Cuts In Hyperfinite Time Lines

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Abstract

In an ω_1–saturated nonstandard universe a cut is an initial segment of the hyperintegers, which is closed under addition. Keisler and Leth in [KL] introduced, for each given cut U, a corresponding U–topology on the hyperintegers by letting O be $U-$open if for any $x \in O$ there is a y greater than all the elements in U such that the interval $[x-y, x+y] \subseteq O$. Let U be a cut in a hyperfinite time line $\mathcal{H}$, which is a hyperfinite initial segment of the hyperintegers. U is called a good cut if there exists a $U-$meager subset of $\mathcal{H}$ of Loeb measure one. Otherwise U is bad. In this paper we discuss the questions of Keisler and Leth about the existence of bad cuts and related cuts. We show that assuming $\underline{b} > \omega_1$, every hyperfinite time line has a cut with both cofinality and coinitiality uncountable. We construct bad cuts in a nonstandard universe under *ZFC*. We also give two results about the existence of other kinds of cuts.

Throughout this paper we work within ω_1-saturated nonstandard universes. We let $\mathcal{M}$ be a nonstandard universe and ${}^*\mathbb{N}$ be the set of all hyperintegers in $\mathcal{M}$ which contains $\mathbb{N}$, the set of all standard positive integers. Let $H \in {}^*\mathbb{N} - \mathbb{N}$; we call $\mathcal{H} = \{n \in {}^*\mathbb{N} : \ltimes \leq \mathbb{H}\}$ a hyperfinite time line or a hyperline for short. From now on we always let H be the largest element of $\mathcal{H}$. Let $[a,b] = \{x \in \mathcal{H} : a \leq x \leq b\}$ be an interval in $\mathcal{H}$ and $[r] = \max\{n \in {}^*\mathbb{N} : \ltimes \leq \diagdown\}$ for any hyperreal r.

A notion of Loeb measure for $\mathcal{H}$, which is the standard part of the countably additive extension of the counting measure on $\mathcal{H}$, was introduced by P. Loeb (cf.[Lo]) as a counterpart of Lebesgue measure for the reals. Recently H. J. Keisler and S. Leth (cf.[KL]) introduced $U-$topology on $\mathcal{H}$ for each cut U as an analogue of the order topology on the reals. They discussed the relationship between $U-$meager sets and Loeb measure zero sets and listed many questions at the end of the paper. In this paper we discuss some of those questions about the existence of bad cuts and related cuts. For background in model theory see Chang and Keisler [CK], for background in nonstandard universes and the Loeb measure see Stroyan and Bayod [SB], and for background about hyperfinite sets see [KKLM] and [KL]. This paper was developed under the supervision of H. J. Keisler, to whom the author is deeply grateful.

Throughout this paper we let $card(A)$ mean external cardinality of A and ${}^*card(A)$ mean internal cardinality of A if A is an internal set. When A is hyperfinite, ${}^*card(A)$ is a hyperinteger.

Let us recall that a cut in $\mathcal{H}$ is an initial segment of $\mathcal{H}$ which is closed under addition. A cut must be external. Let U be a cut in $\mathcal{H}$. A subset O of $\mathcal{H}$ is called $U-$open if for any $x \in O$ there is a $y \in \mathcal{H} - U$ such that $[x-y, x+y] \subseteq O$. All

$U-$open sets form a $U-$topology on $\mathcal{H}$. A subset of $\mathcal{H}$ is called $U-$nowhere dense ($U-$meager) if it is nowhere dense (meager) under the $U-$topology. A cut U in $\mathcal{H}$ is called good if there exists a $U-$meager subset of $\mathcal{H}$ of Loeb measure one. Otherwise U is bad.

Let U be a cut in $\mathcal{H}$ and $x \in \mathcal{H}$. The following are cuts:

$xU = \{y \in \mathcal{H} : \exists z \in U\ (y \leq xz)\}$

$x/U = \{y \in \mathcal{H} : \forall z \in U\ (y \leq x/z)\}$

$x^U = \{y \in \mathcal{H} : \exists z \in U\ (y \leq x^z)\}$

and $M(U) = \{y \in \mathcal{H} : \forall z \in U\ (yz \in U)\}$

$M(U)$ is closed under multiplication.

U is called a type one cut if there exists an $x \in \mathcal{H}$ such that $U = xM(U)$ or $U = x/M(U)$. Otherwise U is called type two.

An internal increasing function f from $[1, x]$ to $\mathcal{H}$ is said to cross a cut U if $U \bigcap range(f)$ is unbounded in U.

$J(U) = \{x \in \mathcal{H} :$ there are no functions from $[1, x]$ to $\mathcal{H}$ which cross $U\}$. $J(U)$ is also a cut closed under multiplication.

Let U be a cut in $\mathcal{H}$; we define

$cf(U)$, the cofinality of U, $= \min\{card(F) : F \subseteq U$ and F is cofinal in $U\}$.

$ci(U)$, the coinitiality of U, $= \min\{card(F) : F \subseteq \mathcal{H} - U$ and F is coinitial in $\mathcal{H} - U\}$.

U is called a (κ, λ) cut ($(\geq \kappa, \geq \lambda)$ cut) if $cf(U) = \kappa$ and $ci(U) = \lambda$ ($cf(U) \geq \kappa$ and $ci(U) \geq \lambda$).

A proposition from [KL] tells the relation between $(\geq \omega_1, \geq \omega_1)$ cuts, type two cuts and bad cuts.

Proposition 1 (Keisler and Leth) *U is a bad cut $\Rightarrow$ U is a type two cut $\Rightarrow$ U is a $(\geq \omega_1, \geq \omega_1)$ cut.*

Proof: See Proposition 6.7. and Corollary 6.12. in [KL]. □

Here are the two questions from [KL]:

(1) Does there exist a nonstandard universe in which there are no $(\geq \omega_1, \geq \omega_1)$ cuts? In which there are no type two cuts? In which there are no bad cuts?

(2) Can it be proved in *ZFC* that there exist nonstandard universes with bad cuts? With $(\geq \omega_1, \geq \omega_1)$ cuts M which are closed under multiplication and $J(M) = \mathbb{N}$?

First we give a consistency result for the first sentence of the first problem. Before that we need more notation.

Let f and g be two functions from ω to ω.

$f \leq_* g$ means that $\{n \in \omega : f(n) > g(n)\}$ is finite.

$\underset{\sim}{b} = \min\{card(F) : F \subseteq \omega^\omega$ and F is up unbounded under $\leq_*\}$.

Let $h \in \mathcal{H} - \mathbb{N}$. $U_h = \{x \in \mathcal{H} : \forall f \in \omega^\omega \ ({}^*f(x) < h)\}$. Here ${}^*f$ is a nonstandard version of the standard function f.

U_h is a cut closed under multiplication and $h \notin U_h$.

Theorem 1 *Assume $\underline{b} > \omega_1$. Then every hyperline has $(\geq \omega_1, \geq \omega_1)$ cuts.*

Theorem 1 is a simple consequence of two lemmas from [Ca].

Lemma 1 *Let $ci(\mathbb{N}) = \kappa$. If U is a cut in $\mathcal{H}$ with $cf(U) = \omega$ ($ci(U) = \omega$), then $ci(U) = \kappa$ ($cf(U) = \kappa$).*

Proof: Let $\langle a_n : n \in \omega \rangle$ be strictly increasing and cofinal in U. Let $\mathcal{F}_n = \{f \subseteq \mathcal{H} \times \mathcal{H} : f$ is an increasing internal function with $f(m) = a_m$ for any $m \leq n\}$. Then $\bigcap_{n \in \omega} \mathcal{F}_n \neq \emptyset$ by ω_1-saturation. Let $g \in \bigcap_{n \in \omega} \mathcal{F}_n$. Then $\exists x \in \mathcal{H} - \mathbb{N}$, $[1, x] \subseteq \mathrm{dom}(g)$. Let $\langle x_\alpha : \alpha < \kappa \rangle$ be a decreasing sequence coinitial in $[1, x] - \mathbb{N}$. Then $\langle g(x_\alpha) : \alpha < \kappa \rangle$ should be a decreasing sequence coinitial in $\mathcal{H} - U$, hence $ci(U) = \kappa$. □

Lemma 2 *Let $h \in \mathcal{H} - \mathbb{N}$, then $ci(U_h) \geq \underline{b}$.*

Proof: Let $X \subseteq \mathcal{H} - U_h$ with $card(X) < \underline{b}$. For each $x \in X$ there exists an $f_x \in \omega^\omega$ such that ${}^*f_x(x) \geq h$. Since $card(\{f_x : x \in X\}) < \underline{b}$, there exists a $g \in \omega_\omega$ such that $f_x \leq_* g$ for every $x \in X$.

Let $x_0 = \min\{y \in \mathcal{H} : {}^*g(y) \geq h\}$ (x_0 exists since the right side is an internal set). Then $x_0 \notin U_h$ and $x_0 \leq x$ for every $x \in X$ since ${}^*g(x) \geq {}^*f_x(x) \geq h$. Hence X cannot be coinitial in $\mathcal{H} - U_h$. □

Proof of Theorem 1:

Case 1. $ci(\mathbb{N}) = \omega_{\nVdash}$.

Let $h \in \mathcal{H} - \mathbb{N}$. By Lemma 2 $ci(U_h) \geq \underline{b} > \omega_1$ and by Lemma 1 $cf(U_h) \neq \omega$. So U_h is a $(\geq \omega_1, \geq \omega_1)$ cut.

Case 2. $ci(\mathbb{N}) \geq \omega_{\nvDash}$.

Let $\langle x_\alpha : \alpha < \omega_1 \rangle$ be a decreasing sequence in $\mathcal{H} - \mathbb{N}$ such that $2x_{\alpha+1} \leq x_\alpha$ (or $x_{\alpha+1}^2 \leq x_\alpha$ if we want the cut to be closed under multiplication) for all $\alpha < \omega_1$. Let

$$U = \{x \in \mathcal{H} : \forall \alpha < \omega_1 (x \leq x_\alpha)\}.$$

Then $ci(U) = \omega_1 \neq ci(\mathbb{N})$. Hence $cf(U) \neq \omega$ by Lemma 1.

This U is a $(\geq \omega_1, \geq \omega_1)$ cut in $\mathcal{H}$. □

Now we give a slight generalization of Proposition 7.8 of [KL].

Theorem 2 *If there exists a $\kappa-$decreasing sequence $\langle x_\alpha : \alpha < \kappa\rangle$ in $\mathcal{H}$ for some cardinal κ such that $x^2_{\alpha+1} \le x_\alpha$ and $card(\mathcal{H}) < 2^\kappa$, then there exists a type two cut in $\mathcal{H}$.*

Proof: Suppose the theorem is not true. We derive a contradiction. For simplicity we can assume $x^5_{\alpha+1} \le x_\alpha$ for all $\alpha < \kappa$ and $x_0^2 \le H$.

Now we construct a family of intervals $T = \{[a_s, b_s] \subseteq \mathcal{H} : s \in 2^{<\kappa}\}$ inductively such that

(1) $\forall\alpha < \kappa\ \forall s \in 2^\alpha\ (a_s x_\alpha = b_s)$.

(2) $\forall\alpha < \kappa\ \forall s \in 2^\alpha\ (a_s x^5_{\alpha+1} \le a_{s\hat{}\langle 0\rangle} x^4_{\alpha+1} \le b_{s\hat{}\langle 0\rangle} x^3_{\alpha+1} \le a_{s\hat{}\langle 1\rangle} x^2_{\alpha+1} \le b_{s\hat{}\langle 1\rangle} x_{\alpha+1} \le b_s)$.

(3) $\forall s, t \in 2^{<\kappa}\ (s \subseteq t \rightarrow [a_t, b_t] \subseteq [a_s, b_s])$.

Let $a_{\langle\rangle} = x_0$ and $b_{\langle\rangle} = x_0^2$. Assume that we have all a_s and b_s for every $s \in 2^{<\alpha}$ for some $\alpha < \kappa$.

Case 1. $\alpha = \beta + 1$.

For any $s \in 2^\beta$, let $a_{s\hat{}\langle 0\rangle} = a_s x_\alpha$, $b_{s\hat{}\langle 0\rangle} = a_s x^2_\alpha$, $a_{s\hat{}\langle 1\rangle} = a_s x^3_\alpha$ and $b_{s\hat{}\langle 1\rangle} = a_s x^4_\alpha$.

Case 2. α is a limit ordinal below κ.

Claim: $\forall s \in 2^\alpha\ (\bigcap_{\beta<\alpha}[a_{s|\beta}, b_{s|\beta}] \ne \emptyset)$, where $s|\beta$ means the restriction of s on β.

Proof of Claim: Assume the claim is not true for some $s \in 2^\alpha$.

Let $U = \{x \in \mathcal{H} : \exists\beta < \alpha\ (x \le a_{s|\beta})\}$ and $M = \{y \in \mathcal{H} : \forall\beta < \alpha\ (y \le x_\beta)\}$. Then we can show that $M(U) = M$ and U is a type two cut.

For any $y \in M$, for any $x \in U$ there exists a $\beta < \alpha$ such that $x \le a_{s|\beta}$. Then $yx \le x_{\beta+1} a_{s|\beta} \le a_{s|\beta+1} \in U$. For any $y \in \mathcal{H} - M$, there is a $\beta < \alpha$ such that $y \ge x_\beta$. Then $y a_{s|\beta} \ge x_\beta a_{s|\beta} = b_{s|\beta} \notin U$. So $M(U) = M$.

For any $x \in U$, there exists a $\beta < \alpha$ such that $x \le a_{s|\beta}$. Then $xM(U) \subseteq [1, a_{s|\beta} x_{\beta+1}] \subseteq [1, a_{s|\beta+1}]$. Hence $U \ne xM(U)$. For any $x \notin U$, there exists a $\beta < \alpha$ such that $x \ge b_{s|\beta}$. Then $x/M(U) \supseteq [1, b_{s|\beta}/x_{\beta+1}] \supseteq [1, b_{s|\beta+1}]$. Hence $U \ne x/M(U)$. So U is a type two cut in $\mathcal{H}$.

But there are no type two cuts in $\mathcal{H}$, a contradiction. This ends the proof of the claim.

Let $a \in \bigcap_{\beta<\alpha}[a_{s|\beta}, b_{s|\beta}]$. Then one of the intervals $[[a/x_\alpha], a]$ and $[a, ax_\alpha]$ is contained in $\bigcap_{\beta<\alpha}[a_{s|\beta}, b_{s|\beta}]$, because otherwise there exists a $\beta < \alpha$ such that $a/x_\alpha < a_{s|\beta}$ and $b_{s|\beta} < ax_\alpha$, which implies $ax_\alpha/(a/x_\alpha) = x^2_\alpha > b_{s|\beta}/a_{s|\beta} = x_\beta \ge x^2_{\beta+1}$. But $x_\alpha < x_{\beta+1}$.

If $[[a/x_\alpha], a] \subseteq \bigcap_{\beta<\alpha}[a_{s|\beta}, b_{s|\beta}]$, then let $a_s = [a/x_\alpha]$ and $b_s = a_s x_\alpha$. Otherwise let $a_s = a$ and $b_s = ax_\alpha$.

Now we have finished the construction.

Since $card(\mathcal{H}) < 2^\kappa$, there exists an $f \in 2^\kappa$ such that $\bigcap_{\alpha<\kappa}[a_{f|\alpha}, b_{f|\alpha}] = \emptyset$. That gives us a type two cut by the same proof of the claim, a contradiction. □

Corollary 1 (Keisler and Leth) *Assume $2^\omega < 2^{\omega_1}$ and $card(\mathcal{H}) < 2^{\omega_1}$. Then there exist type two cuts in $\mathcal{H}$.*

Proof: Since $ci(\mathbb{N}) \geq \omega_{\nVdash}$, there exists an ω_1-decreasing sequence $\langle x_\alpha : \alpha < \omega_1 \rangle$ in $\mathcal{H} - \mathbb{N}$ such that $x^2_{\alpha+1} \leq x_\alpha$ for any $\alpha < \omega_1$. $\Box$

Remark: Every $\mathcal{H}$ in an $\omega-$ultrapower $\mathbb{N}^\omega/\mathcal{F}$ has type two cuts if $2^\omega < 2^{\omega_1}$. ($\mathcal{F}$ is a nonprincipal ultrafilter on ω.)

Corollary 2 *If $card(\mathcal{H}) < 2^{\underline{b}}$, then $\mathcal{H}$ has type two cuts.*

Proof: Let $h \in \mathcal{H} - \mathbb{N}$, then U_h is a cut closed under multiplication and $ci(U_h) \geq \underline{b}$. That implies that there exists a $\underline{b}-$decreasing sequence $\langle x_\alpha : \alpha < \underline{b} \rangle$ in $\mathcal{H} - U_h$ such that $x^2_{\alpha+1} \leq x_\alpha$ for any $\alpha < \underline{b}$. $\Box$

Remark: Assume Martin's Axiom. Then every $\mathcal{H}$ in a $\kappa-$ultrapower $\mathbb{N}^\kappa/\mathcal{F}$ has type two cuts if $\kappa < 2^\omega$. ($\mathcal{F}$ is a nonprincipal ultrafilter on κ.)

Corollary 3 *Let $\mathbb{P}$ be a Suslin tree in a ground model V of ZFC. Then $V^{\mathbb{P}} \models$ Every $\mathcal{H}$ in an $\omega-$ultrapower $\mathbb{N}^\omega/\mathcal{F}$ has type two cuts if $\mathcal{F}$ is in V.*

Proof: Since $\mathcal{F} \in V$, we can build a family of intervals $T = \{[a_s, b_s] : s \in 2^{<\omega_1}\}$ in V as in the construction in the proof of theorem 2. Because there is a new function $f : \omega_1 \to 2$ and there are no new countable sets in the forcing extension, the cut

$$U = \{x \in \mathcal{H} : \exists \alpha < \omega_1 (x \leq a_{f|\alpha})\}$$

is an (ω_1, ω_1) cut. By the same argument as in the proof of theorem 2, we can show that U is a type two cut in $V^{\mathbb{P}}$. $\Box$

Next we construct a bad cut in *ZFC*. Before that we need a result from [KL].

Proposition 2 (Keisler and Leth) *If U is a $(\geq \omega_1, \geq \omega_1)$ cut and $M(U) = \mathbb{N}$, then U is bad.*

Proof: See [KL, Proposition 6.15]. $\Box$

Theorem 3 *There exists a nonstandard universe in which every hyperline $\mathcal{H}$ has bad cuts. $\mathcal{H}$ also has (ω_1, ω_1) cuts M which are closed under multiplication and $J(M) = \mathbb{N}$.*

Proof: Let $\mathcal{F}_\alpha$ be any nonprincipal ultrafilter on ω for an $\alpha < \omega_1$. Let $\mathcal{M}_0$ be a standard superstructure.

Now we build an $\omega_1 + 1-$elementary chain of nonstandard universes by taking

$$\mathcal{M}_{\alpha+1} \cong \mathcal{M}^\omega_\alpha/\mathcal{F}_\alpha$$

and

$$\mathcal{M}_\alpha \cong \bigcup_{\beta<\alpha} \mathcal{M}_\beta$$

if α is a limit ordinal. The ω_1-saturated nonstandard universe we want is $\mathcal{M}_{\omega_1}$.

Let ${}^*\mathbb{N}_\alpha$ be all hyperintegers in $\mathcal{M}_\alpha$, so ${}^*\mathbb{N}^{\nvdash}_{\alpha+\nVdash} \cong \mathbb{N}^\omega/\mathcal{F}_\alpha$ is an initial segment of ${}^*\mathbb{N}_{\alpha+\nVdash} \cong {}^*\mathbb{N}^\omega_\alpha/\mathcal{F}_\alpha$.

Fact: $\forall n \in {}^*\mathbb{N}_\alpha\ (\ltimes \in \mathbb{N} \text{ or } \forall m \in {}^*\mathbb{N}^{\nvdash}_{\alpha+\nVdash}\ (\ltimes \geq \gtrdot))$.

Let $H \in {}^*\mathbb{N}_{\omega_{\nVdash}} - \mathbb{N}$ and $\langle x_\alpha : \alpha < \omega_1\rangle$ be a decreasing sequence in $\mathcal{H} - \mathbb{N}$ such that $x_\alpha \in {}^*\mathbb{N}^{\nvdash}_{\alpha+\nVdash} - \mathbb{N}$ if ${}^*\mathbb{N}^{\nvdash}_{\alpha+\nVdash} \subseteq \mathcal{H}$ or $x_\alpha = H$ otherwise. $\langle x_\alpha : \alpha < \omega_1\rangle$ is coinitial in $\mathcal{H} - \mathbb{N}$.

Now we construct a decreasing sequence of intervals $\langle [a_\alpha, b_\alpha] : \alpha < \omega_1\rangle$ in $\mathcal{H}$ such that $\forall \alpha < \omega_1$

(1) $\forall n \in \mathbb{N}\ (\ltimes\partial_\alpha \leq \partial_{\alpha+\nVdash})$.

(2) $\exists y_\alpha \in \mathcal{H} - \mathbb{N}\ (\curvearrowright_\alpha \leq \curvearrowleft_\alpha$ and $a_\alpha y_\alpha = b_\alpha)$.

(3) $[a_{\alpha+1}, b_{\alpha+1}] \bigcap {}^*\mathbb{N}_\alpha = \emptyset$.

Let a_0 be any nonstandard integer such that $a_0^2 \leq H$, $y_0 = \min\{x_0, a_0\}$ and $b_0 = a_0 y_0$.

Assume we have a_β, b_β and y_β for all $\beta < \alpha$ for some $\alpha < \omega_1$.

Case 1. α is a limit ordinal.

Let $y_\alpha \in \mathcal{H} - \mathbb{N}$ such that $y_\alpha \leq x_\alpha$ and $y_\alpha \leq y_\beta$ for all $\beta < \alpha$. y_α exists since $ci(\mathbb{N}) = \omega_{\nVdash}$. Let , (u, v) be the type of the formulas

$$\{a_\beta \leq u \wedge v \leq b_\beta \wedge u y_\alpha = v : \beta < \alpha\}.$$

Then , (u, v) is countable and finitely satisfiable. Hence there are $a_\alpha = u$ and $b_\alpha = v$ realizing the type , (u, v) by ω_1-saturation.

Case 2. $\alpha = \beta + 1$.

Let $\eta = \min\{\gamma : a_\beta \in {}^*\mathbb{N}_\gamma\}$. Since $\forall n \in \mathbb{N}\ (\ltimes\partial_\beta < {}_\beta)$, there exists a $K_\alpha \in \mathcal{H} - \mathbb{N}$ such that $K_\alpha \leq \min\{x_\alpha, x_\eta\}$ and $a_\beta K_\alpha^2 \leq b_\beta$.

Let $a_\alpha = a_\beta K_\alpha$, $b_\alpha = a_\beta K_\alpha^2$ and $y_\alpha = K_\alpha$.

Claim: $[a_\alpha, b_\alpha] \bigcap {}^*\mathbb{N}_\beta = \emptyset$.

Proof of the claim: Let $\beta_0 = \max\{\beta, \eta\}$. $a_\beta \in {}^*\mathbb{N}_{\beta_\nvdash}$ implies that

$$\forall x \in {}^*\mathbb{N}_{\beta_\nvdash}\ (\forall \ltimes \in \mathbb{N}\ (\curvearrowleft \geq \partial_\beta \ltimes) \to \forall \curvearrowright \in {}^*\mathbb{N}^{\nvdash}_{\beta_\nvdash+\nVdash}\ (\curvearrowleft \geq \partial_\beta \curvearrowright)).$$

Hence

$$\forall x \in {}^*\mathbb{N}_{\beta_\nvdash}\ (\exists \ltimes \in \mathbb{N}\ (\curvearrowleft \leq \partial_\beta \ltimes) \text{ or } x > a_\beta x_{\beta_0}^2)$$

since $x_{\beta_0}^2 \in {}^*\mathbb{N}^{\nvdash}_{\beta_\nvdash+\nVdash}$. Because $\forall n \in \mathbb{N}\ (\partial_\beta \ltimes < \partial_\beta \mathbb{K}_\alpha)$ and $a_\beta K_\alpha^2 \leq a_\beta x_{\beta_0}^2$, $[a_\alpha, b_\alpha] \bigcap {}^*\mathbb{N}_{\beta_\nvdash} = \emptyset$. Hence $[a_\alpha, b_\alpha] \bigcap {}^*\mathbb{N}_\beta = \emptyset$ since $\beta \leq \beta_0$. This ends the proof of the claim.

By Case 1 and Case 2 we have $\bigcap_{\alpha<\omega_1} [a_\alpha, b_\alpha] = \emptyset$.

Let $U = \{x \in \mathcal{H} : \exists\alpha < \omega_1(x \le a_\alpha)\}$, then U is an (ω_1, ω_1) cut.

$\forall x \in \mathcal{H} - \mathbb{N}\ \exists\alpha < \omega_{\not\Vdash}(\curvearrowleft_\alpha < \curvearrowleft)$ since $\langle x_\alpha : \alpha < \omega_1\rangle$ is coinitial in $\mathcal{H} - \mathbb{N}$. Then $xa_\alpha \ge x_\alpha a_\alpha \ge y_\alpha a_\alpha = b_\alpha$. Hence $xa_\alpha \notin U$. So $M(U) = \mathbb{N}$ and this implies U is bad by Proposition 2.

Now we prove the second assertion of the theorem.

Let $h \in \mathcal{H} - \mathbb{N}$ such that $h^h \le H$. Let U be a bad cut in $[1, h]$ from the proof above and let $M = h^U$, then M is an (ω_1, ω_1) cut which is closed under multiplication.

Now we prove that $J(M) = \mathbb{N}$.

Let $x \in \mathcal{H} - \mathbb{N}$, we want an internal function f from $[1, x]$ to $\mathcal{H}$ which crosses M.

Since $M(U) = \mathbb{N}$ there exists an $a \in U$ such that $xa \notin U$. For every $y \in [1, x]$ let $f(y) = h^{ay}$. Then f is internal. f crosses M because $f(x) \notin M$ and $h^{ay} \in M$ implies $h^{a(y+1)} = h^{ay}h^a \in M$, so the range of f is unbounded in M. □

Theorem 4 *Every hyperline $\mathcal{H}$ in an ω_2–saturated nonstandard universe has $(\ge \omega_1, \ge \omega_1)$ cuts M which are closed under multiplication and $J(M) = \mathbb{N}$.*

Proof: Let $\langle x_\alpha : \alpha < \omega_1\rangle$ be a decreasing sequence in $\mathcal{H} - \mathbb{N}$ such that $x_{\alpha+1}^2 \le x_\alpha$ and $M = \{x \in \mathcal{H} : \forall\alpha < \omega_1(x \le x_\alpha)\}$, then M is a cut closed under multiplication and $ci(M) = \omega_1$.

Because $ci(\mathbb{N}) \ge \omega_{\not\models}$, $cf(M) \ne \omega$ by Lemma 1.

For any $x \in \mathcal{H} - \mathbb{N}$ there is a strictly decreasing sequence $\langle y_\alpha : \alpha < \omega_1\rangle$ with $x = y_0$. For each $\alpha < \omega_1$ let

$$\mathcal{F}_\alpha = \{f : f \text{ is an increasing internal function from } [1, x] \text{ to } \mathcal{H} \text{ such that } f(y_\alpha) = x_\alpha\}.$$

For any finite subset $F = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of ω_1 such that $\alpha_1 > \alpha_2 > \ldots > \alpha_n$ we define

$$f_F(y) = \begin{cases} 1 & \text{if } 1 \le y < y_{\alpha_1} \\ x_{\alpha_i} & \text{if } y_{\alpha_i} \le y < y_{\alpha_{i+1}} \text{ for } 1 \le i < n \\ x_{\alpha_n} & \text{if } y_{\alpha_n} \le y \le x \end{cases}$$

Then $f_F \in \bigcap_{\alpha\in F} \mathcal{F}_\alpha$. This implies that $\{\mathcal{F}_\alpha : \alpha < \omega_1\}$ has finite intersection property. Therefore $\bigcap_{\alpha<\omega_1} \mathcal{F}_\alpha \ne \emptyset$ by ω_2–saturation. Let $f \in \bigcap_{\alpha<\omega_1} \mathcal{F}_\alpha$. Then f is an internal function from $[1, x]$ to $\mathcal{H}$, which crosses M. Hence $J(M) = \mathbb{N}$. □

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