The Isomorphism Property Versus The Special Model Axiom

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Abstract

This paper answers some questions of D. Ross in [R]. In §1, we show that some consequences of the $\aleph_0$ or $\aleph_1$–special model axiom in [R] can not be proved by the κ–isomorphism property for any cardinal κ. In §2, we show that with one exception, the $\aleph_0$–isomorphism property does imply the remaining consequences of the special model axiom in [R]. In §3, we improve a result in [R] by showing that the κ–special model axiom is equivalent to the $\aleph_0$–special model axiom plus κ–saturation.

§0. Notation and introduction

Throughout this paper we use $\kappa, \lambda, \ldots$ for infinite cardinals, $\alpha, \beta, \gamma, \ldots$ for ordinals, $\mathcal{L}, \mathcal{L}', \ldots$ for some first–order languages and $\mathcal{A}, \mathcal{B}, \ldots$ for models (or structures) with base sets (or universes) $A, B, \ldots$.

Let $\mathbb{N}$ be the set of all natural numbers. By a standard universe we mean the superstructure $V_\omega(\mathbb{N}) = \bigcup_{\ltimes \in \omega} \mathbb{V}_{\ltimes}$ with the "$\in$" relation, where $V_0 = \mathbb{N}$, a set of urelements, and $V_{n+1} = V_n \bigcup \mathcal{P}(V_n)$. $|V_\omega(\mathbb{N})| = \beth_\omega$. By a nonstandard universe we mean the image of V under Mostowski collapse, where V is an elementary extension of the standard universe truncated at $\in$–rank ω. We use $V, V', \overline{V}, V_\alpha$, etc. for standard or nonstandard universes. Note that each V contains a base set and a binary relation $\in^V$. We will not distinguish a nonstandard universe from its base set.

We let $\mathbb{N}$ ($\mathbb{R}$) be all natural (real) numbers and $\mathbb{N}^{\mathbb{V}}$ ($\mathbb{R}^{\mathbb{V}}$) be all standard and nonstandard natural (real) numbers in V. If V is not explicitly given, we usually use ${}^*\mathbb{N}$ (${}^*\mathbb{R}$) instead of $\mathbb{N}^{\mathbb{V}}$ ($\mathbb{R}^{\mathbb{V}}$).

If P and Q are two linear orders, an order–preserving map $f : P \longmapsto Q$ is called a cofinal (coinitial) embedding if $f[P]$ is upper (lower) unbounded in Q. $cf(Q)$ ($ci(Q)$) is the least cardinal λ such that λ (the reverse order of λ) can be cofinally (coinitially) embedded into Q. We let $cf({}^*\mathbb{N})$ mean the cofinality of ${}^*\mathbb{N}$ with the usual order and $ci({}^*\mathbb{N})$ mean the coinitiality of ${}^*\mathbb{N} - \mathbb{N}$ with the usual order.

A set A is called internal in V if A is an element of V. If V is not explicitly given, A is internal means A is an element of a nonstandard universe. An $\mathcal{L}$–structure $\mathcal{A}$ is called internally presented in V if both the base set and interpretations under $\mathcal{A}$ of every symbol in $\mathcal{L}$ are internal in V. Note that the internally presented structure itself may not be internal if $\mathcal{L}$ is infinite because an infinite sequence of internal sets can be external. Let V and V' be two nonstandard universes. We let $V \lhd V'$ mean that V' is the Mostowski collapse of the truncation at rank ω of an elementary extension of V. Let $j : V \longmapsto V'$ be the corresponding embedding. Then for every formula $\phi(x_1, \ldots, x_n)$ of $\mathcal{L} = \langle \in \rangle$ with only bounded quantifiers and any $a_1, \ldots, a_n \in V$, $V \models \phi(a_1, \ldots, a_n)$ iff $V' \models \phi(a_1, \ldots, a_n)$. For an internal set A in V we use $A^{V'}$ for $j(A)$. If $\mathcal{A} = \langle A, (P_\alpha)_{\alpha \in I} \rangle$ is internally presented in V, we use $\mathcal{A}^{V'}$ for the structure $\langle j(A), (j(P_\alpha))_{\alpha \in I} \rangle$. We use $\equiv$ for elementary equivalence and $\cong$ for isomorphism between two structures. Let $\mathcal{A}$ and $\mathcal{B}$ be two structures. We let $\mathcal{A} \prec \mathcal{B}$ mean that $\mathcal{A}$ can be elementarily embedded into $\mathcal{B}$ and let $\mathcal{A} \preceq \mathcal{B}$ mean that $\mathcal{A}$ is an elementary submodel of $\mathcal{B}$. We call a sequence of models $\{\mathcal{A}_\alpha : \alpha < \lambda\}$ in a nonstandard universe an elementary chain if $\mathcal{A}_0 \preceq \mathcal{A}_1 \preceq \cdots \preceq \mathcal{A}_\alpha \preceq \cdots$. We call a sequence of nonstandard universes $\{V_\alpha : \alpha < \lambda\}$ an elementary chain if $V_0 \lhd V_1 \lhd \ldots \lhd V_\alpha \lhd \ldots$. Note that the elementary chain theorem works for $\lhd$. For example, if $V \lhd V_\alpha$ for all $\alpha < \beta$, then $V \lhd \bigcup_{\alpha<\beta} V_\alpha$. Here $\bigcup$ means direct limit. (See [CK, §4.4].)

We say that a nonstandard universe V satisfies the κ–isomorphism property (*IP*$_\kappa$ for short) if the following statement is true in V:

> For any two internally presented $\mathcal{L}$–structures $\mathcal{A}$ and $\mathcal{B}$ with $|\mathcal{L}| < \kappa$, if $\mathcal{A} \equiv \mathcal{B}$, then $\mathcal{A} \cong \mathcal{B}$.

We should give the definition of the special model axiom next. Before that, let's recall that an $\mathcal{L}$–structure $\mathcal{A}$ is called special if there exists an elementary chain $\{\mathcal{A}_\alpha : \alpha < |A|\}$ such that $\mathcal{A} = \bigcup_{\alpha<|A|} \mathcal{A}_\alpha$ and for every $\alpha < |A|$, $\mathcal{A}_\alpha$ is $|\alpha|^+$–saturated. $\{\mathcal{A}_\alpha : \alpha < |A|\}$ is called a specializing chain for $\mathcal{A}$.

We say that a nonstandard universe V satisfies the κ–special model axiom (*SMA*$_\kappa$ for short) if the following statement is true in V:

> For any internally presented $\mathcal{L}$–structure $\mathcal{A}$ with $|\mathcal{L}| < \kappa$, $\mathcal{A}$ is special.

IP$_\kappa$ was defined by Henson in [H] and recently D. Ross published his paper [R] to investigate the strength of *SMA*$_\kappa$. *IP*$_\kappa$ and *SMA*$_\kappa$ both are very useful tools for nonstandard analysis and both are stronger than κ–saturation. The nonstandard universes satisfying IP_κ and SMA_κ exist for arbitrarily large κ. In [R], Ross shows that SMA_κ implies IP_κ and asks whether the converse is true. He also obtains many interesting results under SMA_κ, which haven't been proved by IP_κ. He also asks whether or not these results can be proved by IP_κ.

Before answering D. Ross's questions, we would like to list some facts from [H] and [R] about IP_κ and SMA_κ. When we say that IP_κ (SMA_κ) implies a property P of nonstandard universes in this paper, we mean that for every nonstandard universe V if IP_κ (SMA_κ) is true in V, then P is true in V.

(1) $\forall\lambda < \kappa\,(IP_\kappa \Longrightarrow IP_\lambda$ and $SMA_\kappa \Longrightarrow SMA_\lambda)$.

(2) $IP_{\aleph_0} \Longrightarrow$ any two infinite internal sets have the same external cardinality. As all nonstandard universes considered in this paper will satisfy $IP_{\aleph_0}$, we denote by Ξ the common external cardinality of every infinite internal set in a nonstandard universe.

(3) $\forall\kappa\,(SMA_\kappa \Longrightarrow IP_\kappa)$.

(4) $\forall\kappa\,(IP_\kappa \Longrightarrow \kappa$–saturation$)$.

The proofs of these facts can be found in [R].

We expect that a copy of [R] is available to the reader. We refer to [CK] for the background in model theory and nonstandard universes. The author is very grateful to H. J. Keisler, A. W. Miller and C. W. Henson for valuable discussion and encouragement.

§1. IP_κ does not imply $SMA_{\aleph_0}$

In this section we show that Theorem 4.1, Corollary 4.2, Corollary 4.3, Corollary 4.4, Corollary 4.7 and Corollary 4.8 of [R] are not the consequences of IP_κ for any cardinal κ. Reaching that goal we construct several nonstandard universes which satisfy IP_κ but do not satisfy some consequences of $SMA_{\aleph_0}$ or $SMA_{\aleph_1}$ in [R]. Since the nonstandard universe of IP_κ in [H] built by the ultralimit construction also satisfies SMA_κ, we have to find another way of construction. Instead of the ultralimit construction we use just elementary extension and elementary chain arguments. Although these constructions take a little bit more time to make IP_κ true, they give us some kind of control of the saturation behavior of the nonstandard universes produced and hence, make some consequences of the special model axiom false.

First we show that IP_κ does not imply Corollary 4.2 of [R]. Recall that Ξ denotes the common external cardinality of every infinite internal set in a given nonstandard universe. Corollary 4.2 of [R] says that there is a sequence $\langle n_\alpha : \alpha < \Xi \rangle$ which is increasing and cofinal in $^*\mathbb{N}$. Obviously, this implies $cf(\Xi) = cf(^*\mathbb{N})$.

Theorem 1 *Let $\kappa > \beth_\omega$ be regular. There exists a nonstandard universe V such that V satisfies IP_κ, $\Xi = 2^\kappa$ and $cf(\mathbb{N}^{\mathbb{V}}) = \kappa$.*

Proof: Let $\overline{V}$ be the standard universe and $\mathcal{F}$ be a κ–regular ultrafilter over κ. Let V_0 be the nonstandard universe which is isomorphic to the truncation at rank ω of $\overline{V}^\kappa/\mathcal{F}$, the ultrapower of $\overline{V}$ over $\mathcal{F}$. (This is not necessary. It is only for simplicity.) V_0 has the properties that $|V_0| = 2^\kappa$ and for any infinite set A in $\overline{V}$, $|A^{V_0}| = 2^\kappa$. We now construct an elementary chain of nonstandard universes $\{V_\alpha : \alpha < \kappa\}$ recursively such that for every $\alpha < \kappa$:

(1) $|V_\alpha| = 2^\kappa$.

(2) for any two internally presented $\mathcal{L}$–structures $\mathcal{A}$ and $\mathcal{B}$ in V_α with $|\mathcal{L}| < \kappa$, if $\mathcal{A} \equiv \mathcal{B}$, then for any β, $\alpha < \beta < \kappa$, there is an isomorphism $i^\beta_{\langle \mathcal{A},\mathcal{B}\rangle} : \mathcal{A}^{V_\beta} \cong \mathcal{B}^{V_\beta}$. Besides, for any β, β', $\alpha < \beta < \beta' < \kappa$, $i^{\beta'}_{\langle \mathcal{A},\mathcal{B}\rangle}$ extends $i^\beta_{\langle \mathcal{A},\mathcal{B}\rangle}$, i.e. for every $a \in A^{V_\beta}$ $i^{\beta'}_{\langle \mathcal{A},\mathcal{B}\rangle} \circ j(a) = j \circ i^\beta_{\langle \mathcal{A},\mathcal{B}\rangle}(a)$, where $j : V_\beta \to V_{\beta'}$ is the natural embedding.

(3) $\exists n \in \mathbb{N}^{\mathbb{V}_{\alpha+\nVdash}}\ \forall \gtrdot \in \mathbb{N}^{\mathbb{V}_\alpha}\ (\gtrdot < \ltimes)$.

If we let V be the union of that chain, then $\Xi = 2^\kappa$ follows from (1), IP_κ follows from (2) and $cf(\mathbb{N}^{\mathbb{V}}) = \kappa$ follows from (3).

Suppose V_β have been found for every $\beta < \alpha < \kappa$.

Case 1: α is a limit ordinal.

Let $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$. If $\mathcal{A}$ and $\mathcal{B}$ are two internally presented $\mathcal{L}$–structures in V_{β_0} for some $\beta_0 < \alpha$ and $\mathcal{A} \equiv \mathcal{B}$, let $i^\alpha_{\langle \mathcal{A},\mathcal{B}\rangle} = \bigcup_{\beta_0<\beta<\alpha} i^\beta_{\langle \mathcal{A},\mathcal{B}\rangle}$, where $\bigcup$ means direct limit.

(1), (2) and (3) are trivially satisfied up to the α–th stage.

Case 2: $\alpha = \beta + 1$.

First we add a new element at the end of $\mathbb{N}_\beta$. Let

$$, (x) = \{x > n : n \in \mathbb{N}^{\mathbb{V}_\beta}\},$$

which is a consistent type of $\langle V_\beta; \in, (n)_{n\in\mathbb{N}^{\mathbb{V}_\beta}} \rangle$. Since $|\mathbb{N}^{\mathbb{V}_\beta}| = \nvDash^\kappa$ and by the downward Löwenheim–Skolem–Tarski theorem, there is a nonstandard universe V'_0 such that $V_\beta \lhd V'_0$, $|V'_0| = 2^\kappa$ and V'_0 realizes the type , (x).

Let $\mathcal{M} = \{\langle \mathcal{A}, \mathcal{B} \rangle : \mathcal{A}, \mathcal{B}$ are two internally presented $\mathcal{L}$–structures in V_β with $|\mathcal{L}| < \kappa$ and $\mathcal{A} \equiv \mathcal{B}\}$. $|\mathcal{M}| \leq (2^\kappa)^{<\kappa}(2^\kappa)^{<\kappa} = 2^\kappa$. We now construct an elementary chain of nonstandard universes $\{V'_n : n \in \omega\}$ starting from V'_0 such that:

(i) $\forall n \in \omega\ (|V'_n| = 2^\kappa)$.

(ii) For each $\langle \mathcal{A}, \mathcal{B} \rangle \in \mathcal{M}$, there are $f^{(2n)}_{\langle \mathcal{A}, \mathcal{B} \rangle}$ and $g^{(2n+1)}_{\langle \mathcal{A}, \mathcal{B} \rangle}$ for $n = 0, 1, \ldots$ such that $f^{(2n)}_{\langle \mathcal{A}, \mathcal{B} \rangle} : \mathcal{A}^{V'_{2n}} \prec \mathcal{B}^{V'_{2n+1}}$ and $g^{(2n+1)}_{\langle \mathcal{A}, \mathcal{B} \rangle} : \mathcal{B}^{V'_{2n+1}} \prec \mathcal{A}^{V'_{2n+2}}$, $(g^{(2n+1)}_{\langle \mathcal{A}, \mathcal{B} \rangle})^{-1}$ extends $f^{(2n)}_{\langle \mathcal{A}, \mathcal{B} \rangle}$ and $f^{(2n+2)}_{\langle \mathcal{A}, \mathcal{B} \rangle}$ extends $(g^{(2n+1)}_{\langle \mathcal{A}, \mathcal{B} \rangle})^{-1}$.

(iii) If there is already an $i^\beta_{\langle \mathcal{A}, \mathcal{B} \rangle} : \mathcal{A}^{V_\beta} \cong \mathcal{B}^{V_\beta}$, we require $f^{(0)}_{\langle \mathcal{A}, \mathcal{B} \rangle}$ extends $i^\beta_{\langle \mathcal{A}, \mathcal{B} \rangle}$.

Suppose we have V'_m for every $m \leq n$ which meets the requirements. Without loss of generality we can assume that n is an even number. For each $\langle \mathcal{A}, \mathcal{B} \rangle \in \mathcal{M}$, let $,_{\langle \mathcal{A}, \mathcal{B} \rangle}(x_{\langle \mathcal{A}, \mathcal{B} \rangle})$ be the type of $\langle V'_n; \in, (v)_{v \in V'_n} \rangle$, that is

> $\{x_{\langle \mathcal{A}, \mathcal{B} \rangle}$ is a function from $A^{V'_n}$ to $B^{V'_n}$ $\wedge$ "$\mathcal{A}^{V'_n} \models \sigma(a_1, \ldots, a_k) \Longleftrightarrow \mathcal{B}^{V'_n} \models \sigma(x_{\langle \mathcal{A}, \mathcal{B} \rangle}(a_1), \ldots, x_{\langle \mathcal{A}, \mathcal{B} \rangle}(a_k))$" $\wedge$ $x_{\langle \mathcal{A}, \mathcal{B} \rangle}(c) = b$: $\sigma(v_1, \ldots, v_k)$ is a formula of $\mathcal{L}$ which is the language of $\mathcal{A}$ and $\mathcal{B}$, $a_1, \ldots, a_k \in A^{V'_n}$, $c \in \text{range}(g^{(n-1)}_{\langle \mathcal{A}, \mathcal{B} \rangle})$ and $c = g^{(n-1)}_{\langle \mathcal{A}, \mathcal{B} \rangle}(b)\}$.

$,_{\langle \mathcal{A}, \mathcal{B} \rangle}(x_{\langle \mathcal{A}, \mathcal{B} \rangle})$ is a consistent type. By the downward Löwenheim–Skolem–Tarski theorem, there is a V'_{n+1} such that $V'_n \lhd V'_{n+1}$, $|V'_{n+1}| = 2^\kappa$ and V'_{n+1} realizes every type $,_{\langle \mathcal{A}, \mathcal{B} \rangle}(x_{\langle \mathcal{A}, \mathcal{B} \rangle})$ since $|\mathcal{M}| = 2^\kappa$.

Let $f'_{\langle \mathcal{A}, \mathcal{B} \rangle}$ be the witness of $,_{\langle \mathcal{A}, \mathcal{B} \rangle}(x_{\langle \mathcal{A}, \mathcal{B} \rangle})$ in V'_{n+1} and $f^{(n)}_{\langle \mathcal{A}, \mathcal{B} \rangle} = f'_{\langle \mathcal{A}, \mathcal{B} \rangle} | A^{V'_n}$. If n is an odd number, everything is same except switching $\mathcal{A}$ and $\mathcal{B}$ and witching f and g. For each $\langle \mathcal{A}, \mathcal{B} \rangle \in \mathcal{M}$, let $i^\alpha_{\langle \mathcal{A}, \mathcal{B} \rangle} = \bigcup_{n \in \omega} f^{(2n)}_{\langle \mathcal{A}, \mathcal{B} \rangle}$. Let $V_\alpha = \bigcup_{n \in \omega} V'_n$.

Obviously (1), (2) and (3) are true up to the α–th stage and we finish the construction

Let $V = \bigcup_{\alpha < \kappa} V_\alpha$. V satisfies IP_κ because for each pair of internally presented $\mathcal{L}$–structures $\mathcal{A}$ and $\mathcal{B}$ in V with $|\mathcal{L}| < \kappa$ there exists an $\alpha_0 < \kappa$ such that $\mathcal{A}$ and $\mathcal{B}$ are already in V_{α_0}. If $\mathcal{A}^V \equiv \mathcal{B}^V$, then $\mathcal{A}^{V_{\alpha_0}} \equiv \mathcal{B}^{V_{\alpha_0}}$. By the construction $i = \bigcup_{\alpha_0 < \alpha < \kappa} i^\alpha_{\langle \mathcal{A}, \mathcal{B} \rangle}$ is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. Since we have built an increasing κ–sequence which is cofinal in $\mathbb{N}^{\mathbb{V}}$, we have $cf(\mathbb{N}^{\mathbb{V}}) = \kappa$. Obviously $\Xi = 2^\kappa$. $\Box$

Remarks: (1) IP_κ does not imply Corollary 4.2 of [R].

(2) IP_κ does not imply Theorem 4.1 of [R] because Theorem 4.1 of [R] implies Corollary 4.2 of [R].

(3) IP_κ does not imply Corollary 4.3 of [R] because the coinitiality of all positive infinitesimals equals $cf({}^*\mathbb{N})$.

Theorem 2 *Let $\kappa > \beth_\omega$ be regular. There exists a nonstandard universe V such that V satisfies IP_κ, $\Xi = 2^\kappa$ and $ci(\mathbb{N}^{\mathbb{V}}) = \kappa$.*

Proof: Almost same as the proof of Theorem 1 except that at the $\alpha + 1$–th stage we add a new infinite integer which is below every old infinite integers instead of adding one above every old integers. $\square$

Remarks: (1) IP_κ does not imply Corollary 4.4 of [R], which is a result under $SMA_{\aleph_1}$, because the cofinality of all positive infinitesimals equals $ci({}^*\mathbb{N})$.

(2) As we will show that SMA_κ is equivalent to $SMA_{\aleph_0}$ plus κ–saturation in §3 and IP_κ implies κ–saturation, $SMA_{\aleph_0}$ is also false in the nonstandard universe built in Theorem 2.

In the nonstandard universes constructed in Theorem 1 and Theorem 2, $\Xi = 2^\kappa$. Since $cf(2^\kappa) > \kappa$, we need only build a chain of length κ to satisfy IP_κ and destroy $SMA_{\aleph_0}$. If we want to lift IP_κ to IP_{κ^+} and still keep $\Xi = 2^\kappa$, then we might need to build a chain of length κ^+. But in this case $cf(2^\kappa)$ may equal κ^+ and cause some trouble in destroying $SMA_{\aleph_0}$. For example, if $2^\kappa = \kappa^+$, then every internally presented $\mathcal{L}$–structure with $|\mathcal{L}| < \kappa^+$ is saturated because IP_{κ^+} implies κ^+–saturation and $\Xi = \kappa^+$. So SMA_{κ^+} is true. Next we show what will happen when $2^\kappa > \kappa^+$.

Theorem 3 *Let $\kappa > \beth_\omega$ be regular and $2^\kappa > \kappa^+$. There exists a nonstandard universe V such that V satisfies IP_{κ^+}, $\Xi = 2^\kappa$ and $SMA_{\aleph_0}$ is false in V.*

Proof: Case 1: $cf(2^\kappa) > \kappa^+$.

We use the same method as in the proof of theorem 1 to construct an elementary chain of nonstandard universes with length κ^+. Then V, the union of the chain, is a desired nonstandard universe.

Case 2: $cf(2^\kappa) = \kappa^+$.

We take the same V as in Case 1. V satisfies IP_{κ^+}. We want to show that $\mathbb{R}^{\mathbb{V}}$ with the usual order is not special.

Suppose that $\mathbb{R}^{\mathbb{V}}$ has a specializing chain $\{\mathbb{R}_\alpha : \alpha < \not\models^\kappa\}$. Since $\kappa^+ < 2^\kappa$, $\mathbb{R}_{\kappa^+}$ is an element of the chain. Since $\mathbb{R}_{\kappa^+}$ is a dense linear order and is κ^{++}–saturated, then $|\mathbb{R}_{\kappa^+}| \geq \not\models^{\kappa^+} = (\not\models^\kappa)^{\kappa^+} = (\not\models^\kappa)^{\mho(\not\models^\kappa)} > \not\models^\kappa$. That contradicts $|\mathbb{R}^{\mathbb{V}}| = \not< = \not\models^\kappa$. $\square$

Next we show that IP_κ does not imply Corollary 4.7 of [R], which says that the set of all infinitesimals can be cofinally embedded into ${}^*\mathbb{N}$. Since we will prove Theorem 4.5 of [R] by $IP_{\aleph_0}$ in §2, and Corollary 4.7 of [R] is a corollary of Theorem 4.5 of [R] in some sense, it seems promising that Corollary 4.7 of [R] is a consequence of the isomorphism property. But Corollary 4.7 of [R] is a result under $SMA_{\aleph_1}$ while Theorem 4.5 of [R] is a result under only $SMA_{\aleph_0}$. The relation between them is not so simple. In the other hand, if Corollary 4.7 of [R] is true, then $cf({}^*\mathbb{N}) = \beth({}^*\mathbb{N})$. But we found that this is not a consequence of IP_κ.

Theorem 4 *For any κ there exists a nonstandard universe V such that V satisfies IP_{κ^+} and $cf(\mathbb{N}^{\mathbb{V}}) \neq \beth(\mathbb{N}^{\mathbb{V}})$.*

Proof: Without loss of generality we assume $\kappa > \beth_\omega$. First we pick a cardinal λ such that $\lambda^\kappa = \lambda$. For example, let $\lambda = 2^\kappa$. We will construct an elementary chain of nonstandard universes $\{V_\alpha : \alpha < \lambda\}$ recursively such that:

(1) $\forall \alpha < \lambda\, (|V_\alpha| = \lambda^+)$.

(2) $cf(\mathbb{N}^{\mathbb{V}_{\nvdash}}) = \lambda^+$ and $\forall \alpha < \lambda$, $\mathbb{N}^{\mathbb{V}_{\nvdash}}$ is cofinal in $\mathbb{N}^{\mathbb{V}_\alpha}$.

(3) $\forall \alpha < \lambda$, $\exists a \in \mathbb{N}^{\mathbb{V}_{\alpha+\nVdash}} - \mathbb{N}$, $\forall \in \mathbb{N}^{\mathbb{V}_\alpha} - \mathbb{N}$, $(\text{Ↄ} <)$.

(4) For any two internally presented $\mathcal{L}$–structures $\mathcal{A}$ and $\mathcal{B}$ in V_α with $|\mathcal{L}| < \kappa^+$, if $\mathcal{A}^{V_\alpha} \equiv \mathcal{B}^{V_\alpha}$, then for any β and β' such that $\alpha < \beta < \beta' < \lambda$, there is an isomorphism $i^\beta_{\langle \mathcal{A},\mathcal{B}\rangle} : \mathcal{A}^{V_\beta} \cong \mathcal{B}^{V_\beta}$ and $i^{\beta'}_{\langle \mathcal{A},\mathcal{B}\rangle}$ extends $i^\beta_{\langle \mathcal{A},\mathcal{B}\rangle}$.

The theorem follows because by (2) and (3) we have $cf(\mathbb{N}^{\mathbb{V}}) = \lambda^+$ and $ci(\mathbb{N}^{\mathbb{V}}) = \lambda$, where $V = \bigcup_{\alpha<\lambda} V_\alpha$, and by (4) and $cf(\lambda) \geq \kappa^+$, V satisfies IP_{κ^+}.

Let V_0 be a nonstandard universe such that $cf(\mathbb{N}^{\mathbb{V}_{\nvdash}}) = \lambda^+$ and $|V_0| = \lambda^+$. (We can do this by adding λ^+ many new constants consecutively to the end of ${}^*\mathbb{N}$ and controlling the cardinalities by the downward Löwenheim–Skolem–Tarski theorem.)

Suppose that we have found V_β for every $\beta < \alpha < \lambda$.

Case 1: α is a limit ordinal.

Let $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$. If $\mathcal{A}$ and $\mathcal{B}$ are two internally presented $\mathcal{L}$–structures in V_{β_0} for some $\beta_0 < \alpha$ with $|\mathcal{L}| < \kappa^+$ and $\mathcal{A} \equiv \mathcal{B}$, then let $i^\alpha_{\langle \mathcal{A},\mathcal{B}\rangle} = \bigcup_{\beta_0<\beta<\alpha} i^\beta_{\langle \mathcal{A},\mathcal{B}\rangle}$.

Case 2: $\alpha = \beta + 1$ for some $\beta < \lambda$. In order to avoid nested recursive construction which might cause confusion, we break the proof of Case 2 into a lemma.

Lemma *There exists a nonstandard universe V_α, ($\alpha = \beta+1$), which is an elementary extension of V_β such that V_α satisfies conditions (1)—(4) in the proof of Theorem 4.*

Proof of Lemma: Let

> $\mathcal{M} = \{\langle \mathcal{A}, \mathcal{B} \rangle : \mathcal{A}, \mathcal{B}$ are two internally presented $\mathcal{L}$–structures in V_β for some $\mathcal{L}$ with $|\mathcal{L}| < \kappa^+$ and $\mathcal{A} \equiv \mathcal{B}\}$.

$$|\mathcal{M}| \leq (\lambda^+)^\kappa (\lambda^+)^\kappa = \lambda^+.$$

Let

> $\mathcal{M}_1 = \{\langle \mathcal{A}, \mathcal{B} \rangle \in \mathcal{M}$: there is a $\gamma < \beta$ such that $\mathcal{A}$ and $\mathcal{B}$ are already internally presented in $V_\gamma\}$,

and

$$\mathcal{M}_2 = \mathcal{M} - \mathcal{M}_1.$$

Note that for any $\langle \mathcal{A}, \mathcal{B} \rangle \in \mathcal{M}_1$, we have already an isomorphism $i^\beta_{\langle \mathcal{A}, \mathcal{B} \rangle} : \mathcal{A} \cong \mathcal{B}$.

We will now construct an elementary chain of nonstandard universes $\{V'_\gamma : \gamma < \lambda^+\}$ and let $V_\alpha = \bigcup_{\gamma<\lambda^+} V'_\gamma$ such that:

(a) $V'_0 = V_\beta$.

(b) $\forall \gamma < \lambda^+ \ (|V'_\gamma| = \lambda^+)$.

(c) $\mathbb{N}^{\mathbb{V}_{\nvdash}}$ is cofinal in $\mathbb{N}^{\mathbb{V}'_\gamma}$ for every $\gamma < \lambda^+$.

(d) $\forall \gamma < \lambda^+$, $\exists a \in \mathbb{N}^{\mathbb{V}'_{\gamma+\nVdash}} - \mathbb{N}$, $\forall \in \mathbb{N}^{\mathbb{V}'_\gamma} - \mathbb{N}$, $(\eth <)$.

(e) For each $\langle \mathcal{A}, \mathcal{B} \rangle \in \mathcal{M}_1$, there is an isomorphism $j^\gamma_{\langle \mathcal{A}, \mathcal{B} \rangle} : \mathcal{A}^{V'_\gamma} \cong \mathcal{B}^{V'_\gamma}$ for every $\gamma < \lambda^+$ such that

> $\forall \gamma < \gamma' < \lambda^+$ ($j^{\gamma'}_{\langle \mathcal{A}, \mathcal{B} \rangle}$ extends $j^\gamma_{\langle \mathcal{A}, \mathcal{B} \rangle}$ and both $j^\gamma_{\langle \mathcal{A}, \mathcal{B} \rangle}$ and $j^{\gamma'}_{\langle \mathcal{A}, \mathcal{B} \rangle}$ extend $i^\beta_{\langle \mathcal{A}, \mathcal{B} \rangle}$).

(f) For each $\langle \mathcal{A}, \mathcal{B} \rangle \in \mathcal{M}_2$ there is an isomorphism $k_{\langle \mathcal{A}, \mathcal{B} \rangle} : \mathcal{A}^{V_\alpha} \cong \mathcal{B}^{V_\alpha}$ such that

$$k_{\langle \mathcal{A}, \mathcal{B} \rangle} = \bigcup_{\gamma<\lambda^+} f^\gamma_{\langle \mathcal{A}, \mathcal{B} \rangle},$$

where $f^\gamma_{\langle \mathcal{A}, \mathcal{B} \rangle}$ is a partial function from $A^{V'_\gamma}$ to $B^{V'_\gamma}$ such that $|f^\gamma_{\langle \mathcal{A}, \mathcal{B} \rangle}| \leq \lambda$ and $f^{\gamma'}_{\langle \mathcal{A}, \mathcal{B} \rangle}$ extends $f^\gamma_{\langle \mathcal{A}, \mathcal{B} \rangle}$ for all $\gamma < \gamma' < \lambda^+$. (This will be shown later by a standard book–keeping argument.)

The lemma follows from the construction because by (b), condition (1) is true; by (c), condition (2) is true; by (d), condition (3) is true; by (e) and (f), condition (4) is true.

The reader will find that (d) follows directly from the construction although it seems too much for condition (3).

Fix a bijection $\varphi : \lambda^+ \times \lambda^+ \longmapsto \lambda^+$ such that $\xi \leq \varphi(\xi, \eta)$, which will be used for book–keeping.

Suppose that we have V'_δ for every $\delta < \gamma < \lambda^+$ and for each $\langle \mathcal{A}, \mathcal{B} \rangle \in \mathcal{M}_2$, we have two enumerations of $A^{V'_\delta} = \{a^{\delta,\eta}_{\langle \mathcal{A},\mathcal{B}\rangle} : \eta < \lambda^+\}$ and $B^{V'_\delta} = \{b^{\delta,\eta}_{\langle \mathcal{A},\mathcal{B}\rangle} : \eta < \lambda^+\}$ for every successor ordinal $\delta < \gamma < \lambda^+$.

Case 1: γ is a limit ordinal.

Let $V'_\gamma = \bigcup_{\delta<\gamma} V'_\delta$. For each $\langle \mathcal{A}, \mathcal{B} \rangle \in \mathcal{M}_1$, let $j^\gamma_{\langle \mathcal{A},\mathcal{B}\rangle} = \bigcup_{\delta<\gamma} j^\delta_{\langle \mathcal{A},\mathcal{B}\rangle}$. For each $\langle \mathcal{A}, \mathcal{B} \rangle \in \mathcal{M}_2$, let $f^\gamma_{\langle \mathcal{A},\mathcal{B}\rangle} = \bigcup_{\delta<\gamma} f^\delta_{\langle \mathcal{A},\mathcal{B}\rangle}$.

Case 2: $\gamma = \delta + 1$ for some $\delta < \lambda^+$.

Let $\mathcal{F}$ be a λ–regular ultrafilter over λ. Let $\overline{V}'_\gamma$ be the nonstandard universe which is isomorphic to the truncation at rank ω of $(V'_\delta)^\lambda/\mathcal{F}$. We let $V'_\delta \lhd \overline{V}'_\gamma$ in a natural way. For each $\langle \mathcal{A}, \mathcal{B} \rangle \in \mathcal{M}_1$, since we have $j^\delta_{\langle \mathcal{A},\mathcal{B}\rangle} : \mathcal{A}^{V'_\delta} \cong \mathcal{B}^{V'_\delta}$, there is an extension $j_{\langle \mathcal{A},\mathcal{B}\rangle} : \mathcal{A}^{\overline{V}'_\gamma} \cong \mathcal{B}^{\overline{V}'_\gamma}$ because both models are the copies of the same ultrapower of two isomorphic models.

For each $\langle \mathcal{A}, \mathcal{B} \rangle \in \mathcal{M}_2$, let $\langle \xi, \eta \rangle \in \lambda^+ \times \lambda^+$ such that $\varphi(\xi, \eta) = \delta$. Let , $_{\langle \mathcal{A},\mathcal{B}\rangle}(g)$ be the type of

$$\langle V'_\delta; (a)_{a \in \mathrm{domain}(f^\delta_{\langle \mathcal{A},\mathcal{B}\rangle})}, (b)_{b \in \mathrm{range}(f^\delta_{\langle \mathcal{A},\mathcal{B}\rangle})}, a^{\xi,\eta}_{\langle \mathcal{A},\mathcal{B}\rangle}, b^{\xi,\eta}_{\langle \mathcal{A},\mathcal{B}\rangle} \rangle,$$

which is

$$\begin{aligned}&\{g \text{ is an internal function from } A^{V'_\delta} \text{ to } B^{V'_\delta} \wedge \text{“}\mathcal{A}^{V'_\delta} \models \sigma(a_1, \ldots, a_k) \Longleftrightarrow \\ &\mathcal{B}^{V'_\delta} \models \sigma(g(a_1), \ldots, g(a_k))\text{”} \wedge g(b) = f^\delta_{\langle \mathcal{A},\mathcal{B}\rangle}(b) : a_1, \ldots, a_k \in \mathrm{domain}(f^\delta_{\langle \mathcal{A},\mathcal{B}\rangle}) \bigcup \{a^{\xi,\eta}_{\langle \mathcal{A},\mathcal{B}\rangle}\} \\ &\text{or } g(a_1), \ldots, g(a_k) \in \mathrm{range}(f^\delta_{\langle \mathcal{A},\mathcal{B}\rangle}) \bigcup \{b^{\xi,\eta}_{\langle \mathcal{A},\mathcal{B}\rangle}\},\ b \in \mathrm{domain}(f^\delta_{\langle \mathcal{A},\mathcal{B}\rangle}) \text{ and } \sigma \text{ is} \\ &\text{a formula of } \mathcal{L} \text{ which is the language of } \mathcal{A} \text{ and } \mathcal{B}\}.\end{aligned}$$

, $_{\langle \mathcal{A},\mathcal{B}\rangle}(g)$ is finitely satisfiable and $|, _{\langle \mathcal{A},\mathcal{B}\rangle}(g)| \leq \lambda$. Since $\overline{V}'_\gamma$ is a λ–regular ultrapower of V'_δ, there exists a $g_{\langle \mathcal{A},\mathcal{B}\rangle} \in \overline{V}'_\gamma$ which witnesses the type , $_{\langle \mathcal{A},\mathcal{B}\rangle}(g)$

Since $\overline{V}'_\gamma$ is an ultrapower over $\mathcal{F}$, $\mathbb{N}^{\mathbb{V}_{\nvdash}}$ is still cofinal in $\mathbb{N}^{\overline{\mathbb{V}}'_\gamma}$ and there exists an $a \in \mathbb{N}^{\overline{\mathbb{V}}'_\gamma} - \mathbb{N}$ which is a lower bound of $\mathbb{N}^{\mathbb{V}'_\delta} - \mathbb{N}$.

Now let V'_γ be a nonstandard universe such that $V'_\delta \lhd V'_\gamma \lhd \overline{V}'_\gamma$, $a \in V'_\gamma$, $|V'_\gamma| = \lambda^+$, $g_{\langle \mathcal{A},\mathcal{B}\rangle} \in V'_\gamma$ for each $\langle \mathcal{A},\mathcal{B}\rangle \in \mathcal{M}_2$ and V'_γ is closed under $j_{\langle \mathcal{A},\mathcal{B}\rangle}$ for each $\langle \mathcal{A},\mathcal{B}\rangle \in \mathcal{M}_1$. V'_γ exists by $|\mathcal{M}| \leq \lambda^+$ and the downward Löwenheim–Skolem–Tarski theorem. Let

$$f^\gamma_{\langle \mathcal{A},\mathcal{B}\rangle} = f^\delta_{\langle \mathcal{A},\mathcal{B}\rangle} \bigcup \{\langle a^{\xi,\eta}_{\langle \mathcal{A},\mathcal{B}\rangle}, g_{\langle \mathcal{A},\mathcal{B}\rangle}(a^{\xi,\eta}_{\langle \mathcal{A},\mathcal{B}\rangle})\rangle, \langle g^{-1}_{\langle \mathcal{A},\mathcal{B}\rangle}(b^{\xi,\eta}_{\langle \mathcal{A},\mathcal{B}\rangle}), b^{\xi,\eta}_{\langle \mathcal{A},\mathcal{B}\rangle}\rangle\}$$

and let

$$j^\gamma_{\langle \mathcal{A},\mathcal{B}\rangle} = j_{\langle \mathcal{A},\mathcal{B}\rangle} | A^{V'_\gamma}.$$

We now finish the construction and $V_\alpha = \bigcup_{\gamma < \lambda^+} V'_\gamma$ is what we want. □

Remark: (1) IP_κ does not imply Corollary 4.8 of [R] by the same reason as above.

(2) In the nonstandard universe constructed in Theorem 4, $SMA_{\aleph_0}$ is also false although Corollary 4.7 and Corollary 4.8 of [R] are the results under $SMA_{\aleph_1}$.

§2. Power of the isomorphism property

As we discovered the limitation of the power of IP_κ in §1, we also found that many results of [R] under $SMA_{\aleph_0}$ can be proved by $IP_{\aleph_0}$. In this section we show that except Theorem 5.5 of [R] and all the results of [R] involved in §1 of this paper, $IP_{\aleph_0}$ does imply all of the remaining consequences of $SMA_{\aleph_0}$ in [R]. Before giving the proofs we should mention two things. First, Corollary 5.3 of [R] is trivially true in every nonstandard universe because we can let the set S be all positive infinitesimals, which is external but for every positive infinitesimal ι, $S \bigcap [0, \iota)$ is internal. Second, Theorem 5.4 of [R] is true in every nonstandard universe in which every infinite internal set has the same external cardinality, which is a consequence of $IP_{\aleph_0}$. This can be shown by a simple diagonal argument.

Next we prove Theorem 4.5 of [R] by $IP_{\aleph 0}$.

Theorem 5 *($IP_{\aleph_0}$) Suppose that P and Q are two internal linear order without endpoints. There is an order–preserving map $f : P \longmapsto Q$ such that $f[P]$ is cofinal in Q.*

Proof: Let P_0 and Q_0 be two countable elementary submodels of P and Q respectively. Both P_0 and Q_0 have no endpoints. Let Q_1 be a countable elementary extension of Q_0 such that Q_1 contains a subset Q_2 which is isomorphic to the order of all rational numbers and is cofinal in Q_1. Q_1 can be built by following way.

Let $\mathbb{Q}$ be the set of all rational numbers and $Q^0 = Q_0$. By induction on n we construct an ω–elementary chain $\langle Q^n : n \in \omega\rangle$ such that Q^{n+1} is a countable elementary extension of Q^n which satisfies all formulas in

$$\{c_p < c_q : p, q \in \mathbb{Q}\bigcap[n, n+1)\}$$

and

$$\{a < c_p : a \in Q^n,\ p \in \mathbb{Q}\bigcap[n, n+1)\}$$

where c_p, c_q for some $p, q \in \mathbb{Q}\bigcap[n, n+1)$ are new constants which are not in Q^n. Then let

$$Q_1 = \bigcup_{n\in\omega} Q^n$$

and

$$Q_2 = \{c_p : p \in \mathbb{Q}\bigcap(0, +\infty)\}.$$

Note that ω–many steps are needed because $\{c_p : p \in \mathbb{Q}\bigcap[n, n+1)\}$ might not be cofinal in Q^{n+1}. We can also build Q_1 by one step with a simple type–omitting argument.

We can now find a cofinal embedding g from P_0 to Q_2, hence, g cofinally embeds P_0 to Q_1, because P_0 is countable and the order of rational numbers is universal for countable orders. Let us consider P_0, Q_1 and g as standard objects in the standard universe. and let ${}^*P_0$, ${}^*Q_1$ and ${}^*g$ be their internal versions. Now ${}^*g[{}^*P_0]$ is cofinal in ${}^*Q_1$ by transfer principle. Since $P \equiv P_0 \equiv {}^*P_0$ and $Q \equiv Q_0 \equiv Q_1 \equiv {}^*Q_1$, there exist isomorphisms $i : P \cong {}^*P_0$ and $j : {}^*Q_1 \cong Q$ by $IP_{\aleph_0}$. $f = j \circ {}^*g \circ i$ is a desired map which maps P cofinally into Q. $\Box$

Remark: Corollary 4.6 of [R] is also a consequence of $IP_{\aleph_0}$.

We need more notations before we go further. In a nonstandard universe an internal linear order $\langle W, <_W\rangle$ is called *well ordered if every non–empty internal subset of W has a $<_W$–least element. We denote W_x for the initial segment $\{w \in W : w <_W x\}$ for every $x \in W$. We call a *well ordered set a regular *cardinal set if for every proper internal initial segment I of W, there are no internal functions f from I to W such that $f[I]$ is unbounded in W.

Theorem 6 ($IP_{\aleph_0}$) *For every regular *cardinal set $\langle W, <_W\rangle$, there exists an external set $S \subseteq W$ such that for every $x \in W$, $S \bigcap W_x$ is internal.*

Proof: Let ${}^*\mathbb{Z}$ be the set of all integers (both standard and nonstandard) with the natural order. Let

$$K = \{f : f \text{ is an internal function from } W \text{ to } {}^*\mathbb{Z}\}$$

with a lexicographic order $<_K$. We now form an internally presented model $\mathcal{A}$ and an internally presented submodel $\mathcal{B}$ of $\mathcal{A}$. Let $A = K \bigcup W$ be the base set of $\mathcal{A}$ and

$$\mathcal{A} = \langle A; K, <_K, <_W, F\rangle$$

where K is a unary relation, $<_K$ and $<_W$ are two binary relations and F is a function from $K \times K$ to W such that

$$F(f,g) = \begin{cases} \min\{x \in W : f(x) \neq g(x)\} & \text{if } f \neq g \\ \min W & \text{if } f = g \end{cases}$$

F gives an "analogue" of a distance function. Let $B = L \bigcup W$, where

$$L = \{f \in K : f \text{ is eventually zero } \}$$

and $\mathcal{B}$ is the submodel of $\mathcal{A}$ with the base set B. We now want to prove that $\mathcal{B}$ is an elementary submodel of $\mathcal{A}$. Before that we want to define an isomorphism $T_{h,w}$ from $\mathcal{A}$ to $\mathcal{A}$ for every $h \in K$ and $w \in W$. Let

(1) $T_{h,w}$ is an identity on W;

(2) if $f \in K$ and $F(f,h) < w$, then $T_{h,w}(f) = f$;

(3) if $f \in K$ and $F(f,h) \geq w$, then

$$T_{h,w}(x) = \begin{cases} f(x) & \text{if } x < w \\ f(x) + h(x) & \text{if } x \geq w \end{cases}$$

It is easy to show that $T_{h,w} : \mathcal{A} \to \mathcal{A}$ is an isomorphism.

Claim 1: $\mathcal{B}$ is an elementary submodel of $\mathcal{A}$.

Proof of Claim 1: It suffices to show that for every formula $\exists x\phi(x, x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in B$, if $\mathcal{A} \models \exists x\phi(x, a_1, \ldots, a_n)$, then there exists a $b \in B$ such that $\mathcal{A} \models \phi(b, a_1, \ldots, a_n)$.

Suppose that $a \in A - B$ such that $\mathcal{A} \models \phi(a, a_1, \ldots, a_n)$. a must be in $K - L$. Let

$$w > \max\{F(a, a_i) : 1 \leq i \leq n \wedge a_i \in L\}$$

and

$$h(x) = \begin{cases} a(x) & \text{if } x < w \\ -a(x) & \text{if } x \geq w \end{cases}$$

and let $T_{h,w} : \mathcal{A} \to \mathcal{A}$ be the isomorphism. Then

$$\mathcal{A} \models \phi(T_{h,w}(a), T_{h,w}(a_1), \ldots, T_{h,w}(a_n)).$$

Since $T_{h,w}(a_i) = a_i$ for $1 \leq i \leq n$ and $T_{h,w}(a) \in B$, there exists a $b = T_{h,w}(a) \in B$ such that

$$\mathcal{A} \models \phi(b, a_1, \ldots, a_n).$$

By applying $IP_{\aleph_0}$ we have $\mathcal{B} \cong \mathcal{A}$. Let $j : \mathcal{B} \to \mathcal{A}$ be the isomorphism.

Let $f_0 \equiv 1$, a constant function, which is in $K - L$ and let $C = \{f \in L : f <_K f_0\}$. C is an initial segment of L with no least upper bound in L. Since j is an isomorphism, $j[C]$ is also an initial segment of K with no least upper bound in K. Let $\lambda = cf(W)$ and $\langle x_\alpha : \alpha < \lambda \rangle$ be a strictly increasing sequence which is cofinal in W. Define $f_\alpha \in L$ such that

$$f_\alpha(x) = \begin{cases} 1 & \text{if } x < x_\alpha \\ 0 & \text{if } x \geq x_\alpha \end{cases}$$

Then $\langle f_\alpha : \alpha < \lambda \rangle$ is a strictly increasing sequence in L which is cofinal in C. Hence $\langle j(f_\alpha) : \alpha < \lambda \rangle$ is cofinal in $j[C]$. Let $g_\alpha = j(f_\alpha)$. Then $F(g_\alpha, g_\beta) = j(x_\alpha)$ for every $\alpha < \beta < \lambda$. So $\langle F(g_\alpha, g_{\alpha+1}) : \alpha < \lambda \rangle$ is cofinal in W

We define a function $H : W \to {}^*\mathbb{Z}$ such that

$$H(x) = g_\alpha(x) \text{ when } x < j(x_\alpha).$$

Claim 2: H is well defined.

Proof of Claim 2: Suppose that is not true. Let $x < j(x_\alpha)$ and $g_\alpha(x) \neq g_\beta(x)$ for some $\alpha < \beta < \lambda$. Since

$$F(g_\alpha, g_\beta) = j(F(f_\alpha, f_\beta)) = j(x_\alpha),$$

$g_\alpha | W_{j(x_\alpha)} = g_\beta | W_{j(x_\alpha)}$ but $x \in W_{j(x_\alpha)}$, a contradiction.

H is external because otherwise H would be the least upper bound of $j[C]$ in K. For each $x \in W$ there exists an $\alpha < \lambda$ such that $x < j(x_\alpha)$, hence $H|W_x = g_\alpha|W_x$ which is internal. Now we have constructed a function $H : W \to {}^*\mathbb{Z}$, which is external but the restriction of which to every initial segment W_x is internal. We next translate the graph of H into an internal subset of W, which is the set we want.

By the transfer principle there exists an internal bijection $I : W \times {}^*\mathbb{Z} \to W$. Let G_H be the graph of H. Since H is external, then G_H is external, hence $I[G_H]$ is external.

Claim 3: For every $x \in W$, $I[G_H] \bigcap W_x$ is internal.

Proof of Claim 3: Since W is a regular *cardinal set, there exists an $x' \in W$ such that $I^{-1}[W_x] \subseteq W_{x'} \times {}^*\mathbb{Z}$. Since $G_H \bigcap (W_{x'} \times {}^*\mathbb{Z})$ is the graph of the internal function $H|W_{x'}$, it is internal too. So

$$I[G_H] \bigcap W_x = I[G_H] \bigcap I[W_{x'} \times {}^*\mathbb{Z}] \bigcap W_x = I[G_H \bigcap (W_{x'} \times {}^*\mathbb{Z})] \bigcap W_x$$

is internal. $I[G_H]$ is the set we want. $\Box$

Remarks: (1) Theorem 5.1 of [R] is the consequence of $IP_{\aleph_0}$ because for every internal partial order $\mathbb{P}$ with no right endpoint, there always exists a regular *cardinal set W, which is internally cofinal in $\mathbb{P}$.

(2) Corollary 5.2 of [R] is a simple conclusion of above theorem.

C. W. Henson in [H] showed that if we assume IP_κ and $cf({}^*\mathbb{N}) = \kappa$, then for any two *infinite sets K and L, there exists a bijection $f : K \to L$ such that $f|C$ and $f^{-1}|D$ are internal for every *finite $C \subseteq K$ and *finite $D \subseteq L$. D. Ross proved the same thing in [R] under $SMA_{\aleph_0}$. Next we prove that under only $IP_{\aleph_0}$.

Theorem 7 ($IP_{\aleph_0}$) *For any two *infinite sets K and L there exists a bijection $f : K \to L$ such that $f|C$ and $f^{-1}|D$ are internal for every *finite $C \subseteq K$ and *finite $D \subseteq L$.*

Proof: Without loss of generality we can assume that L is a copy of ${}^*\mathbb{N}$, $L \subseteq K$, $K - L$ is *infinite and K is disjoint from ${}^*\mathbb{N}$. We now form an internally presented model $\mathcal{A}$ and an internally presented submodel $\mathcal{B}$ of $\mathcal{A}$. Let A, the base set of $\mathcal{A}$, $= {}^*\mathbb{N} \bigcup \mathbb{K} \bigcup \mathbb{F}$, where $F = \{f : f$ is an internal bijection from some *finite subset of ${}^*\mathbb{N}$ to some *finite subset of $K\}$. Let

$$\mathcal{A} = \langle A; {}^*\mathbb{N}, \mathbb{K}, <_{{}^*\mathbb{N}}, \mathbb{R} \rangle,$$

where ${}^*\mathbb{N}$ and K are two unary relations, $<_{{}^*\mathbb{N}}$ is the natural order on ${}^*\mathbb{N}$ and $R \subseteq {}^*\mathbb{N} \times \mathbb{K} \times \mathbb{F}$ is a trinary relation defined by

$$\langle a, b, f \rangle \in R \Longleftrightarrow a \in \mathrm{dom}(f) \wedge f(a) = b.$$

Let $\mathcal{B}$ be the internally presented submodel of $\mathcal{A}$ with the base set $B = {}^*\mathbb{N} \bigcup \mathbb{L} \bigcup \mathbb{G}$, where $G = \{f \in F : \mathrm{range}(f) \subseteq L\}$. Before we show that $\mathcal{B}$ is an elementary submodel of $\mathcal{A}$ we need more notation on isomorphisms from $\mathcal{A}$ to $\mathcal{A}$. Let $\pi : K \to K$ be an internal permutation. We define an isomorphism j_π from $\mathcal{A}$ to $\mathcal{A}$ by

$$j_\pi(x) = \begin{cases} x & \text{if } x \in {}^*\mathbb{N} \\ \pi(x) & \text{if } x \in K \\ f & \text{if } x \in F \end{cases}$$

where $f \in F$ such that $\mathrm{dom}(f) = \mathrm{dom}(x)$ and for every $a \in \mathrm{dom}(f)$, $f(a) = \pi \circ x(a)$.

It is easy to check that j_π is an isomorphism from $\mathcal{A}$ to $\mathcal{A}$.

Claim 1: $\mathcal{B}$ is an elementary submodel of $\mathcal{A}$.

Proof of Claim 1: It suffices to show that for every formula $\exists x \phi(x, x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in B$, if $\mathcal{A} \models \exists x \phi(x, a_1, \ldots, a_n)$, then there exists a $b \in B$ such that $\mathcal{A} \models \phi(b, a_1, \ldots, a_n)$.

Let $a \in A - B$ such that $\mathcal{A} \models \phi(a, a_1, \ldots, a_n)$.

Case 1: $a \in K$. Let

$$I = \{a_i : 1 \leq i \leq n \wedge a_i \in L\} \bigcup (\bigcup \{\mathrm{dom}(a_i) : 1 \leq i \leq n \wedge a_i \in G\}.$$

Then I is a *finite subset of L. Let $b \in L - I$ and define a permutation $\pi : K \to K$ by switching a and b, i.e. π is an identity on $K - \{a, b\}$, $\pi(a) = b$ and $\pi(b) = a$. Let j_π be the isomorphism.

Since $\mathcal{A} \models \phi(j_\pi(a), j_\pi(a_1), \ldots, j_\pi(a_n))$, $j_\pi(a) = b \in L$ and $j_\pi(a_i) = a_i$ for $1 \leq i \leq n$, there exists a $b \in B$ such that $\mathcal{A} \models \pi(b, a_1, \ldots, a_n)$.

Case 2: $a \in F - G$. Let $J = \mathrm{range}(a)$. $J - L$ and $J \bigcap L$ both are *finite. Let I be the *finite set defined in Case 1. Since $L - (J \bigcup I)$ is still *infinite, there exists an internal injection ϕ from $J - L$ into $L - (I \bigcup J)$. Let $\pi : K \to K$ be the internal permutation such that π is an identity on $K - (\mathrm{dom}(\phi) \bigcup \mathrm{range}(\phi))$ and

$$\pi(x) = \begin{cases} \phi(x) & \text{if } x \in \mathrm{dom}(\phi) \\ \phi^{-1}(x) & \text{if } x \in \mathrm{range}(\phi) \end{cases}$$

Now let $j_\pi : \mathcal{A} \to \mathcal{A}$ be the isomorphism. Since $\mathcal{A} \models \phi(j_\pi(a), j_\pi(a_1), \ldots, j_\pi(a_n))$, $j_\pi(a) = b \in G$ and $j_\pi(a_i) = a_i$ for $1 \le i \le n$, then there exists a $b \in B$ such that $\mathcal{A} \models \phi(b, a_1, \ldots, a_n)$.

By applying $IP_{\aleph_0}$ there exists an isomorphism $j : \mathcal{B} \to \mathcal{A}$. Let $\lambda = cf(^*\mathbb{N})$ and $\langle n_\alpha : \alpha < \lambda \rangle$ be an increasing sequence which is cofinal in $^*\mathbb{N}$. Recall that L is a copy of $^*\mathbb{N}$. We can consider the elements of L to be all standard or nonstandard natural numbers (keep in mind that L and $^*\mathbb{N}$ are disjoint). Let f_α be an "identity" function from its domain $\{0, \ldots, n_\alpha - 1\}$ in $^*\mathbb{N}$ to $\{0, \ldots, n_\alpha - 1\}$ in L. Let $g_\alpha = j(f_\alpha)$.

Claim 2: dom(g_α) is an initial segment of $^*\mathbb{N}$.

Proof of Claim 2: Since j should preserve the relations $<_{^*\mathbb{N}}$ and R, then $\mathrm{dom}(f_\alpha) = \{0, \ldots, n_\alpha - 1\}$ implies $\mathrm{dom}(g_\alpha) = \{0, \ldots, j(n_\alpha) - 1\}$.

Claim 3: $g = \bigcup_{\alpha < \lambda} g_\alpha$ is a well defined function.

Proof of Claim 3: Suppose there exist $\alpha < \beta < \lambda$ such that

$$k_1 = g_\alpha(n) \neq g_\beta(n) = k_2$$

for some $n \in \mathrm{dom}(\mathrm{g}_\alpha) \bigcap \mathrm{dom}(\mathrm{g}_\beta)$. Since

$$\langle n, k_1, g_\alpha \rangle \in R \wedge \langle n, k_2, g_\beta \rangle \in R,$$

then

$$\langle j^{-1}(n), j^{-1}(k_1), f_\alpha \rangle \in R \wedge \langle j^{-1}(n), j^{-1}(k_2), f_\beta \rangle \in R,$$

which means

$$f_\alpha(j^{-1}(n)) = j^{-1}(k_1) \wedge f_\beta(j^{-1}(n)) = j^{-1}(k_2),$$

hence $j^{-1}(n) = j^{-1}(k_1) = j^{-1}(k_2)$ because f_α and f_β are "identities" on their domains. Now $k_1 = k_2$ since j is an isomorphism, a contradiction.

Claim 4: $\mathrm{dom}(g) = {}^*\mathbb{N}$, $\mathrm{range}(g) = K$ and g is an bijection.

Proof of Claim 4: $\mathrm{dom}(g) = {}^*\mathbb{N}$ because $\langle j(n_\alpha) : \alpha < \lambda \rangle$ is increasing and cofinal in $^*\mathbb{N}$ and $\mathrm{dom}(g) = \{0, \ldots, j(n_\alpha) - 1\}$.

Let $k \in K$ and $l = j^{-1}(k) \in L$. Then there exists an $\alpha < \lambda$ and $n < n_\alpha$ such that $f_\alpha(n) = l$. That implies $g_\alpha(j(n)) = j(l) = k$, hence $k \in \mathrm{range}(g)$.

g is a bijection because every g_α is a bijection from its domain to its range.

Claim 5: For any *finite $C \subseteq {}^*\mathbb{N}$ and *finite $D \subseteq K$, $g|C$ and $g^{-1}|D$ are internal.

Proof of Claim 5: Since C is *finite, there exists an $\alpha < \lambda$ such that $C \subseteq \{0, \ldots, j(n_\alpha) - 1\}$, hence $g|C = g_\alpha|C$ which is internal.

Since D is *finite, there exists an $n \in {}^*\mathbb{N}$ and an internal bijection h from $\{0, \ldots, n\}$ to D. h now is an element of F. Since j is an isomorphism, there is an $f \in G$ such that $j(f) = h$. Since range(f) is *finite, there exists an $\alpha < \lambda$ such that range$(f) \subseteq$ range(f_α). That implies range$(h) \subseteq$ range(g_α). Hence $g^{-1}|D = g_\alpha^{-1}|D$ which is internal. $\Box$

Remark: The idea of the proof of Claim 1 in this theorem is due to C. W. Henson.

§3. A little improvement

In [R], Ross proved that $SMA_\kappa \iff SMA_{\aleph_1} + \kappa$–saturation. We found that $\aleph_1$ can be replaced by $\aleph_0$.

Theorem 8 *$SMA_\kappa \iff SMA_{\aleph_0} + \kappa$–saturation.*

Proof: "$\Longrightarrow$" is trivial.

"$\Longleftarrow$": Let $\mathcal{A}$ be an internally presented $\mathcal{L}$–structure with $|\mathcal{L}| = \lambda < \kappa$. Without loss of generality we can assume that the all symbols in $\mathcal{L}$ are relation symbols $\{P_\alpha : \alpha < \lambda\}$ and P_α is also the interpretation of itself under $\mathcal{A}$ for every $\alpha < \lambda$. By κ–saturation and Theorem 4.1 in [R], $cf(\Xi) \geq \kappa > \lambda$. We can find an injection $\varphi : \lambda \longmapsto {}^*\mathbb{N}$. Again by κ–saturation there exists an internal function

$$f : {}^*\mathbb{N} \longmapsto \bigcup_{\approx \in {}^*\mathbb{N}} {}^*\mathcal{P}(\mathbb{A}^{\approx})$$

such that $f(\varphi(\alpha)) = P_\alpha$ for every $\alpha < \lambda$. To build n–tuples we need one more internal function

$$G : {}^*\mathbb{N} \times \mathbb{A} \times \bigcup_{\approx \in {}^*\mathbb{N}} \mathbb{A}^{\approx} \longmapsto \bigcup_{\approx \in {}^*\mathbb{N}} \mathbb{A}^{\approx}$$

such that $G(m, a, \langle a_1, \ldots, a_h \rangle) = g_m(a, \langle a_1, \ldots, a_h \rangle)$, where

$$g_m(a, \langle a_1, \ldots, a_h \rangle) = \begin{cases} \langle a_1, \ldots, a_h, a \rangle & \text{if } m = h + 1 \\ \langle a_1, \ldots, a_h \rangle & \text{otherwise} \end{cases}$$

Consider the new structure $\mathcal{B}$ with the base set

$$B = A \bigcup {}^*\mathbb{N} \bigcup \Big(\bigcup_{\approx \in {}^*\mathbb{N}} \mathbb{A}^{\approx} \Big) \bigcup \Big(\bigcup_{\approx \in {}^*\mathbb{N}} {}^*\mathcal{P}(\mathbb{A}^{\approx}) \Big)$$

and a unary relation A, a binary relation $\in$ and two functions f and G. $\mathcal{B}$ is special by $SMA_{\aleph_0}$. Let $\{\mathcal{B}_\beta : \beta < \Xi\}$ be a specializing chain for $\mathcal{B}$. Since $\lambda < cf(\Xi)$, there exists a β_0, $\lambda \leq \beta_0 < \Xi$ such that $\varphi[\lambda] \subseteq {}^*\mathbb{N} \bigcap \mathbb{B}_{\beta_\varkappa}$. Let $\mathcal{A}_\beta = \langle A_\beta; (f(\varphi(\alpha)))_{\alpha<\lambda}\rangle$ if $\beta_0 < \beta < \Xi$ and $\mathcal{A}_\beta = \mathcal{A}_{\beta_0}$ if $\beta \leq \beta_0$.

We claim that $\{\mathcal{A}_\beta : \beta < \Xi\}$ is a specializing chain for $\mathcal{A}$.

First for n free variables $v_1, \ldots, v_n$, we let $\overline{v} = \langle v_1, \ldots, v_n\rangle =$

$$G(n, v_n, G(n-1, v_{n-1}, G(n-2, v_{n-2}, \ \ldots, G(1, v_1, \langle\rangle) \ldots)))$$

For any formula σ of $\mathcal{L}$, we translate to a formula σ' of $\mathcal{L}' \bigcup \{\varphi(\alpha) : \alpha < \lambda\} \bigcup \{n : n \in \mathbb{N}\} \bigcup \{\langle\rangle\}$, where $\mathcal{L}'$ is the language of $\mathcal{B}$, by induction on the complexity of σ. If $\sigma(\overline{x})$ is an atomic formula $P_\alpha(\overline{x})$, we let $\sigma'(\overline{x})$ be "$\overline{x} \in f(\varphi(\alpha))$". If σ is $\pi \wedge \tau$ or $\neg\tau$, let σ' be $\pi' \wedge \tau'$ or $\neg\tau'$. If σ is $\exists x\tau$, we let σ' be $\exists x(x \in A \wedge \tau')$.

Since every formula $\sigma(\overline{x})$ of $\mathcal{L}$ involves only finitely many symbols. we add only finitely many constants from ${}^*\mathbb{N} \bigcup \{\langle\rangle\}$ to σ'. Now it is easy to check by induction on the complexity of $\sigma(\overline{x})$ that for any $\overline{a}$ in A, $\sigma(\overline{a})$ is true in $\mathcal{A}_\beta$ iff $\sigma'(\overline{a})$ is true in $\langle \mathcal{B}_\beta, \varphi(\alpha_1), \ldots, \varphi(\alpha_k), 1, \ldots, n, \text{"}\langle\rangle\text{"}\rangle$ for $\beta \geq \beta_0$ and some finite $n \in \mathbb{N}$, where $\varphi(\alpha_1), \ldots, \varphi(\alpha_k)$ and $1, \ldots, n$ are the new constants in σ'.

Pick an $\beta \geq \beta_0$. We want to show that $\mathcal{A}_\beta$ is $|\beta|^+$–saturated. Let $\{a_\gamma : \gamma < |\beta|\} \subseteq \mathcal{A}_\beta$ and , (x) be a consistent type of $\langle \mathcal{A}_\beta, (a_\gamma)_{\gamma<|\beta|}\rangle$. We let

$$,'(x) = \{\sigma'(x) \wedge (x \in A) : \sigma(x) \in , (x)\},$$

which is a type of $\langle \mathcal{B}_\beta, (a_\gamma)_{\gamma<|\beta|}, (\varphi(\alpha))_{\alpha<\lambda}, (n)_{n\in\mathbb{N}}, \text{"}\langle\rangle\text{"}\rangle$. Since , (x) is finitely satisfiable, , $'(x)$ is also finitely satisfiable. Since $\lambda \leq \beta$ and $\mathcal{B}_\beta$ is $|\beta|^+$–saturated, there exists an $a \in A_\beta$ which satisfies the type , $'(x)$, hence $\langle \mathcal{A}_\beta, a\rangle \models$, (a). $\square$

§4. An open question

We still don't know whether or not Theorem 5.5 of [R] is a consequence of IP_κ.

References

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