Characterizing the structure of $A+B$ when $A+B$ has small upper Banach density

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Abstract

Let A and B be two infinite sets of non-negative integers. Similar to Kneser's Theorem (Theorem 1.1 bellow) we characterize the structure of $A+B$ when the upper Banach density of $A+B$ is less than the sum of the upper Banach density of A and the upper Banach density of B.

1 Introduction

By an interval in this paper, we always mean an interval of integers. For each set A of non-negative integers the Shnirel'man density $\sigma(A)$ and the lower asymptotic density $\underline{d}(A)$ of A are defined by

$$\sigma(A) = \inf_{n \geqslant 1} \frac{A(n)}{n} \text{ and } \underline{d}(A) = \liminf_{n \to \infty} \frac{A(n)}{n}$$

where $A(n) = |A \cap [1,n]|$ is the number of elements in A between 1 and n. Let $A \pm B$ denote the set $\{a \pm b : a \in A \text{ and } b \in B\}$. In order to check whether the celebrated Mann's Theorem (cf. [12]) remains true if Shnirel'man density is replaced by lower asymptotic density, Kneser proved the following Theorem 1.1, which indicates that if Mann's inequality is not true for $\underline{d}$, then $A+B$ must essentially be the union of arithmetic progressions of the same difference. For two infinite sets $A, B \subseteq \mathbb{N}$, $A \sim B$ means that the symmetric difference of A and B is a finite set. For a positive integer

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g and a finite set $G \subseteq [0, g-1]$ let $g\mathbb{N} = \{gn : n \in \mathbb{N}\}$ and $G + g\mathbb{N} = \{a + gn : a \in G \text{ and } n \in \mathbb{N}\}$[1].

Theorem 1.1 (M. Kneser, 1953) *If $\underline{d}(A+B) < \underline{d}(A) + \underline{d}(B)$, then there exist $g > 0$ and $G \subseteq [0, g-1]$ such that*

1. *$\underline{d}(A+B) \geqslant \underline{d}(A) + \underline{d}(B) - \frac{1}{g}$ and*
2. *$A + B \subseteq (G + g\mathbb{N})$ and $(A+B) \sim (G + g\mathbb{N})$.*

The proof of above theorem can be found in [6, page 51–75][2].

Naturally people may wonder whether one can have a similar theorem for upper asymptotic density. The upper asymptotic density $\overline{d}(A)$ of A is defined by

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{A(n)}{n}.$$

Although the definition of upper asymptotic density looks similar to the definition of lower asymptotic density, their behaviors are very different. In fact $\overline{d}(A+B)$ can be much smaller than $\overline{d}(A) + \overline{d}(B)$ without requiring $A + B$ being a large subset of the union of arithmetic progressions of the same difference (cf. [10]).

The next natural candidate to consider is upper Banach density. The upper Banach density $BD(A)$ of a set A is defined by

$$BD(A) = \lim_{k\to\infty} \sup_{n\geqslant 0} \frac{A(n, n+k)}{k+1}$$

where $A(a, b) = |A \cap [a, b]|$. Upper Banach density is popular among mathematicians who work on combinatorial number theory problems using ergodic methods (cf. [1, 5]). Clearly, we have

$$0 \leqslant \sigma(A) \leqslant \underline{d}(A) \leqslant \overline{d}(A) \leqslant BD(A) \leqslant 1.$$

It is easy to see that $\alpha = BD(A)$ iff α is the greatest real number satisfying that there is a sequence of intervals $\{[a_n, b_n] : n \in \mathbb{N}\}$ such that

$$\lim_{n\to\infty} (b_n - a_n) = \infty \text{ and } \lim_{n\to\infty} \frac{A(a_n, b_n)}{b_n - a_n + 1} = \alpha. \tag{1}$$

[1] In some literature gA represents the g-fold sum of A. Since only the sum of two sets is considered in this paper, we would like to write $A + A$ instead of $2A$ so that the term $g\mathbb{N}$ can be reserved for the set of all multiples of g without ambiguity.

[2] Kneser's Theorem actually deals with multiple sum of sets. Here, for simplicity, we state only the version for the sum of two sets.

Although upper Banach density is farther away from lower asymptotic density than upper asymptotic density, the behavior of upper Banach density is much more similar to the behavior of lower asymptotic density than that of upper asymptotic density. In [8] a general scheme is introduced that one can obtain a theorem about upper Banach density parallel to each existing theorem about Shnirel'man density or lower asymptotic density. For example, [9, Theorem 3.8] is derived for upper Banach density parallel to Theorem 1.1. However, a simple application of the scheme in [9] only allow us to characterize the structure of $A+B$ in a very small portion of $\mathbb{N}$, which is far from satisfactory.

As the first attempt Bihani and the author dealt with the sum of two copies of the same set in the following theorem proved in [2].

Theorem 1.2 (P. Bihani and R. Jin) *Let $A \subseteq \mathbb{N}$ be such that $BD(A) = \alpha$ and $BD(A+A) < 2\alpha$. Then there are positive $g \in \mathbb{N}$ and $G \subseteq [0, g-1]$ such that*

1. $BD(A+A) \geqslant 2\alpha - \frac{1}{g}$,
2. $A + A \subseteq G + g\mathbb{N}$, *and*
3. *if $\{[a_n, b_n] : n \in \mathbb{N}\}$ is a sequence of intervals satisfying (1), then there exist $[c_n, d_n] \subseteq [a_n, b_n]$ such that*

$$\lim_{n\to\infty} \frac{d_n - c_n}{b_n - a_n} = 1 \text{ and } (A+A) \cap [2c_n, 2d_n] = (G + g\mathbb{N}) \cap [2c_n, 2d_n] \text{ for all } n \in \mathbb{N}.$$

Remark 1.3 *(1) In Theorem 1.2 the structure of $A+A$ is characterized in $\bigcup_{n\in\mathbb{N}}[2c_n, 2d_n]$, which is, in some sense, the maximal possible portion of $\mathbb{N}$ for characterizing the structure of $A + A$.*

(2) It is usually difficult to generalize this kind of results from the sum of two copies of the same set $A + A$ to the sum of two distinct sets $A + B$. For example, in the case of finite sets, Freiman's $2k-1+b$ Theorem (cf. [14, Theorem 1.16]) for $A+A$ is generalized by Lev and Smeliansky to $A + B$ (cf. [14, Theorem 4.6]) with a much harder proof. Furthermore, Freiman's $3k-3$ Theorem (cf. [4]) and the author's result about $A + A$ (cf. [11, Theorem 1.4]) do not even have counterparts for $A + B$. In order to generalize Theorem 1.2 to the sum of two distinct sets $A+B$, some obstacles need to be overcome.

Since Kneser's Theorem works for $A+B$, the result above about $A+A$ is not parallel to Kneser's Theorem. We have been looking for a theorem about $A+B$ since Theorem 1.2 was proved. We accomplished this goal recently and obtained the following result, which is the main theorem of the paper.

Theorem 1.4 *Let $A, B \subseteq \mathbb{N}$ be such that $BD(A)=\alpha$, $BD(B)=\beta$, and $BD(A+B)<\alpha+\beta$. Then there are positive $g \in \mathbb{N}$ and $G \subseteq [0, g-1]$ such that*

1. $BD(A+B) \geqslant \alpha+\beta-\frac{1}{g}$,
2. $A+B \subseteq G+g\mathbb{N}$,
3. *if* $\left\{\left[a_n^{(i)}, b_n^{(i)}\right] : n \in \mathbb{N}\right\}$ *for* $i=1,2$ *are two sequences of intervals such that*

$$\lim_{n\to\infty}\left(b_n^{(i)}-a_n^{(i)}\right)=\infty \text{ for } i=1,2, \tag{2}$$

$$\lim_{n\to\infty}\frac{A\left(a_n^{(1)}, b_n^{(1)}\right)}{b_n^{(1)}-a_n^{(1)}+1}=\alpha,\ \lim_{n\to\infty}\frac{B\left(a_n^{(2)}, b_n^{(2)}\right)}{b_n^{(2)}-a_n^{(2)}+1}=\beta, \tag{3}$$

and

$$0<\liminf_{n\to\infty}\frac{b_n^{(1)}-a_n^{(1)}}{b_n^{(2)}-a_n^{(2)}} \leqslant \limsup_{n\to\infty}\frac{b_n^{(1)}-a_n^{(1)}}{b_n^{(2)}-a_n^{(2)}}<\infty, \tag{4}$$

then there exist $\left[c_n^{(i)}, d_n^{(i)}\right] \subseteq \left[a_n^{(i)}, b_n^{(i)}\right]$ *for each* $n \in \mathbb{N}$ *and* $i=1,2$ *such that*

$$\lim_{n\to\infty}\frac{d_n^{(i)}-c_n^{(i)}}{b_n^{(i)}-a_n^{(i)}}=1 \tag{5}$$

and

$$(A+B)\cap\left[c_n^{(1)}+c_n^{(2)}, d_n^{(1)}+d_n^{(2)}\right]=(G+g\mathbb{N})\cap\left[c_n^{(1)}+c_n^{(2)}, d_n^{(1)}+d_n^{(2)}\right]. \tag{6}$$

Obviously, Theorem 1.4 is motivated by Theorem 1.1. In order to prove Theorem 1.4 we have to deal with some obstacles which do not occur when $A=B$. We would like to discuss two of the obstacles. The first one is that we need to explain why (4), which is trivial when $A=B$, should be imposed. The following example shows that (4) is necessary.

Example 1.5 *Let*

$$A=\bigcup_{n=1}^{\infty}\left(\left[2^{(2n)^2},1.5\times 2^{(2n)^2}-2^{(2n-1)^2+1}\right]\cup\left[1.5\times 2^{(2n)^2}+2^{(2n-1)^2+1},2\times 2^{(2n)^2}\right]\right)$$

$$B=\bigcup_{n=1}^{\infty}\left(\left[2^{(2n+1)^2},1.5\times 2^{(2n+1)^2}-2^{(2n)^2+1}\right]\cup\left[1.5\times 2^{(2n+1)^2}+2^{(2n)^2+1},2\times 2^{(2n+1)^2}\right]\right).$$

We have $BD(A)=\alpha=BD(B)=\beta=1$. Hence $BD(A+B)<BD(A)+BD(B)$. Let $a_n^{(1)}=2^{(2n)^2}$, $b_n^{(1)}=2\times 2^{(2n)^2}$, $a_n^{(2)}=2^{(2n+1)^2}$, and $b_n^{(2)}=2\times 2^{(2n+1)^2}$. Then (2) and (3) are true. However, (5) and (6) cannot be true for this pair of A and B because $(A+B)\cap\left[a_n^{(1)}+a_n^{(2)},b_n^{(1)}+b_n^{(2)}\right]$ has large gaps in the middle of the interval.

Note that (4) depends on the indexing of the intervals in the sequence. However, no matter how these intervals in Example 1.5 are indexed, (4) can never be true. The problem indicated by Example 1.5 is caused by the difference of magnitude between the length of $\left[a_n^{(1)},b_n^{(1)}\right]$ and the length of $\left[a_n^{(2)},b_n^{(2)}\right]$. This is why we have to impose (4) in order to have the desired structure (6).

Note that the upper Banach density of $A+B$ really measures the "size" of $A+B$ on a sequence of intervals without having any restrictions on how far these intervals can be from each other. It might give us a wrong impression that the structure of $(A+B)\cap\left[a_n^{(1)}+a_n^{(2)},b_n^{(1)}+b_n^{(2)}\right]$ should be determined only by $A\cap\left[a_n^{(1)},b_n^{(1)}\right]$ and $B\cap\left[a_n^{(2)},b_n^{(2)}\right]$.

Example 1.6 *Let $A_1=9\cdot[0,3^n]$, $A_2=9\cdot[3\times 3^n,4\times 3^n]$, $B_1=3\cdot[0,3\times 3^n]$, and $B_2=\{0,1,2\}+9\cdot[3\times 3^n,4\times 3^n]$. We have roughly that the "density" of A_1 and A_2 are $\frac{1}{9}$, the "density" of B_1 and B_2 are $\frac{1}{3}$, and the "density" of $(A_1\cup A_2)+(B_1\cup B_2)$ is $\frac{1}{3}$, which is less than $\frac{1}{9}+\frac{1}{3}$. However, the structure of A_1+B_1 is different from the structure of A_2+B_2. Hence we don't have a uniform structure for $(A_1\cup A_2)+(B_1\cup B_2)$.*

Example 1.6 shows that if the structure of $(A+B)\cap\left[a_n^{(1)}+a_n^{(2)},b_n^{(1)}+b_n^{(2)}\right]$ were determined by $A\cap\left[a_n^{(1)},b_n^{(1)}\right]$ and $B\cap\left[a_n^{(2)},b_n^{(2)}\right]$, then part 2 of Theorem 1.4 together with (5) and (6) in part 3 of Theorem 1.4 would not be simultaneously true. This means that we could only characterize the structure of $A+B$ piecewisely in each $\left[a_n^{(1)}+a_n^{(2)},b_n^{(1)}+b_n^{(2)}\right]$ instead of the uniform structure described in Theorem 1.4. Fortunately, the structure of $(A+B)\cap\left[a_n^{(1)}+a_n^{(2)},b_n^{(1)}+b_n^{(2)}\right]$ is also influenced by the elements of A and B outside of $\left[a_n^{(1)}, b_n^{(1)}\right]$ and $\left[a_n^{(2)}, b_n^{(2)}\right]$, respectively. In fact, the main difficulty in the proof of Theorem 1.4 is to eliminate the possible cases resembling Example 1.6 (cf. Claim 3.2 in the proof of Theorem 1.4).

As observed in the analogous situation in [2, Theorem 1.1], Theorem 1.4 is, in some sense, optimal. Since the upper Banach densities of A and B are determined by the sizes of A and B along two correspondent sequences of intervals $\left\{\left[a_n^{(i)}, b_n^{(i)}\right] : n \in \mathbb{N}\right\}$ for $i = 1, 2$ (cf. (3)), we can only hope to characterize the structure of $A + B$ in $\bigcup_{n\in\mathbb{N}} \left[a_n^{(1)} + a_n^{(2)}, b_n^{(1)} + b_n^{(2)}\right]$. Note that there seems to be some flexibility in labeling intervals in $\left\{\left[a_n^{(1)}, b_n^{(1)}\right] : n \in \mathbb{N}\right\}$ and in $\left\{\left[a_n^{(2)}, b_n^{(2)}\right] : n \in \mathbb{N}\right\}$. For example, if we rename $\left[a_n^{(1)}, b_n^{(1)}\right]$ by $\left[\bar{a}_{n+100}^{(1)}, \bar{b}_{n+100}^{(1)}\right]$ and define $\left[\bar{a}_n^{(1)}, \bar{b}_n^{(1)}\right] = [0, 0]$ for $n = 0, 1, \ldots, 99$, do we still have the conclusion of Theorem 1.4 when $\left\{\left[a_n^{(1)}, b_n^{(1)}\right] : n \in \mathbb{N}\right\}$ is replaced by $\left\{\left[\bar{a}_n^{(1)}, \bar{b}_n^{(1)}\right] : n \in \mathbb{N}\right\}$? The answer is "yes" as long as (4) is satisfied. Can we replace $\left\{\left[c_n^{(i)}, d_n^{(i)}\right] : n \in \mathbb{N}\right\}$ by $\left\{\left[a_n^{(i)}, b_n^{(i)}\right] : n \in \mathbb{N}\right\}$ for $i = 1, 2$ in (6)? The answer is "no" because if we delete small portion of elements from left or right side of $A \cap \left[a_n^{(1)}, b_n^{(1)}\right]$ or $B \cap \left[a_n^{(2)}, b_n^{(2)}\right]$, then the upper Banach densities of A, B, and $A+B$ will not be changed. But the structure of $(A + B) \cap \left[a_n^{(1)} + a_n^{(2)}, b_n^{(1)} + b_n^{(2)}\right]$ will change.

One of the main features of this paper is that the methods from nonstandard analysis are used in the proof in an essential way while the main result is a standard theorem. It is interesting to see whether a shorter and essentially different standard proof of Theorem 1.4 can be found. Nonstandard methods have been proved very useful and efficient in, for example, [2, 8, 10, 11] when dealing with asymptotic arguments. The reader is recommended to consult one of [2, 7, 8, 13] for the basic notation, ideas, and principles in nonstandard analysis. Other introductory texts for nonstandard analysis, which cover Loeb measure, should also be sufficient. If we work within a nonstandard universe, we always assume that the nonstandard universe is countably saturated.

From now on $\mathbb{N}$ denotes the set of all non-negative integers and $\mathbb{Z}$ denotes the set of all integers. Capital letters A, B, C, and D usually represent sets of integers and lower case letters a, b, c, d, e, g, h, etc. usually represent integers or real numbers. Greek letters α, β, and γ are reserved for standard real numbers. For any $r \leqslant s$ we write $[r, s]$ exclusively for the interval of all integers between r and s including r and s if they are also integers. Let $A \pm a$ be the abbreviation for $A \pm \{a\}$ and $a \pm A$ for

$\{a\} \pm A$. Let $A[a,b]$ denote the set $A \cap [a,b]$ and $A(a,b)$ denote the cardinality of $A[a,b]$. Since the two terms $A[a,b]$ and $A(a,b)$ look very similar, the reader should be aware of the distinction when read the rest of the paper.

2 Lemmas

This section contains some existing lemmas and some new lemmas, which will be cited in the proof of Theorem 1.4. Lemmas up to 2.6 do not involve nonstandard analysis. The last four lemmas do involve it. Note that since we do not assign any properties distinguishing α and β, the properties of A and B are symmetric and this will simplify proofs.

For a positive integer g let $\mathbb{Z}/g\mathbb{Z}$ be the additive group of integers modulo g with addition $\oplus_g$. Let $a, b, c \in [0, g-1]$. By $a \oplus_g b = c$ we mean $a + b \equiv c \pmod g$. Let $\pi_g : \mathbb{Z} \mapsto \mathbb{Z}/g\mathbb{Z}$ be the natural homomorphism from $\mathbb{Z}$ onto $\mathbb{Z}/g\mathbb{Z}$. If $d > 0$ and $d|g$, we denote $\pi_{g,d} : \mathbb{Z}/g\mathbb{Z} \mapsto \mathbb{Z}/d\mathbb{Z}$ the natural homomorphism from $\mathbb{Z}/g\mathbb{Z}$ onto $\mathbb{Z}/d\mathbb{Z}$. Note that the kernel of $\pi_{g,d}$ is $\langle d \rangle_g$, which is the cyclic subgroup of $\mathbb{Z}/g\mathbb{Z}$ generated by the factor d of g. In fact, every subgroup of $\mathbb{Z}/g\mathbb{Z}$ has the form $\langle d \rangle_g$ for some factor d of g.

The first lemma is due to Kneser and the proof can be found in [14, page 115].

Lemma 2.1 (M. Kneser, 1953) *Let $(G, +)$ be an Abelian group and A, B be finite subsets of G. Let $S = \{g \in G : g + A + B = A + B\}$ be the stabilizer of $A + B$. If $|A + B| < |A| + |B|$, then $|A + B| = |A + S| + |B + S| - |S|$. In particular, S is non-trivial if $|A + B| < |A| + |B| - 1$.*

Note that the stabilizer S is always a subgroup of G and if $|A+B| < |A|+|B|-1$, then the stabilizer S of $A + B$ is non-trivial, *i.e.*, $|S| > 1$.

Let x be an integer, $g > 0$, and $G \subseteq [0, g-1]$. We now state another version of Theorem 1.1, which is more convenient for us to use in the proof of Theorem 1.4.

Lemma 2.2 *Suppose $\underline{d}(A) = \alpha$, $\underline{d}(B) = \beta$, and $\underline{d}(A + B) < \alpha + \beta$. Then there are $g > 0$, $F, F' \subseteq [0, g-1]$ such that*

1. *$A \subseteq F + g\mathbb{N}$, $B \subseteq F' + g\mathbb{N}$, and*
2. *$\frac{|F|+|F'|}{g} - \frac{1}{g} < \alpha + \beta$.*

We first prove that Theorem 1.1 implies Lemma 2.2.

Proof of Lemma 2.2 by Theorem 1.1: Let $g > 0$ be the least such integer and $G \subseteq [0, g-1]$ be the set in Theorem 1.1. Let $F, F' \subseteq [0, g-1]$ be the minimal sets such that $A \subseteq F + g\mathbb{N}$ and $B \subseteq F' + g\mathbb{N}$, respectively. By the minimality of F and F' we have that $F \oplus_g F' = G$.

If $|F| + |F'| \geqslant |G| + 2$, then $|F \oplus_g F'| < |F| + |F'| - 1$. By Lemma 2.1 the stabilizer S of $F \oplus_g F'$ is non-trivial. Let $S = \langle d \rangle_g$ for some proper factor d of g and $|S| = g/d = s > 1$. Let $\bar{G} = \pi_{g,d}(G)$. Then we have that $A + B \sim \bar{G} + d\mathbb{N}$ and $\underline{d}(A+B) \geqslant \alpha + \beta - \frac{1}{g} > \alpha + \beta - \frac{1}{d}$, which contradicts the minimality of g.

If $|F| + |F'| \leqslant |G|$, then $\underline{d}(A+B) = \frac{|G|}{g} \geqslant \frac{|F|}{g} + \frac{|F'|}{g} \geqslant \underline{d}(A) + \underline{d}(B)$, which contradicts the assumption of the lemma.

Hence we can assume that $|F| + |F'| = |G| + 1$. This implies that $\frac{|F|+|F'|}{g} - \frac{1}{g} = \underline{d}(A+B) < \alpha + \beta$. $\square$

To prove that Lemma 2.2 implies Theorem 1.1 we need the following lemma, which is the pigeonhole principle.

Lemma 2.3 *Let $A, B \subseteq g\mathbb{N}$ be such that $\underline{d}(A) + \underline{d}(B) > \frac{1}{g}$. Then $A + B \subseteq g\mathbb{N}$ and $A + B \sim g\mathbb{N}$.*

Proof: Let $m_0 \in \mathbb{N}$ be such that for any $gn > m_0$,

$$\frac{A(0, gn) + B(0, gn)}{gn + g} > \frac{1}{g}.$$

Given any $gn > m_0$, both $A[0, gn]$ and $ng - B[0, gn]$ are subsets of $[0, gn] \cap g\mathbb{N}$. Since $|[0, gn] \cap g\mathbb{N}| = n + 1$ and $A(0, gn) + B(0, gn) > n + 1$, then

$$A[0, gn] \cap (gn - B[0, gn]) \neq \emptyset,$$

which implies $gn \in A[0, gn] + B[0, gn] \subseteq A + B$. Hence $A + B \supseteq (g\mathbb{N} \smallsetminus [0, m_0])$. $\square$

We now prove that Lemma 2.2 implies Theorem 1.1.

Proof of Theorem 1.1 by Lemma 2.2 Suppose Lemma 2.2 is true. Let $\underline{d}(A) = \alpha$, $\underline{d}(B) = \beta$, and $\underline{d}(A + B) < \alpha + \beta$. Let $g > 0$ be the least in Lemma 2.2 and $F, F' \subseteq [0, g-1]$ be as described in Lemma 2.2. For each $f \in F$ and $f' \in F'$ we have

$$\underline{d}(A \cap (f + g\mathbb{N})) + \underline{d}(B \cap (f' + g\mathbb{N})) \geqslant \alpha - \frac{|F| - 1}{g} + \beta - \frac{|F'| - 1}{g} > \frac{1}{g}$$

by 2. of Lemma 2.2. By Lemma 2.3 we have $A+B \subseteq ((F \oplus_g F') + g\mathbb{N})$ and $A+B \sim ((F \oplus_g F') + g\mathbb{N})$. Let $G = F \oplus_g F'$. If $|G| \geqslant |F| + |F'|$, then

$$\underline{d}(A+B) = \frac{|G|}{g} \geqslant \frac{|F| + |F'|}{g} \geqslant \alpha + \beta.$$

Thus we can assume $|G| \leqslant |F| + |F'| - 1$. If $|G| < |F| + |F'| - 1$, then by Lemma 2.1 we have $|G| = |F+S| + |F'+S| - |S|$ where S is the non-trivial stablizer of G. Let $S = \langle d \rangle_g$. Let $\bar{G} = \pi_{g,d}(G)$, $\bar{F} = \pi_{g,d}(F)$, and $\bar{F}' = \pi_{g,d}(F')$. Then $A \subseteq \bar{F} + d\mathbb{N}$, $B \subseteq \bar{F}' + d\mathbb{N}$, and

$$\alpha + \beta > \underline{d}(A+B) = \frac{|\bar{G}|}{d} \geqslant \frac{|\bar{F}| + |\bar{F}'|}{d} - \frac{1}{d}.$$

This contradicts the minimality of g. Hence we can conclude that $|G| = |F| + |F'| - 1$. This finishes the proof of Theorem 1.1 because

$$\underline{d}(A+B) = \frac{|G|}{g} = \frac{|F| + |F'| - 1}{g} \geqslant \alpha + \beta - \frac{1}{g}.$$

□

Remark 2.4

1. *From the proof of the equivalence between Theorem 1.1 and Lemma 2.2 one can see that the least $g > 0$ satisfying Lemma 2.2 and the least g satisfying Theorem 1.1 are the same.*

2. *If g is the least positive integer satisfying Lemma 2.2, then $|F \oplus_g F'| = |F| + |F'| - 1$.*

3. *When g is the least positive integer in Lemma 2.2 and $f \in [0, g-1] \smallsetminus F$, there always exists $f' \in F'$ such that $f \oplus_g f' \notin F \oplus_g F'$ because otherwise we have $|(\{f\} \cup F) \oplus_g F'| < |(\{f\} \cup F)| + |F'| - 1$, which implies that the stablizer $S = \langle d \rangle_g$ of $(\{f\} \cup F) \oplus_g F'$ is non-trivial by Lemma 2.1. Hence $A + B \subseteq \bar{G} + d\mathbb{N}$ and $A + B \sim \bar{G} + d\mathbb{N}$ where $\bar{G} = \pi_{g,d}(F \oplus_g F')$, which contradicts the minimality of g.*

4. *As an easy consequence of 2. of Lemma 2.2 we have that $\alpha \leqslant \frac{|F|}{g} < \alpha + \frac{1}{g}$. This implies that for each $f \in F$ we have $A \cap (f + g\mathbb{N}) \neq \emptyset$.*

5. *In Lemma 2.2 if d is a proper factor of g and $\bar{F} = \pi_{g,d}(F)$, then $A \subseteq F + g\mathbb{N} \subseteq \bar{F} + d\mathbb{N}$.*

Lemma 2.5 *Let $A, B \subseteq \mathbb{N}$ be such that $\underline{d}(A)=\alpha$, $\underline{d}(B)=\beta$, and $\underline{d}(A+B)<\alpha+\beta$. Let $g>0$ be the least and $F, F' \subseteq [0, g-1]$ be the sets in Lemma 2.2. Suppose C is another set such that $C \subseteq F'' + g\mathbb{N}$ for some $F'' \subseteq [0, g-1]$, $|F| = |F''|$, and $\underline{d}(C)=\alpha$. Let $f \in F''$ and $C_f = C \cap (f+g\mathbb{N})$. Then $\underline{d}(A+B)+\underline{d}(C_f) \geqslant \alpha+\beta$.*

Proof Note that the conditions of the lemma imply $|F \oplus_g F'| = |F|+|F'|-1$. Note also that $\underline{d}(C_f) \geqslant \alpha - \frac{|F|-1}{g}$. Hence

$$\underline{d}(A+B)+\underline{d}(C_f) \geqslant \frac{|F|+|F'|-1}{g} + \alpha - \frac{|F|-1}{g} = \alpha + \frac{|F'|}{g} \geqslant \alpha+\beta.$$

□

Lemma 2.6 *Let $C \subseteq \mathbb{N}$. Suppose $g_1, g_2 > 0$, $G_1 \subseteq [0, g_1-1]$, and $G_2 \subseteq [0, g_2-1]$ such that $C \subseteq (G_1+g_1\mathbb{N})$, $C \sim (G_1+g_1\mathbb{N})$, $C \subseteq G_2+g_2\mathbb{N}$, and $C \sim G_2+g_2\mathbb{N}$. If $d = \gcd(g_1, g_2)$ and $\bar{G} = \pi_{g_1,d}(G_1) = \pi_d(C) = \pi_{g_2,}(G_2)$, then $C \subseteq \bar{G}+d\mathbb{N}$ and $C \sim \bar{G}+d\mathbb{N}$.*

Proof The lemma is trivial if $d=g_1$ or $d=g_2$. Assume $d<\min\{g_1,g_2\}$. Let $s,t \in \mathbb{Z}$ such that $sg_1+tg_2=d$. Without loss of generality let $sg_1>0$. Clearly, $C \subseteq (\bar{G}+d\mathbb{N})$. Let $(G_i+g_i\mathbb{N}) \setminus C \subseteq [0, m_i-1]$ for $i=1,2$. For each $x \in C$ and $x>\max\{m_1,m_2\}$ and for each $k>0$ we want to show $x+kd \in C$. Since $x>m_1$, then $x+ksg_1 \in C$. Since $x+ksg_1+ktg_2 = x+kd > m_2$, then we have $h+kd \in C$. This clearly shows that $C \subseteq \bar{G}+d\mathbb{N}$ and $C \sim \bar{G}+d\mathbb{N}$. □

The remaining lemmas in this section involve nonstandard analysis. For convenience we introduce some notation. Let $r,s \in {}^*\mathbb{R}$. By $r \approx s$ we mean that r is infinitesimally close to s, *i.e.*, $|r-s|$ is less than any positive standard real numbers. By $r \ll s$ we mean that $r<s$ but $r \not\approx s$. By $r \lessapprox s$ we mean that $r<s$ or $r \approx s$. We define $r \gg s$ and $r \gtrapprox s$ in a symmetric way. Let H be a hyperfinite integer, *i.e.*, $H \in {}^*\mathbb{N} \setminus \mathbb{N}$. Let

$$U_H = \bigcap_{n\in\mathbb{N}} \left[0, \frac{H}{n}\right]. \tag{7}$$

Then U_H is an initial segment of $[0,H]$ and closed under addition. U_H is often called an additive cut. Note that if $x \in U_H$, then $\frac{x}{H} \approx 0$, and if $x \in {}^*\mathbb{N} \setminus U_H$, then $\frac{x}{H} \gg 0$. Since the sequence $\left\{\frac{H}{n} : n \in \mathbb{N}\right\}$ is lower unbounded in ${}^*\mathbb{N} \setminus U_H$, then by the countable saturation the cofinality of U_H must be uncountable, *i.e.*, any increasing sequence $\{x_n : n \in \mathbb{N}\}$ in U_H must be upper bounded in U_H.

Let $x \in {}^*\mathbb{N}$. Note that $x + \mathbb{N}$ is a copy of $\mathbb{N}$ in ${}^*\mathbb{N}$. Let $C \subseteq x + \mathbb{N}$, $g \in \mathbb{N}$, and $G \subseteq [0, g-1]$. We write $C \sim (G + g\,{}^*\mathbb{N}) \cap (x + \mathbb{N})$ if $((G + g\,{}^*\mathbb{N}) \cap (x + \mathbb{N}))\Delta C$ is finite where Δ means the symmetric difference.

The first lemma below is a nonstandard equivalence of upper Banach density.

Lemma 2.7 *Given α, for any set $A \subseteq \mathbb{N}$, $BD(A) \geqslant \alpha$ iff there is an interval $I = [n, n+K] \subseteq {}^*\mathbb{N}$ for some hyperfinite integer K such that*

$$\frac{{}^*A(n, n+K)}{K+1} \gtrapprox \alpha.$$

Proof Suppose $BD(A) \geqslant \alpha$. Then for each $m \in \mathbb{N}$, there is an interval $[n, n+k]$ with $k > m$ such that $\frac{A(n,n+k)}{k+1} > \alpha - \frac{1}{m+1}$. By the transfer principle, we can fix any hyperfinite m and find an interval $[n, n+K]$ with $K > m$ such that

$$\frac{{}^*A(n, n+K)}{K+1} > \alpha - \frac{1}{m+1} \approx \alpha.$$

Suppose $\frac{{}^*A(n,n+K)}{K+1} \gtrapprox \alpha$ for some hyperfinite integer K. Then for each $m \in \mathbb{N}$ the statement "there is an interval $[a, a+k]$ with $k > m$ such that $\frac{{}^*A(a,a+k)}{k+1} > \alpha - \frac{1}{m+1}$" is true because the interval $[n, n+K]$ is a witness. By the transfer principle we can find an interval $[a_m, b_m] \subseteq \mathbb{N}$ with $b_m - a_m > m$ such that $\frac{A(a_m,b_m)}{b_m - a_m + 1} > \alpha - \frac{1}{m+1}$. This implies $BD(A) \geqslant \alpha$. $\square$

Let Ω be a hyperfinite set, *i.e.*, Ω is an internal set and the internal cardinality of Ω is a hyperfinite integer. Let Σ_0 be the family of all internal subsets of Ω. For each $A \in \Sigma_0$ we can define the normalized counting measure μ of A by $\mu(A) = st(|A|/H)$ where st is the standard part map. Then the finitely–additive measure space (Ω, Σ_0, μ) can generate a countably–additive, complete, atom–less probability space (Ω, Σ, μ) called Loeb space (generated by the normalized counting measure) on Ω.

By Lemma 2.7, the transfer principle, and Birkhoff Ergodic Theorem, one can derive the following two lemmas, which establish the direct connections between lower asymptotic density and upper Banach density. These two lemmas are actually [2, Lemma 3.5] so that the reader can find the proofs in [2].

Lemma 2.8 *Let $A \subseteq \mathbb{N}$. If there exists $x \in {}^*\mathbb{N}$ such that $\underline{d}(({}^*A - x) \cap \mathbb{N}) \geqslant \alpha$, then $BD(A) \geqslant \alpha$.*

Lemma 2.9 *Suppose* $A \subseteq \mathbb{N}$, $BD(A) = \alpha$, *and* $\{[a_n, b_n] : n \in \mathbb{N}\}$ *is a sequence of intervals of standard non-negative integers satisfying (1). Let* N *be any hyperfinite integer and* μ *be the Loeb measure on the hyperfinite set* $[a_N, b_N]$*. Then* $\underline{d}((^*A - x) \cap \mathbb{N}) = \alpha$ *for* μ*–almost all* $x \in [a_N, b_N]$.

The following lemma is similar to Lemma 2.5 in a nonstandard setting.

Lemma 2.10 *Let* H *be hyperfinite and* $A, B \subseteq [0, H]$ *be internal. Suppose for any hyperfinite interval* $[a, b] \subseteq [0, H]$ *with* $\frac{b-a}{H} \gg 0$ *we have*

$$\frac{A(a,b)}{b-a+1} \approx \alpha \ \text{ and } \ \frac{B(a,b)}{b-a+1} \approx \beta.$$

Suppose also $g > 0$, $F, F' \subseteq [0, g-1]$, $A \subseteq F + g\,^*\mathbb{N}$, $B \subseteq F' + g\,^*\mathbb{N}$, $|F \oplus_g F'| = |F| + |F'| - 1$, *and* $\frac{|F|+|F'|}{g} - \frac{1}{g} < \alpha + \beta$*. If* $C \subseteq [0, H]$ *is another set such that* $C \subseteq F'' + g\,^*\mathbb{N}$ *for some* $F'' \subseteq [0, g-1]$ *with* $|F| = |F''|$, $\frac{|C|}{H+1} \approx \alpha$, *and* $C_{f''} = C \cap (f'' + g\,^*\mathbb{N})$, *for some* $f'' \in F''$*, then*

$$\frac{(A+B)(0,H)}{H+1} + \frac{C_{f''}(0,H)}{H+1} \gtrapprox \alpha + \beta.$$

We would like to remark here that Lemma 2.10 is also true by a symmetric argument if we assume that $C \subseteq F'' + g\,^*\mathbb{N}$ for some $F'' \subseteq [0, g-1]$ with $|F'| = |F''|$, $\frac{|C|}{H+1} \approx \beta$, and $C_{f''} = C \cap (f'' + g\,^*\mathbb{N})$, for some $f'' \in F''$.

Proof Let $f \in F$, $f' \in F'$, $A_f = A \cap (f + g\,^*\mathbb{N})$, $B_{f'} = B \cap (f' + g\,^*\mathbb{N})$, and $U = U_H$ defined in (7). For each $x \in ((f \oplus_g f') + g\,^*\mathbb{N})[0, H] \setminus U$ we have

$$\frac{A_f(0,x)}{x+1} + \frac{B_{f'}(0,x)}{x+1} \gtrapprox \alpha - \frac{|F|-1}{g} + \beta - \frac{|F'|-1}{g} \gg \frac{1}{g}.$$

Hence $A_f[0, x] \cap (x - B_{f'})[0, x] \neq \emptyset$. This implies $x \in A_f + B_{f'}$. Therefore there exists $x_0 \in U$ such that

$$(A+B) \cap [x_0, H] = ((F \oplus_g F') + g\,^*\mathbb{N}) \cap [x_0, H].$$

Hence

$$\frac{(A+B)(0,H)}{H+1} \approx \frac{|F \oplus_g F'|}{g} = \frac{|F|+|F'|}{g} - \frac{1}{g}.$$

Now we have

$$\frac{(A+B)(0,H)}{H+1} + \frac{C_{f''}(0,H)}{H+1} \gtrapprox \frac{|F|+|F'|}{g} - \frac{1}{g} + \alpha - \frac{|F''|-1}{g} = \frac{|F'|}{g} + \alpha \geqslant \alpha + \beta.$$

□

3 Proof of Theorem 1.4

Let $A, B \subseteq \mathbb{N}$ be such that $BD(A) = \alpha$, $BD(B) = \beta$, and $BD(A+B) < \alpha + \beta$. Let $\left\{\left[a_n^{(i)}, b_n^{(i)}\right] : n \in \mathbb{N}\right\}$ for $i = 1, 2$ be two sequences of intervals of non-negative integers such that (2), (3), and (4) are true. For each hyperfinite integer N, let

$$S_N = \left\{x \in \left[a_N^{(1)}, b_N^{(1)}\right] : \underline{d}((^*A - x) \cap \mathbb{N}) = \alpha\right\} \text{ and}$$

$$T_N = \left\{y \in \left[a_N^{(2)}, b_N^{(2)}\right] : \underline{d}((^*B - y) \cap \mathbb{N}) = \beta\right\}.$$

Note that S_N has Loeb measure 1 in $\left[a_N^{(1)}, b_N^{(1)}\right]$ and T_N has Loeb measure 1 in $\left[a_N^{(2)}, b_N^{(2)}\right]$ by Lemma 2.9. Let N, N' be two hyperfinite integers. For each $x \in S_N$ and $y \in T_{N'}$ we have that

$$\underline{d}(((^*A - x) + (^*B - y)) \cap \mathbb{N}) < \alpha + \beta$$

by Lemma 2.8. By Lemma 2.2 there exist the least $g_{x,y} > 0$ and $F_{x,y}, F'_{x,y} \subseteq [0, g_{x,y} - 1]$ such that $^*A \cap (x + \mathbb{N}) \subseteq F_{x,y} + g_{x,y}\, ^*\mathbb{N}$, $^*B \cap (y + \mathbb{N}) \subseteq F'_{x,y} + g_{x,y}\, ^*\mathbb{N}$, $\frac{|F_{x,y}| + |F'_{x,y}| - 1}{g_{x,y}} < \alpha + \beta$, and $|F_{x,y} \oplus_{g_{x,y}} F'_{x,y}| = |F_{x,y}| + |F'_{x,y}| - 1$. Note that we also have

$$(^*A \cap (x + \mathbb{N})) + (^*B \cap (y + \mathbb{N})) \subseteq \left((F_{x,y} \oplus_{g_{x,y}} F'_{x,y}) + g_{x,y}\, ^*\mathbb{N}\right) \cap (x + y + \mathbb{N}),$$

$$(^*A \cap (x + \mathbb{N})) + (^*B \cap (y + \mathbb{N})) \sim \left((F_{x,y} \oplus_{g_{x,y}} F'_{x,y}) + g_{x,y}\, ^*\mathbb{N}\right) \cap (x + y + \mathbb{N}),$$

and $\frac{|F_{x,y} \oplus_{g_{x,y}} F'_{x,y}|}{g_{x,y}} \geqslant \alpha + \beta - \frac{1}{g_{x,y}}$ by Theorem 1.1. Let $G_{x,y} = F_{x,y} \oplus_{g_{x,y}} F'_{x,y}$.

We divide the main part of the proof in two claims. In the first claim we characterize the structure of $^*A\left[a_N^{(1)}, b_N^{(1)}\right]$, $^*B\left[a_N^{(2)}, b_N^{(2)}\right]$, and $^*A\left[a_N^{(1)}, b_N^{(1)}\right] + {}^*B\left[a_N^{(2)}, b_N^{(2)}\right]$ for each hyperfinite integer N. In the second claim we characterize the structure of $(^*A + {}^*B) \setminus \mathbb{N}$. The second claim eliminates the possibility of the sets of A and B similar to the sets in Example 1.6. Then we use the transfer principle to pull down the structural property of $^*A + {}^*B$ in the nonstandard model to the structural property of $A + B$ in the standard world to finish the proof.

Note that there exist two fixed standard positive real numbers γ and γ' such that

$$\gamma \leqslant \frac{b_N^{(1)} - a_N^{(1)}}{b_N^{(2)} - a_N^{(2)}} \leqslant \gamma' \tag{8}$$

for every hyperfinite integer N by (4) and the transfer principle.

Claim 3.1 *Given a hyperfinite integer N, there exist $g_N \in \mathbb{N}$, $F_N, F'_N, G_N \subseteq [0, g_N - 1]$, and $\left[c_N^{(i)}, d_N^{(i)}\right] \subseteq \left[a_N^{(i)}, b_N^{(i)}\right]$ for $i = 1, 2$ such that*

$$^*A\left[a_N^{(1)}, b_N^{(1)}\right] \subseteq (F_N + g_N\, ^*\mathbb{N}),$$

$$^*B\left[a_N^{(2)}, b_N^{(2)}\right] \subseteq (F'_N + g_N\, ^*\mathbb{N}),$$

$$F_N \oplus_g F'_N = G_N \ \textit{ and } \ |G_N| = |F_N| + |F'_N| - 1,$$

$$\alpha + \beta - \frac{1}{g_N} \leqslant \frac{|F_N| + |F'_N| - 1}{g_N} < \alpha + \beta,$$

$$\frac{d_N^{(i)} - c_N^{(i)}}{b_N^{(i)} - a_N^{(i)}} \approx 1 \ \textit{ for } \ i = 1, 2,$$

$$\left(^*A\left[a_N^{(1)}, b_N^{(1)}\right] + {}^*B\left[a_N^{(2)}, b_N^{(2)}\right]\right) \subseteq G_N + g_N\, ^*\mathbb{N}, \ \ \textit{and}$$

$$\left(^*A\left[a_N^{(1)}, b_N^{(1)}\right] + {}^*B\left[a_N^{(2)}, b_N^{(2)}\right]\right) \cap \left[c_N^{(1)} + c_N^{(2)}, d_N^{(1)} + d_N^{(2)}\right]$$
$$= (G_N + g_N\, ^*\mathbb{N}) \cap \left[c_N^{(1)} + c_N^{(2)}, d_N^{(1)} + d_N^{(2)}\right].$$

Proof of Claim 3.1 Let $H = b_N^{(1)} - a_N^{(1)}$ and let $U = U_H$ as defined in (7). Note that if we use $H = b_N^{(2)} - a_N^{(2)}$ or $H = b_N^{(1)} - a_N^{(1)} + b_N^{(2)} - a_N^{(2)}$ to define U we will get exactly the same U due to (8). Note also that by the transfer principle we have

$$\frac{^*A\left(a_N^{(1)}, b_N^{(1)}\right)}{b_N^{(1)} - a_N^{(1)} + 1} \approx \alpha \ \text{ and } \ \frac{^*B\left(a_N^{(2)}, b_N^{(2)}\right)}{b_N^{(2)} - a_N^{(2)} + 1} \approx \beta.$$

Subclaim 3.1.1 There are $x_0 \in a_N^{(1)} + U$ and $y_0 \in a_N^{(2)} + U$ such that for all $x \in \left[x_0, b_N^{(1)}\right]$ with $x - x_0$ being hyperfinite and for all $y \in \left[y_0, b_N^{(2)}\right]$ with $y - y_0$ being hyperfinite we have

$$\frac{^*A(x_0, x)}{x - x_0 + 1} \approx \alpha \ \text{ and } \ \frac{^*B(y_0, y)}{y - y_0 + 1} \approx \beta.$$

Proof of Subclaim 3.1.1 We find x_0, the argument for y_0 is analogous. Let $c_k = a_N^{(1)} + [(b_N^{(1)} - a_N^{(1)})/(k+1)]$ for every $k \in \mathbb{N}$. Note that $\bigcap_{k\in\mathbb{N}}[a_N^{(1)}, c_k] = a_N^{(1)} + U$. For every $k \in \mathbb{N}$ define the internal sets

$$X_k = \left\{ m \in [a_N^{(1)}, c_k] : \forall x \in [m+1, b_N^{(1)}] \ \left(\frac{^*A(m+1, x)}{x - m} > \alpha - \frac{1}{k+1} \right) \right\}.$$

Clearly, $X_k \supseteq X_{k+1}$ for every $k \in \mathbb{N}$. Suppose $X_k = \emptyset$ for some $k \in \mathbb{N}$. Let $x \in [a_N^{(1)}, b_N^{(1)}]$ the largest number such that $\dfrac{{}^*A\left(a_N^{(1)}+1, x\right)}{x - a_N^{(1)}} \leqslant \alpha - \dfrac{1}{k+1}$ if it exists, or $b_N^{(1)}$ otherwise. Clearly, $x \notin b_N^{(1)} - U$ because otherwise we have $\dfrac{{}^*A\left(a_N^{(1)}, b_N^{(1)}\right)}{b_N^{(1)} - a_N^{(1)} + 1} \ll \alpha$. Since $X_k = \emptyset$, we have $x \geqslant c_k$. Since $\dfrac{{}^*A(a_N^{(1)}, x)}{x - a_N^{(1)} + 1} \ll \alpha$, we have $\dfrac{{}^*A(x+1, b_N^{(1)})}{b_N^{(1)} - x} \gg \alpha$. This implies $BD(A) > \alpha$ by Lemma 2.7, which contradicts the assumption that $BD(A) = \alpha$. Hence $X_k \neq \emptyset$ for every $k \in \mathbb{N}$. By countable saturation we can find $m \in \bigcap_{k \in \mathbb{N}} X_k$. Let $x_0 = m + 1$. Since $x_0 < c_k$ for every $k \in \mathbb{N}$, then $x_0 \in a_N^{(1)} + U$. Now for each $x \in [x_0, b_N^{(1)}]$ we have $\dfrac{{}^*A(x_0, x)}{x - x_0 + 1} \gtrapprox \alpha$. When $x - x_0$ is hyperfinite, $\dfrac{{}^*A(x_0, x)}{x - x_0 + 1} \gg \alpha$ would imply $BD(A) > \alpha$. Hence we have $\dfrac{{}^*A(x_0, x)}{x - x_0 + 1} \approx \alpha$ whenever $x - x_0$ is hyperfinite. $\square$

We continue to prove Claim 3.1. By the definition of x_0 and y_0 and Lemma 2.8 we have that $\underline{d}(({}^*A - x_0) \cap \mathbb{N}) = \alpha$ and $\underline{d}(({}^*B - y_0) \cap \mathbb{N}) = \beta$. Hence $x_0 \in S_N$ and $y_0 \in T_N$. Let $g_N = g_{x_0, y_0}$, $F_N = F_{x_0, y_0}$, and $F'_N = F'_{x_0, y_0}$. Note that $|G_N| = |F_N| + |F'_N| - 1$ where $G_N = F_N \oplus_{g_N} F'_N$. We need to show that g_N, F_N, F'_N, G_N are what we are looking for. Since

$$(({}^*A \cap (x_0 + \mathbb{N})) + ({}^*B \cap (y_0 + \mathbb{N}))) \subseteq (G_N + g_N\, {}^*\mathbb{N}) \cap (x_0 + y_0 + \mathbb{N}) \text{ and}$$

$$(({}^*A \cap (x_0 + \mathbb{N})) + ({}^*B \cap (y_0 + \mathbb{N}))) \sim (G_N + g_N\, {}^*\mathbb{N}) \cap (x_0 + y_0 + \mathbb{N}),$$

we conclude, by the overspill principle, that there is a hyperfinite integer K such that

$${}^*A[x_0, x_0 + K] \subseteq F_N + g_N\, {}^*\mathbb{N},\ {}^*B[y_0, y_0 + K] \subseteq F'_N + g_N\, {}^*\mathbb{N},$$

$$\text{and } ({}^*A + {}^*B)[z_0, z_0 + K] = (G + g\, {}^*\mathbb{N}) \cap [z_0, z_0 + K]$$

for some $z_0 \in x_0 + y_0 + \mathbb{N}$.

Subclaim 3.1.2 ${}^*A\left[x_0, b_N^{(1)}\right] \subseteq F_N + g_N\, {}^*\mathbb{N}$ and ${}^*B\left[y_0, b_N^{(2)}\right] \subseteq F'_N + g_N\, {}^*\mathbb{N}$.

Proof of Subclaim 3.1.2 Assume the contrary. Without loss of generality, let x be the least element in ${}^*A\left[x_0, b_N^{(1)}\right] \setminus (F_N + g_N\, {}^*\mathbb{N})$ such that ${}^*B[y_0, y_0 + (x - x_0) - 1] \subseteq F'_N + g_N\, {}^*\mathbb{N}$. By Remark 2.4.3 there is $f' \in F'_N$ such that $(\pi_{g_N}(x) \oplus_{g_N} f') \notin (F_N \oplus_{g_N} F'_N)$; let ${}^*B_{f'} = {}^*B \cap (f' + g_N\, {}^*\mathbb{N})$. Note that $x - x_0 \geqslant K$. Let $d = \left[\frac{x - x_0}{2}\right]$.

By Subclaim 3.1.1 we have $\frac{{}^*A(x_0,x_0+2d)}{2d} \approx \alpha$, which implies that for any $x_0 \leqslant a < b \leqslant x_0+2d$ with $\frac{b-a}{2d} \gg 0$, $\frac{{}^*A(a,b)}{b-a} \approx \alpha$. This is true because of the following argument: if $\frac{{}^*A(a,b)}{b-a} \gg \alpha$, then $BD(A) > \alpha$ by Lemma 2.7, which contradicts $BD(A) = \alpha$; if $\frac{{}^*A(a,b)}{b-a} \ll \alpha$, then $\frac{{}^*A(x_0,x_0+2d)}{2d} \approx \alpha$ implies that either $\frac{a-x_0}{2d} \gg 0$ and $\frac{{}^*A(x_0,a)}{a-x_0} \gg \alpha$ or $\frac{x_0+2d-b}{2d} \gg 0$ and $\frac{{}^*A(b,x_0+2d)}{x_0+2d-b} \gg \alpha$. Note that either case above contradicts $BD(A) = \alpha$ again by Lemma 2.7.

By the same reason we have that for any $y_0 \leqslant a < b \leqslant y_0 + 2d$ with $\frac{b-a}{2d} \gg 0$, $\frac{{}^*B(a,b)}{b-a} \approx \beta$. Then by Lemma 2.10 we have

$$\begin{aligned}&\frac{({}^*A + {}^*B)(x_0 + y_0 + 2d, x_0 + y_0 + 3d)}{d+1} \\ &\quad \gtrapprox \frac{({}^*A[x_0 + d, x_0 + 2d - 1] + {}^*B[y_0 + d, y_0 + 2d - 1])(x_0 + y_0 + 2d, x_0 + y_0 + 3d)}{d+1} \\ &\qquad + \frac{(x + {}^*B_{f'}[y_0, y_0 + d])(x_0 + y_0 + 2d, x_0 + y_0 + 3d)}{d+1} \gtrapprox \alpha + \beta,\end{aligned}$$

which contradicts $BD(A + B) < \alpha + \beta$ by Lemma 2.7. $\square$

Subclaim 3.1.3 ${}^*A\left[a_N^{(1)}, b_N^{(1)}\right] \subseteq F_N + g_N\,{}^*\mathbb{N}$ and ${}^*B\left[a_N^{(2)}, b_N^{(2)}\right] \subseteq F'_N + g_N\,{}^*\mathbb{N}$.

Proof of Subclaim 3.1.3 Assume the contrary. Then there is, without loss of generality, $z \in {}^*A\left[a_N^{(1)}, x_0\right] \setminus (F_N + g_N\,{}^*\mathbb{N})$. Let $f' \in F'_N$ be such that $\pi_{g_N}(z) \oplus_{g_N} f' \notin F_N \oplus_{g_N} F'_N$ and ${}^*B_{f'} = {}^*B \cap (f' + g_N\,{}^*\mathbb{N})$. Choose $x_1 \in \left[x_0, b_N^{(1)}\right]$ such that $x_1 \notin x_0 + U$, $x_0 + 2(x_1 - x_0) < b_N^{(1)}$, and $y_0 + 3(x_1 - x_0) < b_N^{(2)}$. Let $d = x_1 - x_0$. Note that $x_0 - z \in U$ and $d \notin U$, which implies that $\frac{x_0 - z}{d} \approx 0$. By the same reason as in the proof of Subclaim 3.1.2 we can apply Lemma 2.10 to obtain that

$$\begin{aligned}&\frac{({}^*A + {}^*B)(x_0 + y_0 + 2d, x_0 + y_0 + 3d)}{d+1} \\ &\quad \gtrapprox \frac{({}^*A[x_0 + d, x_0 + 2d] + {}^*B[y_0 + d, y_0 + 2d])(x_0 + y_0 + 2d, x_0 + y_0 + 3d)}{d+1} \\ &\qquad + \frac{|z + {}^*B_{f'}[y_0 + 2d + x_0 - z, y_0 + 3d + x_0 - z]|}{d+1} \\ &\quad \approx \frac{({}^*A[x_0 + d, x_0 + 2d] + {}^*B[y_0 + d, y_0 + 2d])(x_0 + y_0 + 2d, x_0 + y_0 + 3d)}{d+1} \\ &\qquad + \frac{{}^*B_{f'}(y_0 + 2d, y_0 + 3d)}{d+1} \gtrapprox \alpha + \beta,\end{aligned}$$

which contradicts $BD(A + B) < \alpha + \beta$ again by Lemma 2.7. $\square$

Subclaim 3.1.4 There are $\left[c_N^{(i)}, d_N^{(i)}\right] \subseteq \left[a_N^{(i)}, b_N^{(i)}\right]$ such that

$$\frac{d_N^{(i)} - c_N^{(i)}}{b_N^{(i)} - a_N^{(i)}} \approx 1 \tag{9}$$

for $i = 1, 2$ and

$$\begin{aligned}\left({}^*A\left[a_N^{(1)}, b_N^{(1)}\right] + {}^*B\left[a_N^{(2)}, b_N^{(2)}\right]\right) \cap \left[c_N^{(1)} + c_N^{(2)}, d_N^{(1)} + d_N^{(2)}\right] \\ = (G_N + g_N\,{}^*\mathbb{N}) \cap \left[c_N^{(1)} + c_N^{(2)}, d_N^{(1)} + d_N^{(2)}\right].\end{aligned}$$

Proof of Subclaim 3.1.4 By (4) we have

$$U_{b_N^{(1)} - a_N^{(1)}} = U_{b_N^{(2)} - a_N^{(2)}} = U_{b_N^{(1)} + b_N^{(2)} - a_N^{(1)} - a_N^{(2)}},$$

where U_H is defined by (7). Let $U = U_{b_N^{(1)} - a_N^{(1)}}$. By the same argument as in the proof of Lemma 2.10 we can prove that for every $z \in (G_N + g_N\,{}^*\mathbb{N}) \cap \left[a_N^{(1)} + a_N^{(2)}, b_N^{(1)} + b_N^{(2)}\right]$, if $z \notin a_N^{(1)} + a_N^{(2)} + U$ and $z \notin b_N^{(1)} + b_N^{(2)} - U$, then $z \in {}^*A\left[a_N^{(1)}, b_N^{(1)}\right] + {}^*B\left[a_N^{(2)}, b_N^{(2)}\right]$. Let u with

$$b_N^{(1)} + b_N^{(2)} \geqslant u > \frac{a_N^{(1)} + b_N^{(1)}}{2} + \frac{a_N^{(2)} + b_N^{(2)}}{2}$$

be the greatest and l with

$$a_N^{(1)} + a_N^{(2)} \leqslant l < \frac{a_N^{(1)} + b_N^{(1)}}{2} + \frac{a_N^{(2)} + b_N^{(2)}}{2}$$

be the least such that

$$\left({}^*A\left[a_N^{(1)}, b_N^{(1)}\right] + {}^*B\left[a_N^{(2)}, b_N^{(2)}\right]\right) \cap [l, u] = (G_N + g_N\,{}^*\mathbb{N}) \cap [l, u].$$

Then $l \in a_N^{(1)} + a_N^{(2)} + U$ and $u \in b_N^{(1)} + b_N^{(2)} - U$. It is now easy to select the desired $c_N^{(i)} \in a_N^{(i)} + U$ and $d_N^{(i)} \in b_N^{(i)} - U$ for $i = 1, 2$ such that $l = c_N^{(1)} + c_N^{(2)}$ and $u = d_N^{(1)} + d_N^{(2)}$. This ends the proof of Claim 3.1. $\square$

Claim 3.2 *There are $g > 0$ in $\mathbb{N}$ and $F, F' \subseteq [0, g-1]$ such that ${}^*A \subseteq F + g\,{}^*\mathbb{N}$, ${}^*B \subseteq F' + g\,{}^*\mathbb{N}$, $|F \oplus_g F'| = |F| + |F'| - 1$, and $\frac{|F|+|F'|-1}{g} < \alpha + \beta$.*

In the applications of Claim 3.2 we always assume that g is the least positive integer satisfying this claim.

Proof of Claim 3.2 Let $BD(A+B) = \gamma < \alpha + \beta$. Note that for every hyperfinite integer N,

$$\frac{\left| {}^*A\left[a_N^{(1)}, b_N^{(1)}\right] + {}^*B\left[a_N^{(2)}, b_N^{(2)}\right]\right|}{b_N^{(1)} + b_N^{(2)} - a_N^{(1)} - a_N^{(2)} + 1} \approx \frac{|G_N|}{g_N} \leqslant \gamma$$

by Lemma 2.7. Therefore, $\gamma \geqslant \alpha + \beta - \frac{1}{g_N}$. This implies that the set $\{g_N : N \in {}^*\mathbb{N} \setminus \mathbb{N}\} \subseteq \mathbb{N}$ is finite. Hence there are $g_0 \in \mathbb{N}$ and $F, F' \subseteq [0, g_0 - 1]$ such that the set

$$X_0 = \{N \in {}^*\mathbb{N} \setminus \mathbb{N} : g_N = g_0, F_N = F, \text{ and } F'_N = F'\}$$

is unbounded in ${}^*\mathbb{N}$. We will prove that g_0, F, and F' are what we want for the claim. It suffices to prove that ${}^*A \subseteq F + g_0\,{}^*\mathbb{N}$ and ${}^*B \subseteq F' + g_0\,{}^*\mathbb{N}$.

Suppose the claim is not true and assume, without loss of generality, ${}^*A \nsubseteq (F + g_0\,{}^*\mathbb{N})$. Fix $x \in {}^*A \setminus (F + g_0\,{}^*\mathbb{N})$ and $N_0 \in X_0$. Since (2) and X_0 is unbounded in ${}^*\mathbb{N}$, there is a hyperfinite integer $N \in X_0$ such that $b_N^{(2)} - a_N^{(2)} > 2\max\left\{b_{N_0}^{(1)}, x\right\}$. Choose any $x_0 \in S_{N_0}$. Since T_N has Loeb measure one in $\left[a_N^{(2)}, b_N^{(2)}\right]$, we can find $y, y' \in T_N$ such that $x + y = x_0 + y'$. Note that $\underline{d}(({}^*A - x_0) \cap \mathbb{N}) = \alpha$, $\underline{d}(({}^*B - y') \cap \mathbb{N}) = \beta$, and $\underline{d}((({}^*A + {}^*B) - x_0 - y') \cap \mathbb{N}) < \alpha + \beta$. By Lemma 2.2 there is a least positive integer $g_{N,1}$ and sets $F_{N,1}, F'_{N,1}, G_{N,1} \subseteq [0, g_{N,1} - 1]$ such that

$${}^*A \cap (x_0 + \mathbb{N}) \subseteq (F_{N,1} + g_{N,1}\,{}^*\mathbb{N}),\ \ {}^*B \cap (y' + \mathbb{N}) \subseteq (F'_{N,1} + g_{N,1}\,{}^*\mathbb{N}),$$

$$\frac{|F_{N,1}| + |F'_{N,1}| - 1}{g_{N,1}} < \alpha + \beta, \text{ and } G_{N,1} = F_{N,1} \oplus_{g_{N,1}} F'_{N,1}.$$

We have that $|G_{N,1}| = |F_{N,1}| + |F'_{N,1}| - 1$,

$$({}^*A \cap (x_0 + \mathbb{N})) + ({}^*B \cap (y' + \mathbb{N})) \subseteq (G_{N,1} + g_{N,1}\,{}^*\mathbb{N}) \cap (x_0 + y' + \mathbb{N}), \text{ and}$$

$$({}^*A \cap (x_0 + \mathbb{N})) + ({}^*B \cap (y' + \mathbb{N})) \sim (G_{N,1} + g_{N,1}\,{}^*\mathbb{N}) \cap (x_0 + y' + \mathbb{N}).$$

By the definition of F and F' we also have

$$({}^*A \cap (x_0 + \mathbb{N})) + ({}^*B \cap (y' + \mathbb{N})) \subseteq ((F \oplus_{g_0} F') + g_0\,{}^*\mathbb{N}) \cap (x_0 + y' + \mathbb{N})) \text{ and}$$

$$({}^*A \cap (x_0 + \mathbb{N})) + ({}^*B \cap (y' + \mathbb{N})) \sim ((F \oplus_{g_0} F') + g_0\,{}^*\mathbb{N}) \cap (x_0 + y' + \mathbb{N})).$$

Let $d = \gcd(g_0, g_{N,1})$. Let $\bar{G} = \pi_{g_0,d}(F \oplus_{g_0} F') = \pi_{g_{N,1},d}(G_{N,1})$. By Lemma 2.6

$$({}^*A \cap (x_0 + \mathbb{N})) + ({}^*B \cap (y' + \mathbb{N})) \subseteq (\bar{G} + d\,{}^*\mathbb{N}) \cap (x_0 + y' + \mathbb{N}) \text{ and}$$

$$({}^*A \cap (x_0 + \mathbb{N})) + ({}^*B \cap (y' + \mathbb{N})) \sim (\bar{G} + d\,{}^*\mathbb{N}) \cap (x_0 + y' + \mathbb{N}).$$

Let $d'|d$ be the least positive integer with $\bar{\bar{G}} = \pi_{g_0,d'}(F \oplus_{g_0} F')$ such that the two statements immediately above are true with d replaced by d' and $\bar{G}$ replaced by $\bar{\bar{G}}$. Let $\bar{\bar{F}} = \pi_{g_0,d'}(F)$ and $\bar{\bar{F}}' = \pi_{g_0,d'}(F')$. By Lemma 2.1 we have $|\bar{\bar{G}}| \geqslant |\bar{\bar{F}}| + |\bar{\bar{F}}'| - 1$ by the minimality of d'. Hence $\frac{|\bar{\bar{F}}|+|\bar{\bar{F}}'|-1}{d'} \leqslant \frac{|\bar{\bar{G}}|}{d'} = \frac{|F \oplus_{g_0} F'|}{g_0} < \alpha + \beta$. This implies that $d' = g_{N,1} \leqslant g_0$ by the minimality of $g_{N,1}$.

If $g_0 = g_{N,1}$, then $|F| = |F_{N,1}|$ and $|F'| = |F'_{N,1}|$. This implies $x \notin (F_{N,1} + g_{N,1}\,{}^*\mathbb{N})$. Let $f'_1 \in F'_{N,1}$ be such that $\left(\pi_{g_{N,1}}(x) \oplus_{g_{N,1}} f'_1\right) \notin \left(F_{N,1} \oplus_{g_{N,1}} F'_{N,1}\right)$ and let ${}^*B_{f'_1} = {}^*B \cap (f'_1 + g_{N,1}\,{}^*\mathbb{N})$. Then

$$\begin{aligned}&\underline{d}(({}^*A + {}^*B) - x_0 - y') \cap \mathbb{N})\\ &\quad \geqslant \underline{d}((({}^*A - x_0) \cap \mathbb{N}) + (({}^*B - y') \cap \mathbb{N})) + \underline{d}(({}^*B_{f'_1} - y) \cap \mathbb{N}) \geqslant \alpha + \beta\end{aligned}$$

by Lemma 2.5, which contradicts $BD(A + B) < \alpha + \beta$ by Lemma 2.8. Hence we conclude that $d = g_{N,1} < g_0$ and $x \in (F_{N,1} + g_{N,1}\,{}^*\mathbb{N})$. Clearly, $F_{N,1} = \pi_{g_0,g_{N,1}}(F)$, $F'_{N,1} = \pi_{g_0,g_{N,1}}(F')$, and $\frac{|F_{N,1}|+|F'_{N,1}|-1}{g_{N,1}} < \alpha + \beta$.

Now we have

$$ {}^*A\left[a^{(1)}_{N_0}, b^{(1)}_{N_0}\right] \subseteq \left((F + g_0\,{}^*\mathbb{N}) \cap \left[a^{(1)}_{N_0}, b^{(1)}_{N_0}\right]\right) \subseteq \left((F_{N,1} + g_{N,1}\,{}^*\mathbb{N}) \cap \left[a^{(1)}_{N_0}, b^{(1)}_{N_0}\right]\right),$$

$$ {}^*B\left[a^{(2)}_{N_0}, b^{(2)}_{N_0}\right] \subseteq \left((F' + g_0\,{}^*\mathbb{N}) \cap \left[a^{(2)}_{N_0}, b^{(2)}_{N_0}\right]\right) \subseteq \left((F'_{N,1} + g_{N,1}\,{}^*\mathbb{N}) \cap \left[a^{(2)}_{N_0}, b^{(2)}_{N_0}\right]\right),$$

and $\frac{|F_{N,1}|+|F'_{N,1}|-1}{g_{N,1}} < \alpha + \beta$, which contradict the minimality of $g_{N_0} = g_0$. This completes the proof of the claim. □

Let g, F, and F' be in Claim 3.2 such that g is the least and let N be a hyperfinite integer. We want to show that $g_N = g$, $F_N = F$, and $F'_N = F$ for all hyperfinite N, which will complete the proof of the theorem.

Since $\frac{|F|+|F'|-1}{g} < \alpha + \beta$, then

$$({}^*A \cap (x_0 + \mathbb{N})) + ({}^*B \cap (y_0 + \mathbb{N})) \subseteq ((F \oplus_g F') + g\,{}^*\mathbb{N}) \cap (x_0 + y_0 + \mathbb{N}) \text{ and}$$

$$({}^*A \cap (x_0 + \mathbb{N})) + ({}^*B \cap (y_0 + \mathbb{N})) \sim ((F \oplus_g F') + g\,{}^*\mathbb{N}) \cap (x_0 + y_0 + \mathbb{N})$$

where x_0 and y_0 are the elements from Subclaim 3.1.1. By the minimality of g_N we have $g \geqslant g_N$ and $g_N | g$. Suppose $g > g_N$. Let $F_N = \pi_{g,g_N}(F)$ and $F'_N = \pi_{g,g_N}(F')$. Then ${}^*A \subseteq F_N + g_N\, {}^*\mathbb{N}$, ${}^*B \subseteq F'_N + g_N\, {}^*\mathbb{N}$, and $\frac{|F_N|+|F'_N|-1}{g_N} < \alpha+\beta$. Since $|F_N \oplus_{g_N} F'_N| = |F_N| + |F'_N| - 1$, we can replace g, F, F' by g_N, F_N, F'_N in Claim 3.2, which contradict the minimality of g. Therefore, we have $g_N = g$. This clearly implies $F_N = F$ and $F'_N = F$. Let $G = F \oplus_g F$. We now translate the nonstandard statements to the standard statements.

Clearly, $(A+B) \subseteq (G + g\mathbb{N})$ and $BD(A+B) \geqslant \frac{|G|}{g} = \frac{|F|+|F'|-1}{g} \geqslant \alpha + \beta - \frac{1}{g}$.

By Claim 3.1 we have that for every hyperfinite integer N there are $\left[c_N^{(i)}, d_N^{(i)}\right]$ for $i = 1, 2$ such that (9) is true and

$$({}^*A + {}^*B) \cap \left[c_N^{(1)} + c_N^{(2)}, d_N^{(1)} + d_N^{(2)}\right] = (G + g\, {}^*\mathbb{N}) \cap \left[c_N^{(1)} + c_N^{(2)}, d_N^{(1)} + d_N^{(2)}\right].$$

For each $m \in \mathbb{N}$ let D_m be the set of all $n \in {}^*\mathbb{N}$ such that there exist $\left[c_n^{(i)}, d_n^{(i)}\right] \subseteq \left[a_n^{(i)}, b_n^{(i)}\right]$ with

$$\frac{d_n^{(i)} - c_n^{(i)}}{b_n^{(i)} - a_n^{(i)}} \geqslant 1 - \frac{1}{m+1}$$

for $i = 1, 2$ and

$$({}^*A + {}^*B) \cap \left[c_n^{(1)} + c_n^{(2)}, d_n^{(1)} + d_n^{(2)}\right] = (G + g\, {}^*\mathbb{N}) \cap \left[c_n^{(1)} + c_n^{(2)}, d_n^{(1)} + d_n^{(2)}\right].$$

Clearly, D_m is an internal set and contains all hyperfinite integers. Let $n_m = \min D_m$. Then $n_m \in \mathbb{N}$, $n_0 = 0$, and $n_m \leqslant n_{m+1}$. For each $m \in \mathbb{N}$ and each $n = n_m, \ldots, n_{m+1}-1$ let $\left[c_n^{(i)}, d_n^{(i)}\right] \subseteq \left[a_n^{(i)}, b_n^{(i)}\right]$ be such that $\frac{d_n^{(i)} - c_n^{(i)}}{b_n^{(i)} - a_n^{(i)}} \geqslant 1 - \frac{1}{m+1}$ for $i = 1, 2$ and

$$(A + B) \cap \left[c_n^{(1)} + c_n^{(2)}, d_n^{(1)} + d_n^{(2)}\right] = (G + g\mathbb{N}) \cap \left[c_n^{(1)} + c_n^{(2)}, d_n^{(1)} + d_n^{(2)}\right].$$

It is now easy to check the sequence $\left\{\left[c_n^{(i)}, d_n^{(i)}\right] : n \in \mathbb{N}\right\}$ is what we want. This ends the proof of Theorem 1.4. □

4 A Question

In [3] the structure of A is characterized when $\underline{d}(A)$ is sufficiently small and $\underline{d}(A+A) \leqslant \sigma \underline{d}(A)$ for some $\sigma \geqslant 2$. When $\underline{d}(A+A) = 2\underline{d}(A)$, the structure of A, as indicated in [3], can be drastically different from the structure of A when $\underline{d}(A+A) < 2\underline{d}(A)$.

For example, as described in [3], let $\epsilon > 0$ be a small real number, α be an irrational number, and

$$A = \left\{ n \in \mathbb{N} : \alpha n \equiv x \,(\mathrm{mod}\, 1) \text{ and } x \in \left(\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon \right) \right\}.$$

Then $\underline{d}(A) = 2\epsilon$ and $\underline{d}(A + A) = 4\epsilon = 2\underline{d}(A)$. Clearly, $A + A$ is not a large subset of the union of arithmetic progressions of the same difference.

Question 4.1 *What should be the structure of A, B, or the structure of $A + B$ if $BD(A) = \alpha > 0$, $BD(B) = \beta > 0$, and $BD(A + B) = \alpha + \beta$?*

It should be easier to consider the special case of the question above for $A = B$.

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