A Theorem On The Isomorphism Property

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Abstract

An $\mathcal{L}$–structure is called internally presented in a nonstandard universe if its base set and interpretation of every symbol in $\mathcal{L}$ are internal. A nonstandard universe is said to satisfy the κ–isomorphism property if for any two internally presented $\mathcal{L}$–structures $\mathfrak{A}$ and $\mathfrak{B}$, where $\mathcal{L}$ has less than κ many symbols, $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{B}$ implies that $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$. In this paper we prove that the $\aleph_1$–isomorphism property is equivalent to the $\aleph_0$–isomorphism property plus $\aleph_1$–saturation.

Throughout this paper we use $\mathcal{L}$ for some first–order language and $\mathfrak{A}$, $\mathfrak{B}$ for models (or structures) of some $\mathcal{L}$ with base sets A, B respectively. By a standard universe we mean the superstructure $V_\omega(\mathbb{N}) = \bigcup_{n\in\omega} V_n$ with the "$\in$" relation, where $V_0 = \mathbb{N}$, a set of all positive integers as urelements, and $V_{n+1} = V_n \bigcup \mathcal{P}(V_n)$. By a nonstandard universe we mean the image of V under Mostowski collapse, where V is an elementary extension of the standard universe truncated at $\in$–rank ω. The author refers to [CK] for details of model theory and nonstandard universes.

In this paper we use ${}^*V$ (together with the "$\in$" relation) for a nonstandard universe and ${}^*\mathbb{N}$ for the nonstandard version of $\mathbb{N}$ in ${}^*V$. A set A is called *internal* in ${}^*V$ if A is an element of ${}^*V$. An $\mathcal{L}$–structure $\mathfrak{A}$ is called *internally presented* in ${}^*V$ if both base set A and the interpretation under $\mathfrak{A}$ of every symbol in $\mathcal{L}$ are internal in ${}^*V$.

A nonstandard universe ${}^*V$ is said to satisfy the *κ–isomorphism property* (IP_κ for short) for an infinite cardinal κ if the following is true in ${}^*V$,

> For any two internally presented $\mathcal{L}$–structures $\mathfrak{A}$ and $\mathfrak{B}$ with $|\mathcal{L}| < \kappa$, if $\mathfrak{A} \equiv \mathfrak{B}$, then $\mathfrak{A} \cong \mathfrak{B}$.

A direct corollary of the definition is that $IP_\kappa \Longrightarrow IP_\lambda$ if $\kappa > \lambda$.

IP_κ was given in [H1] by C. Ward Henson. It is a very strong property for a nonstandard universe. It implies κ–saturation. In Henson's several papers (cf.[H1], [H2], [H3], etc.) he applied the isomorphism property to nonstandard functional analysis and used it to get isometries between nonstandard hulls of Banach spaces.

The natural language for investigating Banach space has countably many function symbols. In order to deal with countable languages, $IP_{\aleph_1}$ should be assumed for the nonstandard universe we work in. But, as Henson [H4] observed, in every specific situation in his papers where $IP_{\aleph_1}$ is applied, one can, by $\aleph_1$–saturation, modify the structures to have only a finite number of relations and thus could apply just $IP_{\aleph_0}$. Is it generally true for all situations?

David Ross in [R] suggested a new property for a nonstandard universe, the κ–*special model axiom* for every infinite cardinal κ. Ross showed that the κ–special model axiom implies the κ–isomorphism property. He also proved that the κ–special model axiom is equivalent to the $\aleph_1$–special model axiom plus κ–saturation for $\kappa \geq \aleph_1$. Recently the author in [J] found that $\aleph_1$ above can be replaced by $\aleph_0$. Since the κ–special model axiom and the κ–isomorphism property are very similar properties, it is natural to ask a question:

Is IP_κ equivalent to $IP_{\aleph_0}$ plus κ–saturation?

In this paper we give a positive answer to the question when $\kappa = \aleph_1$. For arbitrary κ the problem is still open.

We are now going to prove that $IP_{\aleph_1} \Longleftrightarrow IP_{\aleph_0} + \aleph_1$–saturation. The only direction which needs to be proved is from right to left since the other direction is known. Let us fix a nonstandard universe ${}^*V$ which is $\aleph_1$–saturated. We want to show that if ${}^*V$ satisfies $IP_{\aleph_0}$, then ${}^*V$ satisfies $IP_{\aleph_1}$.

We call $\mathcal{L}$ a relational language if $\mathcal{L}$ contains only relation symbols. Let $l(P) = n$ if P is an n–placed relation symbol. We call a countable relational $\mathcal{L}$

(1) fully countable if $\mathcal{L} = \{P_{n,m} : n, m \in \mathbb{N}\}$ and $l(P_{n,m}) = n$ for every $n \in \mathbb{N}$. We use $\mathcal{L}_\infty$ for it;

(2) fully k–dimensional if $\mathcal{L} = \{P_{n,m} : n = 1, \ldots, k; m \in \mathbb{N}\}$ and $l(P_{n,m}) = n$ for $n = 1, \ldots, k$. We use $\mathcal{L}_k$ for it;

(3)diagonal-like if $\mathcal{L} = \{P_n : n \in \mathbb{N}\}$ and $l(P_n) = n$ for every $n \in \mathbb{N}$. We use $\mathcal{L}_D$ for it.

From now on we always assume that the languages mentioned below are relational and at most countable.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two $\mathcal{L}$–structures. For each P in $\mathcal{L}$ we always use Q or R for the interpretation of P under $\mathfrak{A}$ or $\mathfrak{B}$ respectively (for example, $P^{\mathfrak{A}}_{n,m} = Q_{n,m}$ and

$P^{\mathfrak{B}}_{n,m} = R_{n,m}$). Let $\mathfrak{A}$ be any internally presented $\mathcal{L}$–structure. By $\aleph_1$–saturation, every sequence $\langle Q_{n,m} : m \in \mathbb{N}\rangle$ of relations in $\mathfrak{A}$ with $Q_{n,m} \subseteq A^n$ for every $m \in \mathbb{N}$ can be extended to an internal sequence $\langle Q_{n,m} : m \in {}^*\mathbb{N}\rangle$ such that $Q_{n,m} \subseteq A^n$ for every $m \in {}^*\mathbb{N}$, and every sequence $\langle Q_n : n \in \mathbb{N}\rangle$ of relations in $\mathfrak{A}$ with $Q_n \subseteq A^n$ can be extended to an internal sequnce $\langle Q_n : n \in {}^*\mathbb{N}\rangle$ such that $Q_n \subseteq A^n$ for every $n \in {}^*\mathbb{N}$. From now on we fix such an internal extension for every sequence of relations described above. Without loss of generality we need only to verify $IP_{\aleph_1}$ for models of $\mathcal{L}_\infty$ because every language can be translated equivalently to a relational one and every countable relational language is a sublanguage of $\mathcal{L}_\infty$. However, we handle the cases of $\mathcal{L}_k$ and $\mathcal{L}_D$, first.

Lemma 1 *Let $\mathcal{L}$ be a relational language and $\varphi(x_1, \ldots, x_m)$ be a $\mathcal{L}$–formula which is built by using symbols $P_1, \ldots, P_n$ from $\mathcal{L}$ only. Then there exists a $\{\in\}$ (the language of ${}^*V$)–formula $\overline{\varphi}(x_1, \ldots, x_m, y, z_1, \ldots, z_n)$ with only bounded quantifiers such that for every internally presented $\mathcal{L}$–structure $\mathfrak{A}$ and $a_1, \ldots, a_m \in A$, $\mathfrak{A} \models \varphi(a_1, \ldots, a_m)$ iff ${}^*V \models \overline{\varphi}(a_1, \ldots, a_m, A, Q_1, \ldots, Q_n)$.*

Proof: Induction on the complexity of φ.

(1) If φ is $P_i(\overline{x})$ for some $i \leq n$, then $\overline{\varphi}$ is $\overline{x} \in z_i$;

(2) If φ is $\psi_1 \wedge \psi_2$ or $\neg\psi$, then $\overline{\varphi}$ is $\overline{\psi_1} \wedge \overline{\psi_2}$ or $\neg\overline{\psi}$ respectively;

(3) If φ is $\exists x \psi(x)$, then $\overline{\varphi}$ is $(\exists x \in y)\overline{\psi}(x)$. $\square$

Let $\mathfrak{A}$ be an internally presented $\mathcal{L}_k$–structure and let $\overline{\mathcal{L}}_k = \{P_1, \ldots, P_k, \leq\}$ be a finite language with $l(P_n) = n + 1$. Without loss of generality we assume that $A \bigcap {}^*\mathbb{N} = \emptyset$. For some $H \in {}^*\mathbb{N}$ we define an internally presented $\overline{\mathcal{L}}_k$–structure $\mathfrak{A}_H$ as follows. The base set of $\mathfrak{A}_H$ is $A \bigcup [1, H]$, where $[1, H] = \{m \in {}^*\mathbb{N} : 1 \leq m \leq H\}$, $Q_{n,H} = P_n^{\mathfrak{A}_H}$ is defined by letting $\langle a_1, \ldots, a_n, m\rangle \in Q_{n,H}$ iff $m \in [1, H]$ and $\langle a_1, \ldots, a_n\rangle \in Q_{n,m}$ for $n = 1, \ldots, k$ (note that $Q_{n,m}$ for an infinite m is in the fixed internal extension of the sequence $\langle Q_{n,m} : m \in \mathbb{N}\rangle$) and $\leq_H = \leq^{\mathfrak{A}_H}$ is just the natural order of $[1, H]$.

Lemma 2 *Let H be a finite positive integer. For every $\overline{\mathcal{L}}_k$–sentence φ, there exists an $\mathcal{L}_k$–sentence φ_* such that for every internally presented $\mathcal{L}_k$–structure $\mathfrak{A}$, $\mathfrak{A}_H \models \varphi$ iff $\mathfrak{A} \models \varphi_*$.*

Proof: For every $\overline{\mathcal{L}}_k$–formula $\varphi(x_1, \ldots, x_n)$ we construct a string of symbols φ_* inductively on the complexity of φ. Generally φ_* may not be even a formula, but when φ is a sentence, φ_* is a sentence too.

(1) If φ is $P_n(x_1, \ldots, x_n, y)$, then φ_* is $P_{n,y}(x_1, \ldots, x_n)$, which is not a formula when y is a variable;

(2) If φ is $x \leq y$, then φ_* is $I_{x \leq y}$, where $I_{x\leq y}$ is a valid sentence if $x \leq y$ is true and an invalid sentence otherwise. This is also not a formula if x or y is a variable;

(3) If φ is $\psi_1 \wedge \psi_2$ or $\neg\psi$, then φ_* is $(\psi_1)_* \wedge (\psi_2)_*$ or $\neg\psi_*$ respectively;

(4) If φ is $\exists x(\neg(x \leq x) \wedge \psi(x))$, then φ_* is $\exists x \psi_*(x)$. ($\mathfrak{A}_H \models \neg(a \leq a)$ implies $a \in A$);

(5) If φ is $\exists x((x \leq x) \wedge \psi(x))$, then φ_* is $\bigvee_{m=1}^{H} \psi_*(m)$. ($\mathfrak{A}_H \models a \leq a$ implies $a \in [1, H]$.) Note that H is finite.

When φ is a sentence, by (5), y in (1) and x, y in (2) become constants so that φ_* is also a sentence. It is easy to check that for any $a_1, \ldots, a_s \in A$ and $m_1, \ldots, m_t \in [1, H]$, $\varphi(a_1, \ldots, a_s, m_1, \ldots, m_t)$ is true in $\mathfrak{A}_H$ iff $\varphi_*(a_1, \ldots, a_s, m_1, \ldots, m_t)$ is true in $\mathfrak{A}$. Hence if φ is an $\overline{\mathcal{L}}_k$–sentence, then φ_* is an $\mathcal{L}_k$–sentence and $\mathfrak{A}_H \models \varphi$ iff $\mathfrak{A} \models \varphi_*$. □

Lemma 3 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two internally presented $\mathcal{L}_k$–structures such that $\mathfrak{A} \equiv \mathfrak{B}$. Then there exists an infinite integer H such that $\mathfrak{A}_H \equiv \mathfrak{B}_H$.*

Proof: Let φ be an $\overline{\mathcal{L}}_k$–sentence. By Lemma 1, there exists an $\{\in\}$–formula $\overline{\varphi}(y, z_1, \ldots, z_k, z_{k+1})$ with only bounded quantifiers such that for every $H \in {}^*\mathbb{N}$

$$\mathfrak{A}_H \models \varphi \text{ iff } {}^*V \models \overline{\varphi}(A \bigcup [1, H], Q_{1,H}, \ldots, Q_{k,H}, \leq_H)$$

and

$$\mathfrak{B}_H \models \varphi \text{ iff } {}^*V \models \overline{\varphi}(B \bigcup [1, H], R_{1,H}, \ldots, R_{k,H}, \leq_H).$$

Note that $\langle A \bigcup [1, H] : H \in {}^*\mathbb{N}\rangle$, $\langle B \bigcup [1, H] : H \in {}^*\mathbb{N}\rangle$, $\langle Q_{n,H} : H \in {}^*\mathbb{N}\rangle$ and $\langle R_{n,H} : H \in {}^*\mathbb{N}\rangle$ for $n = 1, \ldots, k$ and $\langle \leq_H : H \in {}^*\mathbb{N}\rangle$ are all internal sequences. Let

$$S_\varphi = \{H \in {}^*\mathbb{N} : \mathfrak{A}_H \models \varphi \longleftrightarrow \mathfrak{B}_H \models \varphi\}.$$

Then S_φ is internal. By Lemma 2, S_φ contains all finite positive integers and hence there exists an infinite integer H_φ such that $[1, H_\varphi] \subseteq S_\varphi$. Since there are only countably many $\overline{\mathcal{L}}_k$–sentences, there exists an infinite integer H such that

$$[1, H] \subseteq \bigcap \{S_\varphi : \varphi \text{ is an } \overline{\mathcal{L}}_k\text{–sentence}\}$$

by $\aleph_1$–saturation. This implies $\mathfrak{A}_H \equiv \mathfrak{B}_H$. □

Theorem 1 *Assume that* $^*V$ *satisfies* $IP_{\aleph_0}$*. Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be two internally presented* $\mathcal{L}_k$*–structures. Then* $\mathfrak{A} \equiv \mathfrak{B}$ *implies* $\mathfrak{A} \cong \mathfrak{B}$*.*

Proof: By Lemma 3, there exists an infinite integer H such that $\mathfrak{A}_H \equiv \mathfrak{B}_H$. Since $\overline{\mathcal{L}}_k$ is a finite language, there exists an isomorphism j from $\mathfrak{A}_H$ to $\mathfrak{B}_H$ by applying $IP_{\aleph_0}$. We now want to show that $j \upharpoonright A$ is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

Since A and B are definable in $\mathfrak{A}_H$ and $\mathfrak{B}_H$ respectively by the same $\overline{\mathcal{L}}_k$–formula, $j \upharpoonright A$ is a bijection between A and B. Since every finite $m \in [1, H]$ is definable in both $\mathfrak{A}_H$ and $\mathfrak{B}_H$ by the same formula, then j is an identity on $\mathbb{N} \subseteq [1, H]$. Let $P_{n,m}$ be in $\mathcal{L}_k$ and $a_1, \ldots, a_n \in A$. Then

$$\mathfrak{A} \models P_{n,m}(a_1, \ldots, a_n) \text{ iff}$$

$$\mathfrak{A}_H \models P_n(a_1, \ldots, a_n, m) \text{ iff}$$

$$\mathfrak{B}_H \models P_n(j(a_1), \ldots, j(a_n), m) \text{ iff}$$

$$\mathfrak{B} \models P_{n,m}(j(a_1), \ldots, j(a_n)).$$

Hence $j \upharpoonright A : \mathfrak{A} \cong \mathfrak{B}$. $\square$

We turn now to the case of $\mathcal{L}_D$.

Let $\mathfrak{A}$ be an internally presented $\mathcal{L}_D \bigcup \{P'\}$–structure, where P' is a binary relation symbol which is not in $\mathcal{L}_D$ and let $H \in {}^*\mathbb{N}$. We define an internally presented $\overline{\mathcal{L}}_D$–structure $\mathfrak{A}^H$, where $\overline{\mathcal{L}}_D = \{P, P'', F\}$, $l(P) = 1$, $l(P'') = 1$ and $l(F) = 3$. The base set of $\mathfrak{A}^H$ is $\bigcup_{n=1}^{H} A^n$, $Q^H = P^{\mathfrak{A}^H}$ is defined by letting $\langle a_1, \ldots, a_n \rangle \in Q^H$ iff $\langle a_1, \ldots, a_n \rangle \in Q_n$ for some $n \in [1, H]$ (Q_n for an infinite n is in the fixed internal extension of the sequence $\langle Q_n : n \in \mathbb{N} \rangle$), $Q'' = P''^{\mathfrak{A}^H}$ is defined by letting $a \in Q''$ iff $a = \langle a_1, a_2 \rangle$ and $\mathfrak{A} \models P'(a_1, a_2)$ and $F^{\mathfrak{A}^H}$ is the graph of a function

$$G^{\mathfrak{A}^H} : (\bigcup_{n=1}^{H-1} A^n) \times A \longrightarrow \bigcup_{n=2}^{H} A^n$$

which is defined by letting

$$G^{\mathfrak{A}^H}(\langle a_1, \ldots, a_n \rangle, a) = \langle a_1, \ldots, a_n, a \rangle.$$

For convenience we add to $\overline{\mathcal{L}}_D$ new symbols g_n for $n = 1, \ldots, H$ when H is finite. For any internally presented $\mathcal{L}_D \bigcup \{P'\}$–structure $\mathfrak{A}$, hence $\overline{\mathcal{L}}_D$–structure $\mathfrak{A}^H$, we interpret

g_n under $\mathfrak{A}^H$ to be an n–ary function from A to the base set of $\mathfrak{A}^H$ such that for $a_1, \ldots, a_n \in A$, $g_n(a_1, \ldots, a_n) = \langle a_1, \ldots, a_n \rangle$. So in $\mathfrak{A}^H$

$$g_1(a) = b \text{ iff } \neg\exists x, y F(x, y, a) \wedge a = b$$

and

$$g_{n+1}(a_1, \ldots, a_{n+1}) = G(g_n(a_1, \ldots, a_n), a_{n+1}).$$

Hence g_n for $n = 1, \ldots, H$ are definable in $\mathfrak{A}^H$. Note that $\mathfrak{A}^H \models (b = g_n(a_1, \ldots, a_n))$ implies $a_1, \ldots, a_n \in A$.

Lemma 4 *Let H be a finite positive integer. For every $\overline{\mathcal{L}}_D$–sentence φ there exists an $\mathcal{L}_D \bigcup \{P'\}$–sentence φ^* such that for every internally presented $\mathcal{L}_D \bigcup \{P'\}$–structure $\mathfrak{A}$, $\mathfrak{A}^H \models \varphi$ iff $\mathfrak{A} \models \varphi^*$.*

Proof: For every $\overline{\mathcal{L}}_D$–formula φ we construct a string of symbols φ^* inductively on the complexity of φ. Generally φ^* may not be a formula, but when φ is an $\overline{\mathcal{L}}_D$–sentence, φ^* is also a sentence. Note that for every finite n, A^n is definable in $\mathfrak{A}^H$.

(1) If φ is $P(x)$, then φ^* is $P_n(x_1, \ldots, x_n)$ for some $n \in [1, H]$ if x is a term $g_n(x_1, \ldots, x_n)$ and is an invallid sentence otherwise. φ^* may not be a formula in this case if x is a free variable;

(2) If φ is $P''(x)$, then φ^* is $P'(x_1, x_2)$ if $x = g_2(x_1, x_2)$ and is an invalid sentence otherwise. In this case φ^* may not be a formula;

(3) If φ is $F(x, y, z)$, then φ^* is

$$\bigwedge_{i=1}^{n} x_i = z_i \wedge y_1 = z_{n+1}$$

if $x = g_n(x_1, \ldots, x_n)$, $y = g_1(y_1)$ and $z = g_{n+1}(z_1, \ldots, z_{n+1})$ and is an invalid sentence otherwise. In this case φ^* may also fail to be a formula;

(4) If φ is $\psi_1 \wedge \psi_2$ or $\neg\psi$, then φ^* is $\psi_1^* \wedge \psi_2^*$ or $\neg\psi^*$ respectively;

(5) If φ is $\exists x \psi(x)$, then φ^* is

$$\bigvee_{n=1}^{H} \exists x_1, \ldots, \exists x_n (\psi(g_n(x_1, \ldots, x_n)))^*.$$

Notice that H is finite.

It is easy to check that for any $\overline{\mathcal{L}}_D$–formula $\varphi(\overline{x})$ and any $\langle a_{1,1},\ldots,a_{1,m_1}\rangle,\ldots,$ $\langle a_{n,1},\ldots,a_{n,m_n}\rangle \in \bigcup_{n=1}^{H} A^n$,

$$\mathfrak{A}^H \models \varphi(\langle a_{1,1},\ldots,a_{1,m_1}\rangle,\ldots,\langle a_{n,1},\ldots,a_{n,m_n}\rangle) \text{ iff}$$

$$\mathfrak{A} \models \varphi^*(a_{1,1},\ldots,a_{1,m_1},\ldots,a_{n,1},\ldots,a_{n,m_n}).$$

Note that when φ is an $\overline{\mathcal{L}}_D$–sentence, φ^* is also a sentence by case 5. $\square$

Lemma 5 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two internally presented $\mathcal{L}_D \bigcup\{P'\}$–structures. If $\mathfrak{A} \equiv \mathfrak{B}$, then there exists an infinite integer H such that $\mathfrak{A}^H \equiv \mathfrak{B}^H$.*

Proof: By Lemma 1 and the proof of Lemma 3,

$$S_\varphi = \{H \in {}^*\mathbb{N} : \mathfrak{A}^H \models \varphi \longleftrightarrow \mathfrak{B}^H \models \varphi\}$$

is internal for every $\overline{\mathcal{L}}_D$–sentence φ. By Lemma 5, $\mathbb{N} \subseteq S_\varphi$. So there exists an infinite integer H such that

$$[1,H] \subset \bigcap\{S_\varphi : \varphi \text{ is an } \overline{\mathcal{L}}_D\text{–sentence}\}$$

by $\aleph_1$–saturation. This implies $\mathfrak{A}^H \equiv \mathfrak{B}^H$. $\square$

Theorem 2 *Assume that ${}^*V$ satisfies $IP_{\aleph_0}$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two internally presented $\mathcal{L}_D \bigcup\{P'\}$–structures. If $\mathfrak{A} \equiv \mathfrak{B}$, then $\mathfrak{A} \cong \mathfrak{B}$.*

Proof: By Lemma 6, there exists an infinite integer H such that $\mathfrak{A}^H \equiv \mathfrak{B}^H$. Since both $\mathfrak{A}^H$ and $\mathfrak{B}^H$ are internally presented $\overline{\mathcal{L}}_D$–structures and $\overline{\mathcal{L}}_D$ is finite, there exists an isomorphism j from $\mathfrak{A}^H$ to $\mathfrak{B}^H$ by applying $IP_{\aleph_0}$. We now want to show that $j \upharpoonright A$ is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

Since A and B are definable in $\mathfrak{A}^H$ and $\mathfrak{B}^H$ respectively by the same formula $\neg\exists x, y F(x,y,z)$, $j \upharpoonright A$ is an bijection from A to B. Let $a_1,\ldots,a_n \in A$. Since the n–tuple $\langle a_1,\ldots,a_n\rangle$ can be built from $a_1,\ldots,a_n$ by function $G^{\mathfrak{A}^H}$, then $j(\langle a_1,\ldots,a_n\rangle) = \langle j(a_1),\ldots,j(a_n)\rangle$. So

$$\mathfrak{A} \models P_n(a_1,\ldots,a_n) \text{ iff}$$

$$\mathfrak{A}^H \models P(\langle a_1,\ldots,a_n\rangle) \text{ iff}$$

$$\mathfrak{B}^H \models P(j(\langle a_1,\ldots,a_n\rangle)) \text{ iff}$$

$$\mathfrak{B}^H \models P(\langle j(a_1), \ldots, j(a_n)\rangle) \text{ iff}$$
$$\mathfrak{B} \models P_n(j(a_1), \ldots, j(a_n)).$$

It is easy to check the relation P' by same reason. Hence $j \restriction A$ is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. $\square$

Let $\mathfrak{A}$ be an internally presented $\mathcal{L}_\infty$–structure. Let $H \in {}^*\mathbb{N}$. We define $\mathfrak{A}_H^\infty$ to be an internally presented $\mathcal{L}_D \bigcup \{\leq\}$–structure with the base set $A \bigcup [1, H]$, $Q_{1,H} = A$, $Q_{n+1,H}$ is defined by letting $\langle a_1, \ldots, a_n, m\rangle \in Q_{n+1,H}$ iff $\langle a_1, \ldots, a_n\rangle \in Q_{n,m}$ and $\leq_H$ is the natural order of $[1, H]$.

Theorem 3 *Assume that ${}^*V$ satisfies $IP_{\aleph_0}$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two internally presented $\mathcal{L}_\infty$–structures. If $\mathfrak{A} \equiv \mathfrak{B}$, then $\mathfrak{A} \cong \mathfrak{B}$.*

Proof: Let $\mathfrak{A} \restriction k$ and $\mathfrak{B} \restriction k$ be the reducts of $\mathfrak{A}$ and $\mathfrak{B}$ to $\mathcal{L}_\infty \restriction k = \{P_{n,m} \in \mathcal{L}_\infty : n \leq k\}$. $\mathfrak{A} \equiv \mathfrak{B}$ certainly implies $\mathfrak{A} \restriction k \equiv \mathfrak{B} \restriction k$. By Lemma 3, there exists an infinite integer H_k such that $(\mathfrak{A} \restriction k)_{H_k} \equiv (\mathfrak{B} \restriction k)_{H_k}$. By $\aleph_1$–saturation, there exists an infinite integer H such that $H < H_k$ for every $k \in \mathbb{N}$. So $(\mathfrak{A} \restriction k)_H \equiv (\mathfrak{B} \restriction k)_H$ for every k. This implies $\mathfrak{A}_H^\infty \equiv \mathfrak{B}_H^\infty$.

Since $\mathfrak{A}_H^\infty$ and $\mathfrak{B}_H^\infty$ are internally presented $\mathcal{L}_D \bigcup \{\leq\}$–structures, then there exists an isomorphism j from $\mathfrak{A}_H^\infty$ to $\mathfrak{B}_H^\infty$ by Theorem 7. We now want to show that $j \restriction A$ is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

Again $j \restriction A$ is a bijection from A to B. Since $j : (\mathfrak{A} \restriction k)_H \longrightarrow (\mathfrak{B} \restriction k)_H$ is also an isomorphism, then, by Theorem 4, $j \restriction A : \mathfrak{A} \restriction k \longrightarrow \mathfrak{B} \restriction k$ is an isomorphism for every finite k. This implies $j \restriction A$ is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. $\square$

We would like to thank H. Jerome Keisler, Michael Benedikt and Arnold W. Miller for valuable discussions during the preparation of this paper.

A. W. Miller told the author that Lemma 2, Lemma 3, Lemma 5 and Lemma 6 have simpler proofs if we assume that the nonstandard universe satisfies $IP_{\aleph_0}$ at the very beginning of the paper.

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