambda_\alpha, \lambda_{\alpha+1})$ such that

$$A_{\gamma_\alpha} \cap \Omega_\alpha = A_{\gamma'_\alpha} \cap \Omega_\alpha.$$

Hence we have $A_{\gamma_\alpha} \Delta A_{\gamma'_\alpha} \cap \Omega_\alpha = \emptyset$. □ (Claim 10.1)

For any $\alpha \in \kappa$ let $B_\alpha = A_{\gamma_\alpha} \Delta A_{\gamma'_\alpha}$. It is easy to see that $\langle B_\alpha : \alpha \in \kappa \rangle$ is an independent sequence of measure $\frac{1}{2}$. By inner-regularity of $\mathcal{C}$ we can find $X_{\alpha,n} \in \mathcal{C}$ for each $\alpha$ and each $n \in \omega$ such that $X_{\alpha,n} \subseteq B_\alpha$ and

$$L_P(\bigcup_{n\in\omega} X_{\alpha,n}) = \frac{1}{2}.$$

By a similar method as in the proof of Claim 9.1 we can find an $E \in [\kappa]^\kappa$ and an $n(\alpha) \in \omega$ for each $\alpha \in E$ such that for any $m \in \omega$ and any distinct $\{\alpha_0, \ldots, \alpha_{m_1}\} \subseteq E$ we have

$$\bigcap_{i=0}^{m-1} X_{\alpha_i, n(\alpha_i)} \neq \emptyset.$$

So $\{X_{\alpha,n(\alpha)} : \alpha \in E\} \subseteq \mathcal{C}$ has f.i.p., but

$$\bigcap_{\alpha\in E} X_{\alpha,n(\alpha)} \subseteq \bigcap_{\alpha\in E} B_\alpha = \emptyset.$$

□

**Remark:** Theorem 10 is also a consequence of ZFC. So ZFC implies the existence of non-compact hyperfinite Loeb spaces of arbitrarily large singular cardinalities.

We need Maharam Theorem for next two theorems. Given a complete Boolean algebra $\mathcal{B}$. For any $X \subseteq \mathcal{B}$ recall that $\bar{\mathcal{B}}(X)$ is the complete subalgebra generated by

$X$. Let

$$\tau(\mathcal{B}) = \min\{|X| : X \subseteq \mathcal{B} \wedge \mathcal{B} = \bar{\mathcal{B}}(X)\}.$$

For any $a \in \mathcal{B} \smallsetminus \{0\}$ let $\mathcal{B} \restriction a$ be the Boolean algebra $\{b \wedge a : b \in \mathcal{B}\}$ with $a$ being the largest element 1 in $\mathcal{B} \restriction a$. A complete Boolean algebra $\mathcal{B}$ is called homogeneous if $\tau(\mathcal{B}) = \tau(\mathcal{B} \restriction a)$ for every $a \in \mathcal{B} \smallsetminus \{0\}$. The following is a version of Maharam Theorem (see [F, pp.911 Theorem 3.5]).

> **Maharam Theorem** Let $\mathcal{B}$ be a homogeneous measure algebra of a probability space with $\tau(\mathcal{B}) = \lambda$. Then there is a measure preserving isomorphism $\Phi$ from $\mathcal{B}$ to $\mathcal{B}({}^{\lambda}2)$.

Let $\mu$ be a cardinal. For next two theorems we always denote, for each $\alpha \in \mu$,

$$B_\alpha = \{f \in {}^{\mu}2 : f(\alpha) = 0\}.$$

For any set $X \subseteq {}^{\mu}2$ let $supt(X)$ denote the support of $X$, *i.e.* the smallest $w \subseteq \mu$ such that for any $f, f' \in {}^{\mu}2$ we have $f \restriction w = f' \restriction w$ implies $f \in X$ iff $f' \in X$. Clearly $supt(X)$ is at most countable if $X$ is a Baire set. For any measurable set $X$ in a measure space we denote again $\bar{X}$ for the element in the measure algebra, represented by $X$.

**Theorem 11** *Suppose $\mathcal{M}$ is obtained by adding $\lambda$ random reals to a ZFC model $\mathcal{N}$ for some regular $\lambda > (2^\omega)^{\mathcal{N}}$ with $\lambda^\omega = \lambda$. Suppose $|\Omega| = \lambda$ ($\lambda = 2^\omega$ in $\mathcal{M}$). Then $\Omega$ is not compact.*

**Proof:** Let $\Omega$ be a hyperfinite Loeb space in ${}^*V$ with $|\Omega| = \lambda$. Without loss of generality we assume that $\mathcal{B}(\Omega)$ is homogeneous and $\tau(\mathcal{B}(\Omega)) = \mu$ for some $\mu \geqslant \lambda$. The reason for that is the following. It is easy to see that for any internal subset $a$ of $\Omega$ with positive Loeb measure there exists an $|a|$-independent sequence of measure $\frac{1}{2}L_P(a)$ on $\mathcal{B}(\Omega) \restriction \bar{a}$. Since $|a| = \lambda$, then $\tau(\mathcal{B}(\Omega) \restriction \bar{a}) \geqslant \lambda$. Suppose $\mathcal{B}(\Omega)$ is not homogeneous. Then we can choose an internal subset $a \subseteq \Omega$ such that $L_P(a) > 0$ and $\mu = \tau(\mathcal{B}(\Omega) \restriction \bar{a})$ is the smallest. Hence $\mu \geqslant \lambda$ and $\mathcal{B}(\Omega) \restriction \bar{a}$ is homogeneous. Then we could replace $\Omega$ by $a$.

By Maharam Theorem let $\Phi : \mathcal{B}(\Omega) \cong \mathcal{B}({}^{\mu}2)$ be the measure preserving isomorphism. For each $\alpha \in \lambda \subseteq \mu$ let $A_\alpha \subseteq \Omega$ be measurable such that

$$\Phi(\bar{A}_\alpha) = \bar{B}_\alpha.$$

Suppose $\Omega$ is compact and let $\mathcal{C}$ be the inner-regular compact family on $\Omega$. Again let $A^0_\alpha = A_\alpha$, $A^1_\alpha = \Omega \setminus A_\alpha$, $B^0_\alpha = B_\alpha$ and $B^1_\alpha = {}^\mu 2 \setminus B_\alpha$. Then there exist $X^l_{\alpha,n} \in \mathcal{C}$ such that $X^l_{\alpha,n} \subseteq A^l_\alpha$ and

$$L_P(\bigcup_{n\in\omega} X^l_{\alpha,n}) = \frac{1}{2},$$

where $X^1_{\alpha,n}$ may not be $\Omega \setminus X^0_{\alpha,n}$. For each $\alpha \in \lambda$, $n \in \omega$ and $l = 0, 1$ let $Y^l_{\alpha,n} \subseteq B^l_\alpha$ be a Baire set such that

$$\Phi(\bar{X}^l_{\alpha,n}) = \bar{Y}^l_{\alpha,n}.$$

We want to find an $E \in [\lambda]^\lambda$ and an $n(\alpha, l)$ for each $\alpha \in E$ and $l = 0, 1$ such that for any $m \in \omega$, any distinct $\{\alpha_0, \ldots, \alpha_{m-1}\} \subseteq E$ and any $h \in {}^m 2$

$$\bigwedge_{i=0}^{m-1} \bar{Y}^{h(i)}_{\alpha_i, n(\alpha_i, h(i))} \neq 0.$$

This is enough to prove the theorem because by Maharam's isomorphism we have a family

$$\mathcal{F}_f = \{X^{f(\alpha)}_{\alpha, n(\alpha, f(\alpha))} : \alpha \in E\}$$

with f.i.p. for every $f \in {}^E 2$. But $|\Omega| = \lambda$. So there must be a family $\mathcal{F}_f$ for some $f \in {}^E 2$ with empty intersection, which contradicts that $\mathcal{C}$ is a compact family.

Let $\mathbb{P} = \mathcal{B}({}^\lambda 2)$ be the forcing in $\mathcal{N}$ for adding $\lambda$ random reals. Since $\mathbb{P}$ has c.c.c., then for each $\alpha \in \lambda$ there exists a countable set $v_\alpha \subseteq \lambda$ in $\mathcal{N}$ such that

$$\{Y^l_{\alpha,n} : n \in \omega \wedge l = 0, 1\} \subseteq \mathcal{N}^{\mathcal{B}({}^{v_\alpha} 2)}.$$

Work in $\mathcal{N}$. Let $\dot{u}_\alpha$ be a $\mathbb{P}$-name for

$$\bigcup_{n\in\omega, l=0,1} supt(Y^l_{\alpha,n}).$$

Again since $\mathbb{P}$ has c.c.c. there exists a countable $w_\alpha \subseteq \mu$ such that

$$\Vdash_{\mathbb{P}} \dot{u}_\alpha \subseteq w_\alpha.$$

Note that $\alpha \in w_\alpha$ for every $\alpha \in \lambda \subseteq \mu$. Since $\lambda > 2^\omega \geqslant \omega_1$, we can find a $\bar{v} \subseteq \mu$ with $|\bar{v}| < \lambda$ and an $E \in [\lambda]^\lambda$ such that for any two different $\alpha, \beta \in E$

$$w_\alpha \cap w_\beta \subseteq \bar{v}.$$

So in $\mathcal{M}$ for any two different $\alpha, \beta \in E$ we have

$$supt(Y^l_{\alpha,n}) \cap supt(Y^{l'}_{\beta,m}) \subseteq \bar{v}.$$

Without loss of generality we assume that $\bar{v} < \lambda$ is an ordinal, $\bar{v} \cap E = \emptyset$ and

$$|\lambda \setminus \bigcup_{\alpha \in E} v_\alpha| = \lambda.$$

For any $X \subseteq {}^{\mu}2$ and $\eta \in {}^{\bar{v}}2$ let

$$X(\eta) = \{\xi \in {}^{\mu \setminus \bar{v}}2 : \eta\hat{\ }\xi \in X\}.$$

Now we work in $\mathcal{M}$. For each $\alpha \in E$, $n \in \omega$ and $l = 0, 1$ let

$$C^l_{\alpha,n} = \{\eta \in {}^{\bar{v}}2 : \nu_{\mu \setminus \bar{v}}(Y^l_{\alpha,n}(\eta)) > 0\}.$$

**Claim 11.1** $\nu_{\bar{v}}(\bigcup_{n\in\omega} C^l_{\alpha,n}) = 1$ for each $\alpha \in E$ and $l = 0, 1$.

Proof of Claim 11.1: For any $\eta \in {}^{\bar{v}}2$ we have

$$\nu_{\mu \setminus \bar{v}}(\bigcup_{n\in\omega} Y^l_{\alpha,n}(\eta)) = \nu_{\mu \setminus \bar{v}}((\bigcup_{n\in\omega} Y^l_{\alpha,n})(\eta)) \leqslant \frac{1}{2}$$

because $(\bigcup_{n\in\omega} Y^l_{\alpha,n})(\eta) \subseteq B^l_\alpha(\eta)$ and $\nu_{\mu \setminus \bar{v}}(B^l_\alpha(\eta)) = \frac{1}{2}$. So by Fubini Theorem we have

$$\frac{1}{2} = \nu_\mu(\bigcup_{n\in\omega} Y^l_{\alpha,n})$$
$$= \int_{{}^{\bar{v}}2} \nu_{\mu \setminus \bar{v}}(\bigcup_{n\in\omega} Y^l_{\alpha,n}(\eta)) d\nu_{\bar{v}}(\eta)$$
$$\leqslant \frac{1}{2}\nu_{\bar{v}}(\{\eta : \nu_{\mu \setminus \bar{v}}(\bigcup_{n\in\omega} Y^l_{\alpha,n}(\eta)) > 0\}).$$

This implies

$$\nu_{\bar{v}}(\{\eta : \nu_{\mu \setminus \bar{v}}(\bigcup_{n\in\omega} Y^l_{\alpha,n}(\eta)) > 0\}) = 1.$$

But

$$\bigcup_{n\in\omega} C^l_{\alpha,n} = \{\eta : \nu_{\mu \setminus \bar{v}}(\bigcup_{n\in\omega} Y^l_{\alpha,n}(\eta)) > 0\}.$$

□ (Claim 11.1)

We now divide the proof of the theorem into two cases.

Case 1: $\bar{v} = k$ for a finite $k \in \omega$.

Fix an $\eta_0 \in {}^{\bar{v}}2$. For any $\alpha \in E$ and $l = 0, 1$ the fact $\nu_{\bar{v}}(\bigcup_{n\in\omega} C^l_{\alpha,n}) = 1$ implies that there exists an $n(\alpha, l)$ such that $\eta_0 \in C^l_{\alpha,n(\alpha,l)}$. So for any $m \in \omega$, any distinct $\{\alpha_0, \ldots, \alpha_{m-1}\} \subseteq E$ and any $h \in {}^m2$ we have, by Fubini Theorem and independence,

$$\nu_\mu(\bigcap_{i=0}^{m-1} Y^{h(i)}_{\alpha_i,n(\alpha_i,h(i))})$$

$$= \int_{{}^{\bar{v}}2} \nu_{\mu\setminus\bar{v}}(\bigcap_{i=0}^{m-1} Y^{h(i)}_{\alpha_i,n(\alpha_i,h(i))}(\eta))d\nu_{\bar{v}}(\eta)$$

$$= \int_{{}^{\bar{v}}2} \prod_{i=0}^{m-1} \nu_{\mu\setminus\bar{v}}(Y^{h(i)}_{\alpha_i,n(\alpha_i,h(i))}(\eta))d\nu_{\bar{v}}(\eta)$$

$$\geqslant (2^{-k})(\prod_{i=0}^{m-1} \nu_{\mu\setminus\bar{v}}(Y^{h(i)}_{\alpha_i,n(\alpha_i,h(i))}(\eta_0))) > 0.$$

Case 2: $\bar{v} \geqslant \omega$. Let

$$S \subseteq \lambda \setminus ((\bigcup_{\alpha\in E} v_\alpha) \cup \bar{v})$$

be such that $|S| = \bar{v}$. We can factor the forcing $\mathbb{P}$ to $\mathbb{P}_1 * \dot{\mathbb{P}}_2$ such that

$$\mathcal{N}^{\mathbb{P}} = \mathcal{N}^{\mathbb{P}_1 * \dot{\mathbb{P}}_2},$$

where $\mathbb{P}_1 = \mathcal{B}({}^{\lambda\setminus S}2)$ and $\mathbb{P}_2 = (\mathcal{B}({}^S2))^{\mathcal{N}^{\mathbb{P}_1}}$ (see [K2, Theorem 3.13]). Let $r \in {}^S2$ be a random function over $\mathcal{N}^{\mathbb{P}_1}$. Since $|S| = |\bar{v}|$ we can assume that $r$ is a random function from $\bar{v}$ to 2. Since $C^l_{\alpha,n} \in \mathcal{N}^{\mathbb{P}_1}$ for any $\alpha \in E$, $n \in \omega$ and $l = 0, 1$, and $\nu_{\bar{v}}(\bigcup_{n\in\omega} C^l_{\alpha,n}) = 1$, then there exists an $n(\alpha, l)$ for any $\alpha \in E$ and $l = 0, 1$ such that

$$r \in C^l_{\alpha,n(\alpha,l)}.$$

Now for any $m \in \omega$, any distinct $\{\alpha_0, \ldots, \alpha_{m-1}\} \subseteq E$ and any $h \in {}^m2$ we have $r \in C$, where

$$C = \bigcap_{i=0}^{m-1} C^{h(i)}_{\alpha_i,n(\alpha_i,h(i))}.$$

This implies $\nu_{\bar{v}}(C) > 0$. Hence

$$\nu_\mu(\bigcap_{i=0}^{m-1} Y^{h(i)}_{\alpha_i,n(\alpha_i,h(i))})$$

$$\geqslant \int_C \nu_{\mu\setminus\bar{v}}(\bigcap_{i=0}^{m-1} Y^{h(i)}_{\alpha_i,n(\alpha_i,h(i))}(\eta))d\nu_{\bar{v}}(\eta)$$

$$= \int_C (\prod_{i=0}^{m-1} \nu_{\mu \smallsetminus \bar{v}}(Y^{h(i)}_{\alpha_i, n(\alpha_i, h(i))}(\eta))) d\nu_{\bar{v}}(\eta) > 0.$$

□

**Remark:** Theorem 11 is complementary to Theorem 2 and its corollaries. Combining those results we conclude that the compactness of a hyperfinite Loeb space of size $2^\omega$ is undecidable under ZFC.

**Theorem 12** *Suppose $\mathcal{M}$ is obtained by adding $\kappa$ random reals to a ZFC model $\mathcal{N}$ for some regular $\kappa > \omega$. Suppose $\lambda$ is a strong limit cardinal such that $cf(\lambda) \leqslant \kappa$. Suppose $|\Omega| = \lambda$. Then $\Omega$ is not compact.*

**Proof:** Let $\langle \lambda_\alpha : \alpha \in cf(\lambda) \rangle$ be an increasing sequence such that $\lambda = \bigcup_{\alpha \in cf(\lambda)} \lambda_\alpha$ and $2^{\lambda_\alpha} < \lambda_{\alpha+1}$ for each $\alpha \in cf(\lambda)$. By similar arguments in the proof of theorem 11 we can assume that $\mathcal{B}(\Omega)$ is homogeneous and $\tau(\mathcal{B}(\Omega)) = \lambda$. Note that the cardinality of any positive measure internal subset of $\Omega$ is $\lambda$.

By Maharam Theorem there is a measure preserving isomorphism $\Phi$ from $\mathcal{B}(\Omega)$ to $\mathcal{B}({}^\lambda 2)$. Using the same notation as in Theorem 11 let $A_\gamma \subseteq \Omega$ be measurable such that $\Phi(\bar{A}_\gamma) = \bar{B}_\gamma$ for each $\gamma \in \lambda$. By the same argument as in Claim 10.1 we can find a $cf(\lambda)$-independent sequence

$$\langle C_\alpha \subseteq \Omega : \alpha \in cf(\lambda) \rangle$$

of measure $\frac{1}{2}$ such that for any $Z \in [cf(\lambda)]^{cf(\lambda)}$ we have $\bigcap_{\alpha \in Z} C_\alpha = \emptyset$, where $C_\alpha = A_{\gamma_\alpha} \Delta A_{\gamma'_\alpha}$ for some different $\gamma_\alpha, \gamma'_\alpha \in [\lambda_\alpha, \lambda_{\alpha+1})$. Suppose $\Omega$ is compact and assume $\mathcal{C}$ is an inner-regular compact family on $\Omega$. Let $X_{\alpha,n} \subseteq C_\alpha$ be such that

$$L_P(\bigcup_{n \in \omega} X_{\alpha,n}) = \frac{1}{2}$$

and let $Y_{\alpha,n} \subseteq {}^\lambda 2$ be Baire sets such that $\Phi(\bar{X}_{\alpha,n}) = \bar{Y}_{\alpha,n}$. It suffices now to find an $E \in [cf(\lambda)]^{cf(\lambda)}$ and an $n(\alpha)$ for each $\alpha \in E$ such that for any $m \in \omega$ and for any distinct $\{\alpha_0, \ldots, \alpha_{m-1}\} \subseteq E$ we have

$$\bigwedge_{i=0}^{m-1} \bar{Y}_{\alpha_i, n(\alpha_i)} \neq 0.$$

This is enough to prove the theorem because we have a family

$$\{X_{\alpha, n(\alpha)} : \alpha \in E\}$$

with f.i.p. but

$$\bigcap\{X_{\alpha,n(\alpha)} : \alpha \in E\} \subseteq \bigcap_{\alpha\in E} C_\alpha = \emptyset.$$

For any $\alpha \in cf(\lambda)$ there exists a countable set $v_\alpha \subseteq \kappa$ in $\mathcal{N}$ such that

$$\{Y_{\alpha,n} : n \in \omega\} \subseteq \mathcal{N}^{\mathcal{B}({}^{v_\alpha}2)}.$$

Choose a $D \in [cf(\lambda)]^{cf(\lambda)}$ such that

$$|\kappa \setminus \bigcup_{\alpha\in D} v_\alpha| = \kappa.$$

Let $S \subseteq \kappa \setminus \bigcup_{\alpha\in D} v_\alpha$ be such that $|S| = cf(\lambda)$. Again we factor the forcing $\mathbb{P} = \mathcal{B}({}^\kappa 2)$ to $\mathbb{P}_1 * \dot{\mathbb{P}}_2$ such that $\mathbb{P}_1 = \mathcal{B}({}^{\kappa\setminus S}2)$ and $\mathbb{P}_2 = (\mathcal{B}({}^S 2))^{\mathcal{N}^{\mathbb{P}_1}}$. Note that $Y_{\alpha,n} \in \mathcal{N}^{\mathbb{P}_1}$ for any $\alpha \in D$ and $n \in \omega$. Let

$$R = \bigcup_{\alpha\in D, n\in\omega} supt(Y_{\alpha,n}).$$

Then $R \subseteq \lambda$ and $|R| = cf(\lambda)$. It is clear that $R$ is unbounded in $\lambda$. Recall that

$$B_\gamma = \{f \in {}^\lambda 2 : f(\gamma) = 0\}$$

for each $\gamma \in \lambda$ and

$$\bigvee_{n\in\omega} \bar{Y}_{\alpha,n} = \bar{B}_{\gamma_\alpha} \Delta \bar{B}_{\gamma'_\alpha}$$

for each $\alpha \in D$. Notice also that $\gamma_\alpha, \gamma'_\alpha \in R$ for all $\alpha \in D$. Let $G \subseteq \mathbb{P}_2$ be a $\mathcal{N}^{\mathbb{P}_1}$ generic filter. Without loss of generality we assume that $\mathbb{P}_2 = \mathcal{B}({}^R 2)$ since $|S| = |R|$. Now we define a dense subset $D_\alpha \subseteq \mathbb{P}_2$ in $\mathcal{N}^{\mathbb{P}_1}$ for each $\alpha \in D$. For any $Z \subseteq {}^R 2$ let

$$Z^+ = \{f \in {}^\lambda 2 : f \restriction R \in Z\}$$

and for any $Z \subseteq {}^\lambda 2$ let

$$Z^- = \{f \in {}^R 2 : f = g \restriction R \text{ for some } g \in Z\}.$$

Define

$$D_\alpha = \{\bar{Z} : \nu_R(Z) > 0 \wedge (\exists\beta \in [\alpha, cf(\lambda)) \cap D)(\bar{Z^+} \leqslant \bar{B}_{\gamma_\beta} \Delta \bar{B}_{\gamma'_\beta})\}.$$

**Claim 12.1** In $\mathcal{N}^{\mathbb{P}_1}$, the set $D_\alpha$ is dense in $\mathbb{P}_2$.

Proof of Claim 12.1: Given any $\bar{X} \in \mathbb{P}_2$ for some Baire set $X \subseteq {}^R 2$ with $\nu_R(X) > 0$. Since $supt(X)$ is at most countable, there exists a $\beta \in [\alpha, cf(\lambda)) \cap D$ such that $\gamma_\beta, \gamma'_\beta \in R \setminus supt(X)$. Let $Y = X^+ \cap B_{\gamma_\beta} \Delta B_{\gamma'_\beta}$. Then

$$\nu_R(Y^-) = \nu_R(X)\nu_R((B_{\gamma_\beta}\Delta B_{\gamma'_\beta})^-) > 0$$

and $\bar{Y} \leqslant \bar{B}_{\gamma_\beta}\Delta\bar{B}_{\gamma'_\beta}$. □ (Claim 12.1)

Let

$$E = \{\alpha \in D : \bar{B}^-_{\gamma_\alpha}\Delta\bar{B}^-_{\gamma'_\alpha} \in G\}.$$

By Claim 12.1 we have $|E| = cf(\lambda)$. For each $\alpha \in E$ since

$$\bigvee_{n\in\omega} \bar{Y}^-_{\alpha,n} = \bar{B}^-_{\gamma_\alpha}\Delta\bar{B}^-_{\gamma'_\alpha},$$

then there exists an $n(\alpha) \in \omega$ such that $\bar{Y}^-_{\alpha,n(\alpha)} \in G$. We are done because $G$ is a filter and $supt(Y_{\alpha,n(\alpha)}) \subseteq R$. □

**Remarks:** (1) Theorem 12 is complementary to Theorem 6 and Corollary 7.

(2) In this theorem we didn't require $\kappa \geqslant 2^\omega$ in $\mathcal{N}$.

**Theorem 13** *Suppose $\lambda > |V|$, where $V$ is the standard universe, and $\lambda^\omega = \lambda$. Then there exists a ${}^*V$, in which every hyperfinite Loeb space $\Omega$ has cardinality $\lambda$ and is not compact.*

**Proof:** Note that $|V| = \beth_\omega > 2^\omega$. Construct a continuous elementary chain of nonstandard universes

$$\langle {}^*V_\alpha : \alpha \leqslant (2^\omega)^+ \rangle$$

such that for every $\alpha \in (2^\omega)^+$,

(1) every hyperfinite Loeb space in ${}^*V_\alpha$ has cardinality $\lambda$,

(2) ${}^*V_\alpha$ is $\omega_1$-saturated when $\alpha$ is a successor ordinal.

(3) for any hyperfinite Loeb space $\Omega$ in ${}^*V_{(2^\omega)^+}$ if $\Omega \in {}^*V_\alpha$, then $\Omega \cap {}^*V_\alpha$ has Loeb measure zero in ${}^*V_{\alpha+1}$.

The elementary chain of nonstandard universes satisfying (1), (2) and (3) exists because at each step one need only to realize $\leqslant \lambda^\omega$ types. We want to show that nonstandard universe ${}^*V = {}^*V_{(2^\omega)^+}$ is the one we want.

Obviously, ${}^*V_{(2^\omega)^+}$ is $\omega_1$-saturated. Suppose there is a compact hyperfinite Loeb space $\Omega$ in ${}^*V$. Let $\mathcal{C}$ be an inner-regular compact family on $\Omega$. For every $\alpha \in (2^\omega)^+$ such that $\Omega \in {}^*V_\alpha$ there is an internal set $Z_\alpha \in {}^*V_{\alpha+1}$ such that

$$L_P(Z_\alpha) > \frac{1}{2}$$

and

$$Z \cap (\Omega \cap {}^*V_\alpha) = \emptyset.$$

Now we can find $X_{\alpha,n} \in \mathcal{C}$ such that $X_{\alpha,n} \subseteq Z_\alpha$ and

$$L_P(\bigcup_{n\in\omega} X_{\alpha,n}) = L_P(Z_\alpha).$$

Again using the same method as in the proof of Claim 9.1 we can find an $E \subseteq (2^\omega)^+$ with $|E| = (2^\omega)^+$ and an $n(\alpha) \in \omega$ for each $\alpha \in E$ such that $\{X_{\alpha,n(\alpha)} : \alpha \in E\}$ has f.i.p. But

$$\bigcap_{\alpha\in E} X_{\alpha,n(\alpha)} \subseteq \bigcap_{\alpha\in E} Z_\alpha = \emptyset.$$

□

We would like to end this section by making a conjecture.

**Conjecture:** It is consistent with ZFC that there are no compact hyperfinite Loeb spaces in any nonstandard universes.

The reader might notice that all results in §1 are not proved by ZFC. But Theorem 8 assumes only ZFC plus a consequence of Singular Cardinal Hypothesis. So the non-existence of any compact hyperfinite Loeb spaces would have to violate Singular Cardinal Hypothesis, which implies the existence of pretty large cardinals.

# References


[AFHL] Albeverio, S., Fenstad, J. E., Hoegh-Krohn, R., and Lindstrom, T., *Nonstandard methods in stochastic analysis and mathematical physics*, Academic Press, Orlando, 1986.

[A] Aldaz, J., *On compactness and Loeb measures*, Proceedings of American Mathematical Society, **123** (1995), 173—175.

[CK] Chang, C. C. and Keisler, H. J., *Model Theory*, North–Holland, 1973. (Third edition, 1990.)

[F] Fremlin, D., *Measure algebra*, Chapter 22 in Volume 3 of **the Handbook of Boolean Algebra**, ed. by J. D. Monk with R. Bonnet, North-Holland, 1989.

[J] Jin, R., *The isomorphism property versus the special model axiom*, The Journal of Symbolic Logic, **57**, No. 3 (1992), 975—987.

[K1] Kunen, K., *Set Theory–an introduction to independence proofs*, North–Holland, Amsterdam, 1980.

[K2] ________, *Random and Cohen reals*, in **the Handbook of Set Theoretic Topology**, ed. by K. Kunen and J. E. Vaughan, North–Holland, Amsterdam, 1984.

[L] Lindstrom, T., *An invitation to nonstandard analysis*, in **Nonstandard Analysis and Its Application**, ed. by N. Cutland, Cambridge University Press 1988.

[R1] Ross, D. A., *Measurable transformations in saturated models of analysis*, Ph. D. Thesis, University of Wisconsin-Madison, 1983.

[R2] ________, *Compact measures have Loeb preimages*, Proceedings of American Mathematical Society, **115** (1992), 365—370.

[R3] ________, *Unions of Loeb nullsets: the context*, Developments in Nonstandard Mathematics (N. J. Cutland et al., eds.), Pitman Research Notes in Mathematics **336** (1995), 178—185.

[R4] ________, *Unions of Loeb nullsets*, Proceedings of American Mathematical Society, **124** (1996), 1883—1888.

[R5] ________, *The special model axiom in nonstandard analysis*, the Journal of Symbolic Logic, **55** (1990), 1233—1242.

[Sh92] Shelah, S., *Remarks on Boolean algebra*, Algebra Universalis, **11** (1980), 77—89.

[Sh575] ———, *Cellularity of free products of Boolean algebra*, Fundamenta Mathematicae, to appear.

[SB] Stroyan K. D. and Bayod J. M., *Foundation of Infinitesimal Stochastic Analysis*, North–Holland, 1986.

Department of Mathematics, College of Charleston
Charleston, SC 29424

Department of Mathematics, Rutgers University
New Brunswick, NJ 08903
*e-mail: rjin@@math.rutgers.edu*

Institute of Mathematics, The Hebrew University
Jerusalem, Israel

Department of Mathematics, Rutgers University
New Brunswick, NJ 08903

Department of Mathematics, University of Wisconsin
Madison, WI 53706

*Sorting:* The first two addresses are the first author's and the last three are the second author's.