# BETTER NONSTANDARD UNIVERSES WITH APPLICATIONS

R. JIN
*Rutgers University*
*New Brunswick, NJ 08903 USA*
*and*
*College of Charleston*
*Charleston, SC 29424 USA*
*email:* `rjin@math.rutgers.edu`

## 1. Introduction

There are various reasons why some nonstandard universes are considered to be better than others. Different people may have different opinions and may choose different nonstandard universes to work within for different purposes. We think it is reasonable to consider that a nonstandard universe which possesses stronger power for deriving results in either standard or nonstandard mathematics, or which supplies more convenient tools so that, in practice, some complicated derivation procedures admit significant simplifications, is better than the one which doesn't.

In the early time of nonstandard analysis what people needed from a nonstandard universe is basically the existence of an infinitesimal. Since the introduction of $\kappa$-saturation ($\kappa$-saturation was first singled out in [23]), nonstandard analysis has experienced great prosperity. Given an infinite regular cardinal $\kappa$, a nonstandard universe $\mathcal{V}$ is said to be $\kappa$-saturated if every family of less then $\kappa$ internal sets in $\mathcal{V}$ with finite intersection property has non-empty intersection. $\aleph_1$-saturation is also called countable saturation. Countable saturation is one of the most commonly used properties in nonstandard analysis. Countable saturation gives us great convenience in handling countable sequences of internal sets or countable constructions with each step internal. For example, countable saturation laid down a foundation for the invention of Loeb measure construction by P. Loeb in [22], and guarantees the completeness of the nonstandard hull of a metric space. Part of the reason why countable saturation offers so much was made clear by C. W. Henson and H. J. Keisler in [10], where they showed that: second-order nonstandard arithmetic with countable saturation implies exactly the same sentences of standard second-order arithmetic as are implied by the system of standard third-order arithmetic.[1] Bearing those reasons in mind we consider that a nonstandard universe with countable saturation is better than the one without it.

Could we find even a better property for nonstandard universes? By that we mean to find a saturation-like property essentially stronger and evidently more useful than countable saturation for dealing with countable sequences of internal objects.

In §2 we introduce a property called *the isomorphism property* suggested by C. W. Henson in [7]. Henson's isomorphism property is elegant, stronger than countable saturation and very useful. In [8] Henson pointed out that the nonstandard universe satisfying the isomorphism property could be an ultrapower[2] of the standard superstructure by a result of S. Shelah [30]. In both [7] and [8] Henson studied nonstandard hulls of Banach spaces in a nonstandard universe satisfying the isomorphism property. The reader should see there that the isomorphism property makes the subject simple and clear. With three more sophisticated applications we illustrate more of the strength of the isomorphism property.

In §3 we push the issue further towards the direction of §2. We introduce a property called *the special model axiom* suggested by D. Ross in [26] and a property called *full saturation*. The special model axiom is even stronger than the isomorphism property and full saturation is the strongest among them. Ross mentioned in [26] that the nonstandard universe satisfying the special model axiom could be an ultralimit of the standard superstructure. But it couldn't be an ultrapower without extra set theoretic assumptions beyond ZFC. The existence of a fully saturated nonstandard universe is even questionable. In fact, the existence of a fully saturated nonstandard universe is undecidable in ZFC. In §3 we present one application of the special model axiom and one application of full saturation. With those two applications the reader should have a taste of the power of those two properties.

Besides the reasons we mentioned above, there might be some other reasons why some nonstandard universes are better than others. For example, one may consider that a nonstandard universe is better if the hy-

[1]The complete result in [10] is: Let $k \geqslant 2, n \in \mathbb{N}$ and $1 \leqslant m \leqslant \infty$. Then ${}^*PA^{(k)} + \Pi_m^{k-1} - CA + \beth_n^+$-saturation and $PA^{(k+n+1)} + \Pi_m^{k+n} - CA$ have the same consequences in the language of $PA^{(k)}$.

[2]By an ultrapower or an ultralimit in this chapter we always mean a nonstandard universe of the standard superstructure obtained by a bounded ultrapower or a bounded ultralimit construction, respectively.

perreal field in it bears a stronger resemblance to the standard real field. In §4 we introduce such a property called the $\lambda$-Bolzano-Weierstrass property for some uncountable regular cardinal $\lambda$. By a result of H. J. Keisler and J. H. Schmerl in [21] one can construct a countably saturated nonstandard universe satisfying the $\lambda$-Bolzano-Weierstrass property for some $\lambda$ *e.g.* $\lambda = (2^{\aleph_0})^+$. But the $\lambda$-Bolzano-Weierstrass property is inconsistent with the isomorphism property. This shows that the better-ness of nonstandard universes in §4 takes a different direction from the better-ness in the sense of §§2,3.

Finally, we would like to mention, but not to include the details, one more kind of nonstandard universes called *Minimal Nonstandard Universe*. A minimal nonstandard universe is an ultrapower of the standard superstructure modulo a selective ultrafilter on a countable set. Note that such nonstandard universes are countably saturated. Some interesting applications of minimal nonstandard universes to measure theory were obtained by C. W. Henson and B. Wattenberg in [11] and by M. Benedikt in [2] and [3]. But the existence of a selective ultrafilter is undecidable in ZFC. For further study on this subject the reader should consult [11],[2] and [3].

In the end of each section we include some exercises. Star symbols $*$ are added to indicate hard-ness.

**Remark** The definitions of the isomorphism property and the special model axiom could be trivially generalized to the $\kappa$-isomorphism property and the $\kappa$-special model axiom for any infinite regular cardinal $\kappa$. In fact, it was the $\kappa$-forms of those properties which were originally defined. In this chapter we need only the forms with $\kappa = \aleph_1$ for all applications. Hence we drop the cardinal $\kappa$ in the definitions for simplicity. Similar to the generalization of countable saturation to $\kappa$-saturation the reader should be able to generalize those properties, without any difficulties, to the $\kappa$-forms for future applications.

We assume the reader has a basic training in model theory. A one-semester graduate level model theory course should be more than enough. Our notation in model theory are consistent with that in [5]. By a nonstandard universe in this chapter we mean always a triple $(\mathbb{V}(\mathbb{N}), \mathbb{V}({}^*\mathbb{N}), *)$, where $\mathbb{V}(\mathbb{N})$ is the superstructure on $\mathbb{N}$, the set of all standard natural numbers as urelements, $\mathbb{V}({}^*\mathbb{N})$ is the superstructure on ${}^*\mathbb{N}$, and $*$ is a nonstandard extension from $\mathbb{V}(\mathbb{N})$ to $\mathbb{V}({}^*\mathbb{N})$ defined in the first chapter by C. W. Henson in this volume. We call $\mathbb{V}(\mathbb{N})$ the standard superstructure. Our restriction on the set of urelements for the standard superstructure to be $\mathbb{N}$ instead of an arbitrary set $S$ is just for simplicity. Note that the standard superstructure $\mathbb{V}(\mathbb{N})$ contains a copy of the standard real field. In order to avoid writing the whole expression $(\mathbb{V}(\mathbb{N}), \mathbb{V}({}^*\mathbb{N}), *)$ we always denote $\mathcal{V}$

for a nonstandard universe. We write $\alpha, \beta, \gamma, \ldots$ for ordinal numbers and denote $\omega$ for the first infinite ordinal. For any set $S$ we denote $card(S)$ for the (external) cardinality of $S$. For a model $\mathfrak{A}$ with base set $A$ we often write $card(\mathfrak{A})$ for the cardinality of $A$. We reserve the notion $|A|$ for the internal cardinality of $A$ when $A$ is an internal set. For each nonstandard universe $\mathcal{V}$ we denote $\Xi_{\mathcal{V}}$ for the cardinality of the family of all internal sets in $\mathcal{V}$. Note that $\Xi_{\mathcal{V}} = \bigcup_{n\in\omega} card({}^*\mathbb{V}_n(\mathbb{N}))$. By a language we always mean a first-order language.

In order to be coherent we would like to give equivalent forms of countable saturation and $\kappa$-saturation in next exercise, in terms of internally presented structures. First we define internally presented structures.

**Definition 1.1** *Given a nonstandard universe $\mathcal{V}$ and given a language $\mathcal{L}$. An $\mathcal{L}$-structure $\mathfrak{A}$ is called internally presented in $\mathcal{V}$ iff the base set and every $\mathcal{L}$-relation or $\mathcal{L}$-function in $\mathfrak{A}$ are internal in $\mathcal{V}$.*

Note that an internally presented structure itself may not be internal when $\mathcal{L}$ contains infinitely many symbols.

### 1.1. EXERCISES

**Exercise 1.2** *Show that*

*(1) a nonstandard universe $\mathcal{V}$ is countably saturated iff for any countable language $\mathcal{L}$, for every internally presented $\mathcal{L}$-structure $\mathfrak{A}$, and for any set of first-order $\mathcal{L}$-formulas , $(x)$ with one free variable $x$ consistent with $Th(\mathfrak{A})$, the set , $(x)$ is realizable (or satisfiable) in $\mathfrak{A}$.*

*(2) for any infinite regular cardinal $\kappa$ a nonstandard universe $\mathcal{V}$ is $\kappa$-saturated iff for any language $\mathcal{L}$ with $card(\mathcal{L}) < \kappa$, for every internally presented $\mathcal{L}$-structure $\mathfrak{A}$, and for any set of first-order $\mathcal{L}$-formulas , $(x)$ with one free variable $x$ consistent with $Th(\mathfrak{A})$, the set , $(x)$ is realizable in $\mathfrak{A}$.*

## 2. The Isomorphism Property

**Definition 2.1** *A nonstandard universe $\mathcal{V}$ is said to satisfy the isomorphism property iff any two elementarily equivalent, internally presented $\mathcal{L}$-structures for some countable language $\mathcal{L}$, are isomorphic.*

**Remark** Given two structures of some language, people might check the elementary equivalence between them by playing an Ehrenfeucht-Fraissé game or constructing a set of partial isomorphisms with back-forth property [1]. So if one want to show that two internally presented structures of some countable language in a nonstandard universe satisfying the isomorphism property are isomorphic, he might need only to show that he could win the related Ehrenfeucht-Fraissé game or construct a set of partial isomorphisms with back-forth property.

From now on we will write *IP* for the isomorphism property.

**Proposition 2.2** *(C. W. Henson [7]) If $\mathcal{V}$ satisfies IP, then every infinite internal set has cardinality $\Xi_{\mathcal{V}}$.*

**Proof:** It suffices to prove that any two infinite internal sets have same cardinality. Let $A$ and $B$ be two internal sets. Then $A$ and $B$ are two elementarily equivalent structures of the empty language. By *IP* there is a bijection between them. $\square$

**Proposition 2.3** *(C. W. Henson [7]) If $\mathcal{V}$ satisfies IP, then $\mathcal{V}$ is countably saturated.*

Proposition 2.3 is a trivial corollary of Lemma 2.6. See the remark right after Lemma 2.6.

**Proposition 2.4** *(C. W. Henson [7] and S. Shelah [30]) Suppose $\kappa$ is an infinite cardinal such that $\kappa^{\beth_\omega} = \kappa$.[3] Then there is an ultrafilter $\mathcal{F}$ on $\kappa$ such that the ultrapower of the standard superstructure modulo $\mathcal{F}$ satisfies IP.*

The first two applications need an equivalent form of *IP* in terms of the realizability of a "second-order type" due to S. Shelah and the author [16].

Let $\mathcal{L}$ be a countable language and $X$ be a new $n$-ary relation symbol not in $\mathcal{L}$. Suppose , $(X)$ is a set of first-order $\mathcal{L}\cup\{X\}$-sentences and $\mathfrak{A}$ is an $\mathcal{L}$-structure with base set $A$. We say that , $(X)$ is realizable in $\mathfrak{A}$ iff there exists $R \subseteq A^n$ such that

$$(\mathfrak{A}, R) \models \varphi(R)$$

for every $\varphi(X) \in$ , $(X)$.

**Definition 2.5** *A nonstandard universe $\mathcal{V}$ is said to satisfy the resplendency property iff for any countable language $\mathcal{L}$, for any $n$-ary new relation symbol $X$ not in $\mathcal{L}$, for any internally presented $\mathcal{L}$-structure $\mathfrak{A}$, and for any set of first-order $\mathcal{L} \cup \{X\}$-sentences , $(X)$, if , $(X) \cup Th(\mathfrak{A})$ is consistent, then , $(X)$ is realizable in $\mathfrak{A}$.*

**Lemma 2.6** *(R. Jin and S. Shelah [16]) A nonstandard universe $\mathcal{V}$ satisfies IP iff $\mathcal{V}$ satisfies the resplendency property.*

**Remark** Comparing with Exercise 1.2, it is easy to see that the resplendency property is a natural generalization of countable saturation. To get

[3] Note that $\beth_0 = \aleph_0$, $\beth_{n+1} = 2^{\beth_n}$ and $\beth_\omega = \bigcup_{n<\omega} \beth_n$. Note also that the cardinality of the standard superstructure is $\beth_\omega$.

countable saturation simply view the new relation symbol $X$ in the definition as an individual variable. This justifies why we call *IP* a saturation-like property.

### 2.1. UNLIMITED LOEB MEASURE SPACES

Given an internal *finite additive measure space $(\Omega, \mathcal{A}, \mu)$ such that $\mathcal{A}$ contains every singleton set, $\mu(\Omega) > n$ for every $n \in \mathbb{N}$, and $\mu(\{x\}) < \frac{1}{n}$ for every $x \in \Omega$ and $n \in \mathbb{N}$. Following the Loeb construction we can extend the standard finitely additive measure ${}^\circ\mu$ on $\mathcal{A}$ to a countably additive measure $L_\mu$ on the $\sigma$-algebra generated by $\mathcal{A}$. But this measure is not complete. In order to make the measure complete, we have to throw in the subsets of all $L_\mu$-measure zero sets. We need to define a $\sigma$-algebra of all "measurable" sets for this. For each $S \subseteq \Omega$ let the outer measure of $S$ be

$$\overline{{}^\circ\mu}(S) = \inf\{{}^\circ\mu(A) : A \in \mathcal{A} \wedge S \subseteq A\}$$

and the inner measure of $S$ be

$$\underline{{}^\circ\mu}(S) = \sup\{{}^\circ\mu(A) : A \in \mathcal{A} \wedge A \subseteq S\}.$$

If $L_\mu(\Omega)$ were finite, then it would be immediate that a set $S$ is measurable iff its outer measure and inner measure coincide. But now we have $L_\mu(\Omega) = \infty$ (this is what the word "unlimited" means). By a conventional method we define the $\sigma$-algebra $\mathcal{B}$ of all "measurable" sets as following:

$$\mathcal{B} = \{S \subseteq \Omega : (\forall A \in \mathcal{A})\ ({}^\circ\mu(A) < \infty \rightarrow \overline{{}^\circ\mu}(S \cap A) = \underline{{}^\circ\mu}(S \cap A))\}.$$

Note that $\mathcal{B}$ is a $\sigma$-algebra, $\mathcal{A} \subseteq \mathcal{B}$, and every subset of an $L_\mu$-measure zero set is in $\mathcal{B}$.

When trying to extend $L_\mu$ to a complete measure on $\mathcal{B}$ one encounters a problem. For an $S \in \mathcal{B}$ should one let $L_\mu(S) = \overline{{}^\circ\mu}(S)$ or let $L_\mu(S) = \underline{{}^\circ\mu}(S)$? There would have no problem if every $S \in \mathcal{B}$ had same outer measure and inner measure. But this may not be the case. So one has at least two different ways to extend $L_\mu$ to a complete countably additive measure on $\mathcal{B}$ if there are some sets in $\mathcal{B}$ having different outer measure and inner measure.

**Theorem 2.7** *(R. Jin and S. Shelah [16]) Suppose $\mathcal{V}$ satisfies IP. Then every unlimited, non-atomic Loeb measure space has a measurable subset with infinite outer measure and zero inner measure.*

**Proof:** Given an unlimited, non-atomic Loeb measure space $(\Omega, \mathcal{B}, L_\mu)$ generated by an internal space $(\Omega, \mathcal{A}, \mu)$ as above. We need to find a set $S \subseteq \Omega$ such that for any $A \in \mathcal{A}$, if $\mu(A)$ is finite, then $\overline{{}^\circ\mu}(A \cap S) = 0$, and if $S \subseteq A$, then $L_\mu(A) = \infty$.

We will use the resplendency property in the proof. The main idea is the following: First we write a set of sentences , $(X)$ to express that $X$ is the desired set, *i.e.* a measurable set with infinite outer measure and zero inner measure. Then we prove that , $(X)$ is consistent with our internally presented structure. By the resplendency property one could find a set realizing , $(X)$ in our structure. Therefore, the desired set exists.

We first form an internally presented structure $\mathfrak{A}$ of a finite language $\mathcal{L}_{\mathfrak{A}}$. Let

$$\mathfrak{A} = (\Omega \cup \mathcal{A} \cup {}^*\mathbb{R}; \Omega, \mathcal{A}, {}^*\mathbb{R}, \in, \mu, \cap, \smallsetminus, +, \cdot, <, 0, 1),$$

where $\Omega, \mathcal{A}$ and ${}^*\mathbb{R}$ are unary relations, $\in$ is a membership relation between $\Omega$ and $\mathcal{A}$, $\mu$ is the measure function from $\mathcal{A}$ to ${}^*\mathbb{R}$, $\cap$ and $\smallsetminus$ are Boolean operators on $\mathcal{A}$, and $({}^*\mathbb{R}; +, \cdot, <, 0, 1)$ is the hyperreal field in $\mathcal{V}$. Notice that we abuse the notation here. Rigorously, we should use ${}^*\in, {}^*\cap, {}^*+$, etc. We often omit ${}^*$ when the meaning is obvious. Let $X$ be a new unary relation symbol not in $\mathcal{L}_{\mathfrak{A}}$. We now express $X$ as a desired set by a set of $\mathcal{L} \cup \{X\}$-sentences , $(X)$. Let

$$, (X) = \{\phi(X)\} \cup \{\psi_n(X) : n \in \mathbb{N}\} \cup \{\chi_n(X) : n \in \mathbb{N}\},$$

where

$$\phi(X) =: \forall x (X(x) \to \Omega(x)),$$

$$\begin{aligned} \psi_n(X) =: \quad & \forall U (\mathcal{A}(U) \wedge \mu(U) < n \to \\ & \exists V (\mathcal{A}(V) \wedge \forall x (X(x) \wedge x \in U \to x \in V) \wedge \mu(V) < \tfrac{1}{n})), \end{aligned}$$

$$\chi_n(X) =: \forall U (\mathcal{A}(U) \wedge \forall x (X(x) \to x \in U) \to \mu(U) > n).$$

Note that in $\mathfrak{A}$, the elements $n$ and $\frac{1}{n}$ are definable. The sentence $\phi(X)$ says that $X$ is a subset of $\Omega$, the sentence $\psi_n(X)$ says that if $U \in \mathcal{A}$ has measure $< n$, then the outer measure of $U \cap X$ is $< \frac{1}{n}$, and $\chi_n(X)$ says that if $U \in \mathcal{A}$ and $X \subseteq U$, then $U$ has measure $> n$. So if the set , $(X)$ is realized by some set $S$ in $\mathfrak{A}$, then $S$ is clearly the desired set.

By the resplendency property we need only to prove that , $(X)$ is consistent with $Th(\mathfrak{A})$.

**Claim** , $(X) \cup Th(\mathfrak{A})$ is consistent.

Proof of Claim: By Downward Löwenheim-Skolem theorem we can find a countable $\mathcal{L}_{\mathfrak{A}}$-structure $\mathfrak{A}_0$ such that $\mathfrak{A}_0 \models Th(\mathfrak{A})$. Let $\Omega_0 \cup \mathcal{A}_0 \cup \mathbb{R}_0$ be the base set of $\mathfrak{A}_0$. It suffices to prove that , $(X)$ is realizable in $\mathfrak{A}_0$. So we want to find a set $S_0 \subseteq \Omega_0$ such that $S_0$ realizes , $(X)$ in $\mathfrak{A}_0$. Let $\{A_n : n \in \mathbb{N}\}$ be an enumeration of the set

$$\{A \in \mathcal{A}_0 : \exists n \in \mathbb{N} (\mu(A) < n)\}.$$

We choose $x_n$ inductively such that

$$x_n \in \Omega_0 \smallsetminus ((\bigcup_{k=0}^{n-1} A_k) \cup \{x_k : k < n\}).$$

The set $\Omega_0 \smallsetminus ((\bigcup_{k=0}^{n-1} A_k) \cup \{x_k : k < n\})$ is non-empty because it has unlimited measure. Let $S_0 = \{x_n : n \in \mathbb{N}\}$. It is easy to check that $(\mathfrak{A}_0, S_0)$ realizes , $(X)$. $\square$

**Remark:** The result in the theorem was first proved by Henson [9] in a nonstandard universe called a polyenlargement. Then Ross in [26] proved the result by assuming the special model axiom. It is still open if one could settle the result by countable saturation or $\kappa$-saturation for any infinite regular cardinal $\kappa$.

## 2.2. THE EXISTENCE OF BAD CUTS

Let $H$ be a hyperfinite integer and let $\mathcal{H} = \{0, 1, \ldots, H-1\}$. $\mathcal{H}$ is called a hyperfinite time line in [20]. Let $L_\mu$ be the Loeb probability measure on $\mathcal{H}$ generated by the internal normalized uniform counting measure $\mu$. Through a standard part map $x \mapsto {}^{\circ}(\frac{x}{H})$ the hyperfinite time line $\mathcal{H}$ together with $L_\mu$ could be closely associated with the standard unit interval $[0,1]$ together with Lebesgue measure on it. What about the order topology? Could one define a topology on $\mathcal{H}$ which resembles the usual order topology on $[0,1]$? Note that the natural order topology on $\mathcal{H}$ is discrete, hence not interesting. H. J. Keisler and S. C. Leth in [20] gave the definition of a $U$-topology on $\mathcal{H}$ for each cut $U \subseteq \mathcal{H}$.

An infinite initial segment $U$ of ${}^*\mathbb{N}$ is called a cut (or additive cut in some literature) if $a + b \in U$ for any $a, b \in U$. Given a cut $U \subseteq \mathcal{H}$, a set $O \subseteq \mathcal{H}$ is $U$-open iff for any $a \in O$ there exists a $d \in \mathcal{H} \smallsetminus U$ such that $[a-d, a+d] \subseteq O$, where $[a-d, a+d] = \{x \in \mathcal{H} : a - d \leqslant x \leqslant a + d\}$. All $U$-open sets form a $U$-topology. Given a point $x \in \mathcal{H}$, by the $U$-monad of $x$ we mean the set $\{y \in \mathcal{H} : |y - x| \in U\}$. Note that no $U$-topology could be Hausdorff because it could not separate two points within a $U$-monad. But if we view each $U$-monad as a "point" and consider the natural order of those monads, then the resulting "order" topology is just the $U$-topology.

With a $U$-topology we can define $U$-nowhere dense sets and $U$-meager sets (recall that a meager set is a countable union of nowhere dense sets).

**Definition 2.8** *(H. J. Keisler and S. C. Leth [20]) A cut $U$ is called a good cut if there exists, in $\mathcal{H}$, a $U$-meager set of Loeb measure one.*

Recall that the standard unit interval has a meager set of Lebesgue measure one. So a good cut $U$ will make $\mathcal{H}$ together with Loeb measure and the $U$-topology much like $[0,1]$ together with Lebesgue measure and the order topology.

Keisler and Leth proved in [20] that almost all cuts are good. They were also able to construct a bad cut in some nonstandard universe under an assumption beyond ZFC in set theory, *e.g.* $2^{\aleph_0} < 2^{\aleph_1}$. In next theorem we construct a bad cut in every nonstandard universe satisfying *IP* in ZFC.

**Theorem 2.9** *([14]) If $\mathcal{V}$ satisfies IP, then every hyperfinite time line has bad cuts.*

In the proof we need an equivalent form of bad-ness from [20]. Given a cut $U$ in ${}^*\mathbb{N}$, an internal increasing function $f : \{0, 1, \ldots, L_f\} \mapsto {}^*\mathbb{N}$ for some $L_f \in {}^*\mathbb{N}$ is called a $U$-crossing sequence iff the set $U \cap \{f(n) : n \leqslant L_f\}$ is upper unbounded in $U$.

**Lemma 2.10** *(H. J. Keisler and S. C. Leth [20], Proposition 4.5) For any cut $U \subseteq \mathcal{H}$ the following are equivalent.*

*(1) There is a $U$-meager set of Loeb measure one.*

*(2) There is a $U$-crossing sequence $f$ such that*

$$\sum_{n<L_f} \frac{f(n)}{f(n+1)} < 1.$$

**Proof of Theorem 2.9:** By Lemma 2.10 it suffices to construct a cut $U \subseteq \mathcal{H}$ such that for every $U$-crossing sequence $f$ the sum

$$\sum_{n<L_f} \frac{f(n)}{f(n+1)}$$

is not finite. The main idea of the proof is similar to the idea in the proof of Theorem 2.7. Let $\mathcal{F}$ be the set of all internal increasing functions $f$ from $\{0, 1, \ldots, L_f\}$ for some $L_f \in \mathcal{H}$, to $\mathcal{H}$. Then $\mathcal{F}$ is internal. We form an internally presented structure $\mathfrak{A}$. Let

$$\mathfrak{A} = (\mathcal{H} \cup \mathcal{F} \cup {}^*\mathbb{R}; \mathcal{H}, \mathcal{F}, {}^*\mathbb{R}, R, S, +, \cdot, <, 0, 1),$$

where $\mathcal{H}, \mathcal{F}$ and ${}^*\mathbb{R}$ are unary relations, $R$ is a ternary relation such that $\langle a, b, f\rangle \in R$ iff $f \in \mathcal{F}$, $a \in dom(f)$ and $f(a) = b$, $S$ is a function from $\mathcal{F}$ to ${}^*\mathbb{R}$ such that for any $f \in \mathcal{F}$

$$S(f) = \sum_{n<L_f} \frac{f(n)}{f(n+1)},$$

and $({}^*\mathbb{R}; +, \cdot, <, 0, 1)$ is the hyperreal field in $\mathcal{V}$. Let $\mathcal{L}$ be the language of $\mathfrak{A}$. Note that the following first-order $\mathcal{L}$-sentences are true in $\mathfrak{A}$.

$$\theta_n =: \exists x(\mathcal{H}(x) \wedge x \geqslant n \wedge \forall y(\mathcal{H}(y) \rightarrow y \leqslant x))$$

for each $n \in \mathbb{N}$, and

$$\begin{aligned} \eta =: \quad & \forall f \forall x \forall y (\mathcal{F}(f) \wedge \mathcal{H}(x) \wedge \mathcal{H}(y) \wedge x < y \rightarrow \\ & \exists g (\mathcal{F}(g) \wedge range(g) = range(f) \cap [x, y])), \end{aligned}$$

where the formula $range(g) = range(f) \cap [x, y]$ is an abbreviation of the first-order $\mathcal{L}$-formula

$$\forall z (\exists u R(u, z, g) \leftrightarrow x \leqslant z \wedge z \leqslant y \wedge \exists u R(u, z, f)).$$

Let $X \notin \mathcal{L}$ be a unary predicate symbol. We define , $(X)$ to be the set of $\mathcal{L} \cup \{X\}$-sentences which contains exactly the following:

$$\varphi_1(X) =: \forall x (X(x) \rightarrow \mathcal{H}(x))$$

$$\varphi_2(X) =: \forall x \forall y (x \leqslant y \wedge \mathcal{H}(x) \wedge X(y) \rightarrow X(x))$$

$$\varphi_3(X) =: \forall x \forall y (X(x) \wedge X(y) \rightarrow X(x + y))$$

$$\begin{aligned} \psi_n =: \quad & \forall f (\mathcal{F}(f) \wedge \forall x (X(x) \rightarrow \\ & \exists y \exists z (R(y, z, f) \wedge X(z) \wedge x \leqslant z)) \rightarrow S(f) \geqslant n) \end{aligned}$$

for each $n \in \mathbb{N}$.

Note that the sentences $\varphi_1(X)$, $\varphi_2(X)$ and $\varphi_3(X)$ say that $X$ is a cut in $\mathcal{H}$. The sentences $\psi_n(X)$ for $n \in \mathbb{N}$ say that if $f$ is a crossing sequence of $X$, then the internal sum $S(f)$ is not finite. So , $(X)$ describes that $X$ is a bad cut by Lemma 2.10. So if , $(X)$ is realizable in $\mathfrak{A}$, then $\mathcal{H}$ must contain a bad cut. By Lemma 2.6 it suffices to show that , $(X) \cup Th(\mathfrak{A})$ is consistent.

Let $\mathfrak{A}'$ be a countable elementary submodel of $\mathfrak{A}$. Then $Th(\mathfrak{A}') = Th(\mathfrak{A})$. If we show that , $(X)$ is realizable in $\mathfrak{A}'$, then it is clear that $Th(\mathfrak{A}) \cup$ , $(X)$ is consistent.

**Claim** , $(X)$ is realizable in $\mathfrak{A}'$.

Proof of Claim: Let $A' = \mathcal{H}' \cup \mathcal{F}' \cup \mathbb{R}'$ be the base set of $\mathfrak{A}'$ and let $\mathcal{F}' = \{f_i : i \in \mathbb{N}\}$. We now inductively construct an increasing sequence $\langle a_i : i \in \mathbb{N} \rangle$ and a decreasing sequence $\langle b_i : i \in \mathbb{N} \rangle$ in $\mathcal{H}'$ such that for each $i \in \mathbb{N}$

(a) $a_i < b_i$,

(b) $2a_i < a_{i+1}$,

(c) $b_i / a_i$ is not finite in $\mathbb{R}'$,

(d) if $f \in \mathcal{F}'$ such that

$$range(f) = range(f_i) \cap \{x \in \mathcal{H}' : a_i \leqslant x \leqslant b_i\},$$

if $S(f)$ is finite in $\mathbb{R}'$ and if $L_f$ is not finite, then there is a $k \in \{0, 1, \ldots, L_f - 1\} \cap \mathcal{H}'$ such that $f(k) \leqslant a_{i+1}$ and $f(k+1) \geqslant b_{i+1}$ (*i.e.*, $f$ has a jump across the interval $(a_{i+1}, b_{i+1})$).

We show first that the claim follows from the construction. Let

$$U = \{x \in \mathcal{H}' : (\exists i \in \mathbb{N})\,(x \leqslant a_i)\}.$$

Then $\varphi_1(U)$ and $\varphi_2(U)$ are trivially true in $(\mathfrak{A}', U)$. The sentence $\varphi_3(U)$ is true in $(\mathfrak{A}', U)$ by condition (b). Given any $f_i \in \mathcal{F}'$ such that $f_i$ is a crossing sequence of $U$. To show that $\psi_n(U)$ is true in $(\mathfrak{A}', U)$ for any $n \in \mathbb{N}$ we need only to show that $S(f_i)$ is not finite. Suppose $S(f_i)$ is finite. By the fact that $\eta$ is true in $\mathfrak{A}'$ there exists a $g \in \mathcal{F}'$ such that

$$range(g) = range(f_i) \cap \{x \in \mathcal{H}' : a_i \leqslant x \leqslant b_i\}.$$

Then $S(g)$ is also finite because $S(g) \leqslant S(f_i)$. Since $f_i$ is a crossing sequence of $U$, $a_i \in U$ and $b_i \notin U$, then $g$ is also a crossing sequence of $U$. Hence $L_g$ is not finite (since no finite sequence could be a crossing sequence of any cut). By condition (d) we know that $g$ has a jump from $a_{i+1}$ to $b_{i+1}$, *i.e.* $g(k) \leqslant a_{i+1}$ and $g(k+1) \geqslant b_{i+1}$ for some $k \in dom(g)$. So $g$ can't be a crossing sequence of $U$, a contradiction.

We now do the inductive construction. Choose any $a_1$ and $b_1$ in $\mathcal{H}'$ such that $b_1/a_1$ is not finite (for example, $a_1 = 1$ and $b_1 = H$). Suppose we have found $\langle a_i : i < k\rangle$ and $\langle b_i : i < k\rangle$ for some $k > 1$ such that they satisfy the conditions (a)—(d). We need to find $a_k$ and $b_k$. Let $g \in \mathcal{F}'$ be such that

$$range(g) = range(f_{k-1}) \cap \{x \in \mathcal{H}' : a_{k-1} \leqslant x \leqslant b_{k-1}\}.$$

Case 1: $S(g)$ is not finite or $L_g$ is finite. Simply let $a'_k = a_{k-1}$ and $b'_k = b_{k-1}$.

Case 2: $S(g)$ is finite and $L_g$ is not finite. Let $m \in \mathbb{N}$ be such that $S(g) < m$. Since $g$ is an element in $\mathfrak{A}'$ and $\mathfrak{A}' \preceq \mathfrak{A}$, then there is a $t$ in $\mathfrak{A}'$ such that

$$t = \min\{\frac{g(n)}{g(n+1)} : n \in \mathcal{H}' \wedge n < L_g\}.$$

Let $n_0 \in \mathcal{H}'$ and $n_0 < L_g$ be such that $t = g(n_0)/g(n_0+1)$. Then

$$tL_g \leqslant \sum_{n<L_g} \frac{g(n)}{g(n+1)} = S(g) < m.$$

So we have $g(n_0+1)/g(n_0) \geqslant L_g/m$. Now let $a'_k = g(n_0)$ and $b'_k = g(n_0+1)$.

Clearly, we have that $b'_k/a'_k$ is not finite. Let $a_k = 2a'_k$ and $b_k = b'_k - 1$. Then it is easy to see that $b_k/a_k$ is still not finite. Now it is obvious that the sequences

$$\langle a_i : i < k+1 \rangle \text{ and } \langle b_i : i < k+1 \rangle$$

satisfy conditions (a)—(d). □

## 2.3. DEFINABILITY IN CONSTRAINT QUERY LANGUAGES

Although the following application is from database theory, we will present it in a purely model theoretic way. No knowledge on database theory is assumed. The application is due to M. Benedikt *et al.* [4].

Let $\mathcal{L}$ be the language of ordered fields and let $\mathcal{L}' = \{X_1, \ldots, X_n\}$, where $X_i$ is an $n_i$-ary relation symbol not in $\mathcal{L}$. Let $\mathcal{R}$ be the standard real field.

By an $\mathcal{L}'$-finite expansion of $\mathcal{R}$ we mean an expansion of $\mathcal{R}$ to an $\mathcal{L} \cup \mathcal{L}'$-structure $\mathcal{R}_e = (\mathcal{R}, R_1, \ldots, R_n)$ such that the interpretation $e(X_i) = R_i$ of every symbol $X_i$ in $\mathcal{L}'$ is a finite relation in $\mathcal{R}$.

For an $\mathcal{L}'$-finite expansion $\mathcal{R}_e$ let the *carrier* of $\mathcal{R}_e$ be the set

$$\begin{aligned} C_e = \; & \{r \in \mathbb{R} : \exists X_i \in \mathcal{L}' \, \exists (a_1, \ldots, a_{n_i}) \in \mathbb{R}^{n_i} \\ & (r \in \{a_1, \ldots, a_{n_i}\} \wedge (a_1, \ldots, a_{n_i}) \in e(X_i))\}. \end{aligned}$$

Clearly, $C_e$ is finite.

Two $\mathcal{L} \cup \mathcal{L}'$-sentences $\phi$ and $\psi$ are equivalent over $\mathcal{R}$ iff for any $\mathcal{L}'$-finite expansion $\mathcal{R}_e$,

$$\mathcal{R}_e \models \phi \text{ iff } \mathcal{R}_e \models \psi.$$

Given an $\mathcal{L}'$-finite expansion $\mathcal{R}_e$ and an order-preserving injection

$$\Phi : C_e \mapsto \mathbb{R},$$

let $\mathcal{R}_{\Phi(e)}$ be another $\mathcal{L}'$-finite expansion such that $C_{\Phi(e)} = \Phi[C_e]$ and

$$\Phi(e)(X_i) = \{(\Phi(a_1), \ldots, \Phi(a_{n_i})) : (a_1, \ldots, a_{n_i}) \in e(X_i)\}.$$

It is easy to see that $\mathcal{R}_e \restriction \mathcal{L}' \cup \{<\}$ and $\mathcal{R}_{\Phi(e)} \restriction \mathcal{L}' \cup \{<\}$, the reducts of $\mathcal{R}_e$ and $\mathcal{R}_{\Phi(e)}$ on $\mathcal{L}' \cup \{<\}$, are isomorphic.

**Definition 2.11** *An $\mathcal{L} \cup \mathcal{L}'$-sentence $\phi$ is order-invariant in $\mathcal{R}$ iff for any $\mathcal{L}'$-finite expansion $\mathcal{R}_e$ and for any order-preserving injection $\Phi : C_e \mapsto \mathbb{R}$*

$$\mathcal{R}_e \models \phi \textit{ iff } \mathcal{R}_{\Phi(e)} \models \phi.$$

**Theorem 2.12** *(M. Benedikt et al. [4]) For any order-invariant* $\mathcal{L} \cup \mathcal{L}'$*-sentence* $\phi$ *there exists an* $\mathcal{L}' \cup \{<\}$*-sentence* $\psi$ *such that* $\phi$ *and* $\psi$ *are equivalent over* $\mathcal{R}$.[4]

The proof of this theorem is long. The main idea of the proof is like the following: By assuming the contrary we first find two $\mathcal{L}'$-$^*$finite expansions $^*\mathcal{R}_e$ and $^*\mathcal{R}_{e'}$ so that they agree on all $\mathcal{L}' \cup \{<\}$-sentences but disagree on $\phi$, *i.e*

$$^*\mathcal{R}_e \models \varphi \text{ iff } ^*\mathcal{R}_{e'} \models \varphi$$

for every $\mathcal{L}' \cup \{<\}$-sentence $\varphi$ and

$$^*\mathcal{R}_e \models \phi \text{ iff } ^*\mathcal{R}_{e'} \models \neg\phi.$$

Then we show, by the order-invariance of $\phi$, that $^*\mathcal{R}_e$ and $^*\mathcal{R}_{e'}$ could be chosen so that $C_e$ and $C_{e'}$ are subsets of an $\mathcal{L}$-indiscernible sequence in $^*\mathbb{R}$. Finally, one could derive a contradiction by showing that $^*\mathcal{R}_e$ and $^*\mathcal{R}_{e'}$ agree on $\phi$.

**Proof of Theorem 2.12** Suppose the theorem is not true and let $\phi$ be a witness. Let $\{\varphi_n : n \in \mathbb{N}\}$ be an enumeration of all $\mathcal{L}' \cup \{<\}$-sentences.

**Claim 1** For any $m \in \mathbb{N}$ there are two $\mathcal{L}'$-finite expansions $\mathcal{R}_e$ and $\mathcal{R}_{e'}$ such that they agree on every $\varphi_n$ for $n < m$ and disagree on $\phi$.

Proof of Claim 1: Otherwise $\phi$ will be equivalent to some Boolean combination of those $\varphi_n$'s for $n < m$. $\square$(Claim 1)

Let $\mathcal{V}$ be a nonstandard universe satisfying $IP$.

By Claim 1 and Transfer Principle we can find two $\mathcal{L}'$-$^*$finite expansions $^*\mathcal{R}_e$ and $^*\mathcal{R}_{e'}$ such that they agree on every $\varphi_n$ for $n < m$ and disagree on $\phi$. By countable saturation one can let $m$ be $\infty$. So we have now two $\mathcal{L}'$-$^*$finite expansions $^*\mathcal{R}_e$ and $^*\mathcal{R}_{e'}$ such that they agree on $\{\varphi_n : n \in \mathbb{N}\}$ and disagree on $\phi$. Let $|C_e| = H_e$ and let $|C_{e'}| = H_{e'}$.

Again by Transfer Principle and countable saturation one can find an $\mathcal{L}$-indiscernible increasing internal sequence $\langle r_i : i \in {}^*\mathbb{N}\rangle$ in $^*\mathbb{R}$.

**Claim 2** We can assume that $C_e$ and $C_{e'}$ are the sets $\{r_i : i < H_e\}$ and $\{r_i : i < H_{e'}\}$, respectively.

Proof of Claim 2: For $^*\mathcal{R}_e$ there is an internal order-preserving bijection $\Phi$ from $C_e$ to $\{r_i : i < H_e\}$. By Transfer Principle and the order-invariance of $\phi$ one has that $^*\mathcal{R}_e$ and $^*\mathcal{R}_{\Phi(e)}$ agree on $\phi$. $^*\mathcal{R}_e$ and $^*\mathcal{R}_{\Phi(e)}$ agree also on $\{\varphi_n : n \in \mathbb{N}\}$ because the reducts of $^*\mathcal{R}_e$ and $^*\mathcal{R}_{\Phi(e)}$ on $\mathcal{L}' \cup \{<\}$ are isomorphic. Same for $^*\mathcal{R}_{e'}$. $\square$(Claim 2)

[4]The original theorem in [4] is more general with $\mathcal{R}$ replaced by any infinite o-minimal structure.

**Claim 3** There is an order-preserving bijection $I$ from $C_e$ to $C_{e'}$ such that

$$(a_1, \ldots, a_{n_i}) \in e(X_i) \text{ iff } (I(a_1), \ldots, I(a_{n_i})) \in e'(X_i)$$

for $i = 1, \ldots, n$.

Proof of Claim 3: Note that "$x \in C_e$" could be written as an $\mathcal{L}'$-formula. Because the submodel of ${}^*\mathcal{R}_e \restriction \mathcal{L}' \cup \{<\}$ generated by $C_e$ and the submodel of ${}^*\mathcal{R}_{e'} \restriction \mathcal{L}' \cup \{<\}$ generated by $C_{e'}$ are elementarily equivalent, by $IP$, there is an isomorphism $I$ between them. (Here is the place we use $IP$.) $\square$(Claim 3)

Let $\mathcal{L}_C$ be the language $\mathcal{L} \cup \{c_i : i < H_e\}$, where $c_i$ is a constant symbol not in $\mathcal{L}$. Note that ${}^*\mathcal{R}_C = ({}^*\mathcal{R}, r_i)_{i<H_e}$ and ${}^*\mathcal{R}_{C'} = ({}^*\mathcal{R}, I(r_i))_{i<H_e}$ are two $\mathcal{L}_C$-structures by interpreting $c_i$ as $r_i$ or as $I(r_i)$, respectively.

**Claim 4** ${}^*\mathcal{R}_C$ and ${}^*\mathcal{R}_{C'}$ are elementarily equivalent.

Proof of Claim 4: It is a straightforward consequence of indiscernibility of $r_i$'s. $\square$(Claim 4)

Let $\mathcal{L}_C^m$ be the language $\mathcal{L}_C \cup \{d_1 \ldots, d_m\}$ and let $\mathcal{L}^m$ be the language $\mathcal{L} \cup \{d_1 \ldots, d_m\}$, where $d_i$'s are new constant symbols.

We need a property, called $o$-minimality, of real fields in next claim. Let $\mathcal{F}$ be either a standard real field or a hyperreal field. for any $\mathcal{L}$-formula $\chi(x, \bar{y})$ and any $\bar{r}$ in $\mathcal{F}$ the set defined by $\chi(x, \bar{r})$ in $\mathcal{F}$ is a finite union of intervals. Let those intervals be maximal and let $E_\chi^{\bar{r}}$ be the set of all endpoints of those intervals. Clearly, $E_\chi^{\bar{r}}$ is finite and every point in $E_\chi^{\bar{r}}$ is definable by some $\mathcal{L}$-formula with parameters from $\bar{r}$.

**Claim 5** Suppose $\bar{u} = (u_1, \ldots, u_m)$ and $\bar{v} = (v_1, \ldots, v_m)$ are in ${}^*\mathbb{R}^m$ such that $({}^*\mathcal{R}_C, \bar{u})$ and $({}^*\mathcal{R}_{C'}, \bar{v})$ are elementarily equivalent in $\mathcal{L}_C^m$. Then

(1) for every $u_{m+1} \in {}^*\mathbb{R}$ there is a $v_{m+1} \in {}^*\mathbb{R}$ such that $({}^*\mathcal{R}_C, \bar{u}, u_{m+1})$ and $({}^*\mathcal{R}_{C'}, \bar{v}, v_{m+1})$ are elementarily equivalent in $\mathcal{L}_C^{m+1}$,

(2) for every $v_{m+1} \in {}^*\mathbb{R}$ there is a $u_{m+1} \in {}^*\mathbb{R}$ such that $({}^*\mathcal{R}_C, \bar{u}, u_{m+1})$ and $({}^*\mathcal{R}_{C'}, \bar{v}, v_{m+1})$ are elementarily equivalent in $\mathcal{L}_C^{m+1}$.

Proof of Claim 5: By symmetry we need only to prove (1). Given $u_{m+1}$, let $\Lambda(x)$ be the set all $\mathcal{L}_C^m$-formulas realized in $({}^*\mathcal{R}_C, \bar{u})$ by $u_{m+1}$. If $\Lambda(x)$ were countable, then $v_{m+1}$ could be chosen easily by countable saturation. Unfortunately, $\Lambda(x)$ is not countable. To overcome this difficulty we use $o$-minimality. We want to choose a countable set $, (x) \subseteq \Lambda(x)$ so that for any $\sigma(x) \in \Lambda(x)$ there is a $\tau(x) \in , (x)$ such that

$$({}^*\mathcal{R}_C, \bar{u}) \models \forall x(\tau(x) \to \sigma(x)).$$

Given any $\mathcal{L}^m$-formula $\chi(x, \bar{y})$, where $\bar{y} = \{y_1, \ldots, y_n\}$. Let

$$l_\chi = \max(\{a : \exists \bar{r} \in C_e^n (a \in E_\chi^{\bar{r}} \wedge a \leqslant u_{m+1})\} \cup \{-\infty\})$$

and let

$$r_\chi = \min(\{a : \exists \bar{r} \in C_e^n (a \in E_\chi^{\bar{r}} \wedge a \geqslant u_{m+1})\} \cup \{+\infty\}).$$

Note that (1) max and min above exist because the sets are hyperfinite, (2) $l_\chi \leqslant u_{m+1} \leqslant r_\chi$, (3) if $l_\chi < u_{m+1} < r_\chi$, then for any $\bar{r} \in C_e^n$ the formula $\chi(x, \bar{r})$ has a constant truth value in $(l_\chi, r_\chi)$, (4) $l_\chi$ and $r_\chi$ are definable from some $\bar{r} \in C_e^n$.

Case 1: $u_{m+1} = l_\chi$ or $u_{m+1} = r_\chi$ for some $\mathcal{L}^m$-formula $\chi$. Then $u_{m+1}$ is definable by some $\mathcal{L}_C^m$-formula $\theta(x)$. Let , $(x) = \{\theta(x)\}$.

Case 2: $l_\chi < u_{m+1} < r_\chi$ for any $\mathcal{L}^m$-formula $\chi$. Let $\theta_\chi(x)$ be an $\mathcal{L}_C^m$-formula saying that $l_\chi < x < r_\chi$ and let

$$, (x) = \{\theta_\chi : \chi \text{ is an } \mathcal{L}^m\text{-formula.}\}.$$

Then , $(x)$ is countable because there are only countably many $\mathcal{L}^m$-formulas. Obviously , $(x)$ is finitely realizable in $({}^*\mathcal{R}_{C'}, \bar{v})$. By countable saturation , $(x)$ is realized in $({}^*\mathcal{R}_{C'}, \bar{v})$ by some $v_{m+1} \in {}^*\mathbb{R}$.

It is easy now to check that $({}^*\mathcal{R}_C, \bar{u}, u_{m+1})$ and $({}^*\mathcal{R}_{C'}, \bar{v}, v_{m+1})$ are elementarily equivalent in $\mathcal{L}_C^{m+1}$. $\square$(Claim 5)

Note that in Claim 5 $u_{m+1} \in C_e$ iff $v_{m+1} = I(u_{m+1}) \in C_{e'}$.

**Claim 6** Suppose $({}^*\mathcal{R}_C, \bar{u})$ and $({}^*\mathcal{R}_{C'}, \bar{v})$ are elementarily equivalent. Suppose $\psi(\bar{x})$ is a quantifier-free $\mathcal{L} \cup \mathcal{L}'$-formula. Then

$${}^*\mathcal{R}_e \models \psi(\bar{u}) \text{ iff } {}^*\mathcal{R}_{e'} \models \psi(\bar{v}).$$

Proof of Claim 6: It suffices to assume that $\psi$ is an atomic formula.

Case 1: $\psi$ contains no symbol from $\mathcal{L}'$. Then it is trivial by the elementary equivalence.

Case 2: $\psi$ contains symbols from $\mathcal{L}'$. Then $\psi =: X_i(\bar{\tau}(\bar{x}))$ for an $X_i \in \mathcal{L}'$ and an $n_i$-tuple of $\mathcal{L}$-terms $\bar{\tau}$. Note that $\mathcal{L}'$ contains no function symbols. If ${}^*\mathcal{R}_e \models \psi(\bar{u})$, then $\bar{\tau}(\bar{u}) \in e(X_i)$. Hence $\bar{\tau}(\bar{u}) \in C_e^{n_i}$. Since $({}^*\mathcal{R}_C, \bar{u})$ and $({}^*\mathcal{R}_{C'}, \bar{v})$ are elementarily equivalent, then $I(\bar{\tau}(\bar{u})) = \bar{\tau}(\bar{v})$. Note that $I$ is an isomorphism from the submodel of ${}^*\mathcal{R}_e \restriction \mathcal{L}' \cup \{<\}$ generated by $C_e$, to the submodel of ${}^*\mathcal{R}_{e'} \restriction \mathcal{L}' \cup \{<\}$ generated by $C_{e'}$. Then $I(\bar{\tau}(\bar{u})) = \bar{\tau}(\bar{v}) \in e'(X_i)$. Hence ${}^*\mathcal{R}_{e'} \models \psi(\bar{v})$. $\square$(Claim 6)

**Claim 7** ${}^*\mathcal{R}_e \models \phi$ iff ${}^*\mathcal{R}_{e'} \models \phi$.

Proof of Claim 7: Without loss of generality we assume that $\phi$ has the form $\forall \bar{x}_n \exists \bar{x}_{n-1} \ldots \psi(\bar{x}_n, \ldots, \bar{x}_1)$, where $\psi$ is a quantifier-free $\mathcal{L} \cup \mathcal{L}'$-formula. Suppose ${}^*\mathcal{R}_e \models \phi$. By a back-and-forth argument using Claim 5 we are able to find $\bar{u}_n, \ldots, \bar{u}_1$ and $\bar{v}_n, \ldots, \bar{v}_1$ such that $({}^*\mathcal{R}_C, \bar{u}_n, \ldots, \bar{u}_1)$ is elementarily equivalent to $({}^*\mathcal{R}_{C'}, \bar{v}_n, \ldots, \bar{v}_1)$, and

$$({}^*\mathcal{R}_e, \bar{u}_n, \ldots, \bar{u}_1) \models \psi(\bar{u}_n, \ldots, \bar{u}_1).$$

By Claim 6 we have

$$({}^*\mathcal{R}_{e'}, \bar{v}_n, \ldots, \bar{v}_1) \models \psi(\bar{v}_n, \ldots, \bar{v}_1).$$

This shows that for any $\bar{v}_n$ there exists $\bar{v}_{n-1}$ such that for any $\bar{v}_{n-2}$ there exists $\bar{v}_{n-3}$...... such that

$$({}^*\mathcal{R}_{e'}, \bar{v}_n, \ldots, \bar{v}_1) \models \psi(\bar{v}_n, \ldots, \bar{v}_1).$$

So it is clear that ${}^*\mathcal{R}_{e'} \models \phi$. $\square$

## 2.4. EXERCISES

**Exercise 2.13** *Let $\mathcal{F} = (F; +, \cdot, <, 0, 1)$ be an ordered field. An upper bounded initial segment $I$ of $F$ is called a regular gap iff it has no least upper bound and for any positive $\epsilon \in F$ there exists an $r \in I$ such that $r + \epsilon \notin I$. $\mathcal{F}$ is called Scott complete iff $\mathcal{F}$ has no regular gap. Show the following:*

*(1) Suppose $\mathcal{F}$ is a subfield of $\mathcal{F}'$. If an initial segment $I$ in $F$ has a least upper bound in $F' \smallsetminus F$, then $I$ is a regular gap in $\mathcal{F}$.*

*(2) If $\mathcal{V}$ satisfies IP, then the hyperreal field in $\mathcal{V}$ is not Scott complete.*
Hint: *Write a set of formulas , $(X)$ to express that $X$ is a regular gap.*

**Exercise 2.14** *Show that IP implies the existence of an external set $S \subseteq {}^*\mathbb{N}$ such that $S \cap \{0, 1, \ldots, H\}$ is internal for every $H \in {}^*\mathbb{N}$. This means that the nonstandard universe satisfying IP can't be classless.*

**Exercise 2.15** *Show that IP implies the existence of a bijection $f$ between two ${}^*$infinite sets $A$ and $B$ such that for any ${}^*$finite sets $a \subseteq A$ and $b \subseteq B$ the restrictions $f \restriction a$ and $f^{-1} \restriction b$ are internal.*

${}^*$**Exercise 2.16** *Let $IP_0$ be IP restricted only on finite languages in Definition 2.1. Prove that IP is equivalent to $IP_0$ plus countable saturation. See [13] for help.*

${}^{**}$**Exercise 2.17** *Find a nonstandard universe $\mathcal{V}$ satisfying IP such that*

$$cf({}^*\mathbb{N}) \neq coin({}^*\mathbb{N} \smallsetminus \mathbb{N}),$$

*where $cf({}^*\mathbb{N})$ is the smallest cardinality of some set $S \subseteq {}^*\mathbb{N}$ cofinal in ${}^*\mathbb{N}$ and $coin({}^*\mathbb{N} \smallsetminus \mathbb{N})$ is the smallest cardinality of some set $S \subseteq {}^*\mathbb{N} \smallsetminus \mathbb{N}$ coinitial in ${}^*\mathbb{N} \smallsetminus \mathbb{N}$. See [12] for help.*

**Exercise 2.18** *(M. Benedikt et al. [4]) Let $\mathcal{L}$ be any language including $<$ and $\mathcal{L}'$ be same as in §2.3. Let $\mathfrak{A}$ be any infinite totally-ordered $\mathcal{L}$-structure (not necessarily o-minimal). By an $\mathcal{L}'$-bounded quantifier sentence we mean an $\mathcal{L} \cup \mathcal{L}'$-sentence built up from atomic formulas in $\mathcal{L} \cup \mathcal{L}'$ via the usual* *logical connectives and the quantifications* $\forall \bar{x} \in X_i$ *and* $\exists \bar{x} \in X_j$ *for some* $X_i, X_j \in \mathcal{L}'$.

*Show that for any* $\mathcal{L}'$*-bounded quantifier sentence* $\phi$*, if* $\phi$ *is order-invariant in* $\mathfrak{A}$*, then there is an* $\mathcal{L}' \cup \{<\}$*-sentence* $\psi$ *such that* $\phi$ *and* $\psi$ *are equivalent over* $\mathfrak{A}$.

## 3. The Special Model Axiom and Full Saturation

Given a language $\mathcal{L}$, an $\mathcal{L}$-structure $\mathfrak{A}$ is called a special model if there is a sequence $\langle \mathfrak{A}_\alpha : \alpha < card(\mathfrak{A}) \rangle$ of $\mathcal{L}$-structures such that

(1) for any $\alpha < \beta < card(\mathfrak{A})$, $\mathfrak{A}_\alpha$ is an elementary submodel of $\mathfrak{A}_\beta$,

(2) $\mathfrak{A} = \bigcup_{\alpha < card(\mathfrak{A})} \mathfrak{A}_\alpha$, and

(3) for any $\alpha < card(\mathfrak{A})$, $\mathfrak{A}_{\alpha+1}$ is $(card(\alpha))^+$-saturated.

The sequence $\langle \mathfrak{A}_\alpha : \alpha < card(\mathfrak{A}) \rangle$ is called a specializing chain for $\mathfrak{A}$.

**Definition 3.1** *A nonstandard universe* $\mathcal{V}$ *is said to satisfy the special model axiom iff any internally presented structure of some countable language is a special model.*

We will write *SMA* for the special model axiom.

**Proposition 3.2** *(D. Ross [26]) Suppose* $\mathcal{V}$ *satisfies SMA. Then every infinite internal set in* $\mathcal{V}$ *has cardinality* $\Xi_{\mathcal{V}}$.

**Proof:** It suffices to show that any two infinite internal sets have same cardinality. Given two infinite internal sets $C$ and $D$, we want to derive a contradiction by assuming that $card(C) = \kappa < card(D) = \lambda$. We form an internally presented structure

$$\mathfrak{A} = (C \cup D; C, D),$$

where $C, D$ are unary relations. Then one has $card(\mathfrak{A}) = \lambda$. Suppose $\langle \mathfrak{A}_\alpha : \alpha < \lambda \rangle$ is a specializing chain for $\mathfrak{A}$. Let $C_\alpha \cup D_\alpha$ be the base set of $\mathfrak{A}_\alpha$. Note that $C_\alpha \subseteq C$ for every $\alpha < \lambda$. Since $\mathfrak{A}_{\kappa+1}$ is $\kappa^+$-saturated, then $card(C_{\kappa+1}) \geqslant \kappa^+$. This contradicts the fact $C_{\kappa+1} \subseteq C$. So $card(D) \leqslant card(C)$. By symmetry we have $card(C) = card(D)$. $\square$

**Proposition 3.3** *(D. Ross [26]) If* $\mathcal{V}$ *satisfies SMA, then* $\mathcal{V}$ *satisfies IP.*

**Proof:** It is proved in [5] that any two elementarily equivalent special models with same cardinality are isomorphic. $\square$

**Proposition 3.4** *For any strong limit*[5] *cardinal* $\kappa$ *with* $cf(\kappa) > \aleph_0$ *there is an ultralimit* $\mathcal{V}$ *of the standard superstructure such that* $\mathcal{V}$ *satisfies SMA and* $\Xi_{\mathcal{V}} = \kappa$.

[5] A cardinal $\kappa$ is called a strong limit iff for every $\lambda < \kappa$ one has $2^\lambda < \kappa$.

Consult [5] for the proof.

**Definition 3.5** *A nonstandard universe $\mathcal{V}$ is fully saturated iff every internally presented $\mathcal{L}$-structure $\mathfrak{A}$ for some countable language $\mathcal{L}$ is a saturated model,* i.e. *$\mathfrak{A}$ is $card(\mathfrak{A})$-saturated.*

**Proposition 3.6** *If $\mathcal{V}$ is fully saturated, then $\mathcal{V}$ satisfies SMA.*

**Proof:** A saturated model is trivially a special model. $\square$

**Proposition 3.7** *For any cardinal $\kappa$ such that $\kappa > \beth_\omega$ and $\kappa^{<\kappa} = \kappa$ there exists a fully saturated nonstandard universe $\mathcal{V}$ such that $\Xi_{\mathcal{V}} = \kappa$.*

Consult [5] for the proof.

Next we give one application of *SMA* and one application of full saturation.

### 3.1. COMPACTNESS OF LOEB PROBABILITY SPACES

Given any probability space $(\Omega, \mathcal{B}, P)$, a family $\mathcal{C} \subseteq \mathcal{B}$ is called compact iff for any $\mathcal{D} \subseteq \mathcal{C}$, $\mathcal{D}$ has f.i.p., *i.e.* every finite subfamily of $\mathcal{D}$ has non-empty intersection, implies $\bigcap \mathcal{D} \neq \emptyset$. A family $\mathcal{C} \subseteq \mathcal{B}$ is called inner-regular iff for any $B \in \mathcal{B}$

$$P(B) = \sup\{P(C) : C \in \mathcal{C} \wedge C \subseteq B\}.$$

**Definition 3.8** *(D. Ross [27]) A probability space $(\Omega, \mathcal{B}, P)$ is called compact iff there is a compact, inner-regular family $\mathcal{C} \subseteq \mathcal{B}$.*

A Radon space is an example of a compact space. Ross showed in [27] that a compact probability space could be topologized so that the resulting topological measure space is Radon. In [27] a question whether a Loeb probability space is compact, is posed. The following theorem is one of many results in [17] concerning the compactness of Loeb probability spaces.

**Theorem 3.9** *(R. Jin and S. Shelah [17]) Assume CH (Continuum Hypothesis). Suppose $\mathcal{V}$ satisfies SMA and $cf(\Xi_{\mathcal{V}}) = \aleph_1$. Then every non-atomic Loeb probability space is compact.*

We need more notation in the proof. For any set $S$ we write $2^S$ for the set of all functions from $S$ to $\{0,1\}$. Let $2^{<\mathbb{N}}$ be the set $\bigcup_{n\in\mathbb{N}} 2^n$, where $n$ could be viewed as a set $\{0, \ldots, n-1\}$. A set $t \subseteq 2^{<\mathbb{N}}$ is called a tree iff for any $s, s' \in 2^{<\mathbb{N}}$, $s' \subseteq s$ and $s \in t$ imply $s' \in t$. By a branch of $t$ we mean a maximal totally ordered subset of $t$ with order $\subseteq$. We denote $T$ for the trees without maximal nodes and $[T]$ for all branches of $T$. Let's consider $2^{\mathbb{N}}$ as a Cantor space with the usual probability measure $\nu$, *i.e.*

$$\nu(\{f \in 2^{\mathbb{N}} : f(n) = 0\}) = \frac{1}{2}$$

for every $n \in \mathbb{N}$. Note that every closed subset of $2^{\mathbb{N}}$ could be written as $[T]$ for some tree $T \subseteq 2^{<\mathbb{N}}$. By *CH* we can fix an enumeration $\{f_\gamma : \gamma < \aleph_1\}$ of $2^{\mathbb{N}}$. For each $\beta < \aleph_1$ and $m \in \mathbb{N}$ we choose a tree $T_{\beta,m}$ such that

$$[T_{\beta,m}] \cap \{f_\gamma : \gamma < \beta\} = \emptyset \text{ and } \nu([T_{\beta,m}]) > \frac{m}{m+1}.$$

This can be done because $\nu(\{f_\gamma : \gamma < \beta\}) = 0$.

Given a probability space $(\Omega, \mathcal{B}, P)$ and a sequence of measurable sets $\langle A_n : n \in \mathbb{N}\rangle$. The sequence $\langle A_n : n \in \mathbb{N}\rangle$ is called independent iff for any $m \in \mathbb{N}$ and for any $h \in 2^m$

$$P(\bigcap_{n=0}^{m-1} A_n^{h(n)}) = \prod_{n=0}^{m-1} P(A_n^{h(n)}),$$

where $A_n^0 = A_n$ and $A_n^1 = \Omega \smallsetminus A_n$.

**Proof of Theorem 3.9:** Given a non-atomic Loeb probability space $(\Omega, \mathcal{B}, L_\mu)$ generated by an internal probability space $(\Omega, \mathcal{A}, \mu)$. Choose an independent sequence $\langle A_n : n \in \mathbb{N}\rangle$ in $\mathcal{A}$ such that $L_\mu(A_n) = \frac{1}{2}$ for each $n \in \mathbb{N}$. For any tree $T \subseteq 2^{<\mathbb{N}}$ let

$$A_T = \bigcap_{n\in\mathbb{N}} \bigcup_{h \in 2^n \cap T} \bigcap_{i=0}^{n-1} A_i^{h(i)}.$$

It is easy to check that $L_\mu(A_T) = \nu([T])$. Note that $A_T$ is a countable intersection of internal sets. We are now ready to construct an inner-regular, compact family $\mathcal{C}$. Note that one needs only to deal with the inner-regularity for all sets in $\mathcal{A}$.

Let $\mathfrak{A}$ be the internally presented structure same as the one in the proof of Theorem 2.7, *i.e.*

$$\mathfrak{A} = (\Omega \cup \mathcal{A} \cup {}^*\mathbb{R}; \Omega, \mathcal{A}, {}^*\mathbb{R}, \in, \mu, \cap, \smallsetminus, +, \cdot, <, 0, 1).$$

By *SMA* there is a specializing chain $\langle \mathfrak{A}_\alpha : \alpha < \Xi_\mathcal{V}\rangle$ for $\mathfrak{A}$. Let $\{\kappa_\beta : \beta < \aleph_1\}$ be an increasing sequence of regular cardinals cofinal in $\Xi_\mathcal{V}$. Let $\mathfrak{B}_\beta = \bigcup_{\alpha<\kappa_\beta} \mathfrak{A}_\alpha$. Suppose the base set of $\mathfrak{B}_\beta$ is $\Omega_\beta \cup \mathcal{A}_\beta \cup \mathbb{R}_\beta$. We can choose an enumeration $\{a_\alpha : \alpha < \Xi_\mathcal{V}\}$ of $\mathcal{A}$ such that for every $\beta < \aleph_1$

$$\{a_\alpha : \alpha < \kappa_\beta\} \subseteq \mathcal{A}_\beta.$$

Without loss of generality we assume $A_n \in \mathcal{A}_0$ for every $n \in \mathbb{N}$. For any $\alpha < \Xi_\mathcal{V}$ let

$$g(\alpha) = \min\{\beta < \aleph_1 : \alpha < \kappa_\beta\}.$$

For each $a_\alpha \in \mathcal{A}$ and each $m \in \mathbb{N}$ we choose $b_{\alpha,m} \subseteq a_\alpha \cap A_{T_{g(\alpha),m}}$ such that $b_{\alpha,m} \in \mathcal{A}_{g(\alpha)+1}$ and $L_\mu(b_{\alpha,m}) = L_\mu(a_\alpha \cap A_{T_{g(\alpha),m}})$. Note that $b_{\alpha,m}$ exists by countable saturation of $\mathfrak{B}_{g(\alpha)+1}$. Now let

$$\mathcal{C} = \{b_{\alpha,m} : \alpha < \Xi_{\mathcal{V}} \wedge m \in \mathbb{N}\} \cup \{A_n : n \in \mathbb{N}\}.$$

**Claim** $\mathcal{C}$ is compact and inner-regular.

Proof of Claim: Clearly, $\mathcal{C}$ is inner-regular. Given any $\mathcal{D} \subseteq \mathcal{C}$ with f.i.p., we want to show that $\bigcap \mathcal{D} \neq \emptyset$. Without loss of generality we assume that $\mathcal{D}$ is maximal. For each $n \in \mathbb{N}$ one has either $A_n^0 \in \mathcal{D}$ or $A_n^1 \in \mathcal{D}$. So there is a function $h' \in 2^{\mathbb{N}}$ such that $A_n^{h'(n)} \in \mathcal{D}$ for every $n \in \mathbb{N}$. Let

$$\delta = \bigcup\{g(\alpha) : \exists m(b_{\alpha,m} \in \mathcal{D})\}.$$

Case 1: $\delta < \aleph_1$. Then $\mathcal{D} \subseteq \mathcal{A}_{\delta+1}$ and $card(\mathcal{D}) \leqslant \kappa_\delta$. Since $\mathfrak{B}_{\delta+1}$ is $(\kappa_\delta)^+$-saturated, then $\bigcap \mathcal{D} \neq \emptyset$.

Case 2: $\delta = \aleph_1$. For each $b_{\alpha,m} \in \mathcal{D}$ and for every $n \in \mathbb{N}$ we have

$$(\bigcap_{i=0}^{n-1} A_i^{h'(i)}) \cap b_{\alpha,m} \neq \emptyset.$$

But that means

$$(\bigcap_{i=0}^{n-1} A_i^{h'(i)}) \cap A_{T_{g(\alpha),m}} \neq \emptyset.$$

Note that

$$A_{T_{g(\alpha),m}} = \bigcap_{n\in\mathbb{N}} \bigcup_{h\in 2^n \cap T_{g(\alpha),m}} \bigcap_{i=0}^{n-1} A_i^{h(i)}.$$

By a careful check one can see that $h' \restriction n \in T_{g(\alpha),m}$ for every $n \in \mathbb{N}$. So $h' \in [T_{g(\alpha),m}]$. But $h' = f_\gamma$ for some $\gamma < \aleph_1$ (recall that we have a fixed enumeration of $2^{\mathbb{N}}$). So when $g(\alpha) > \gamma$ one has $h' \notin [T_{g(\alpha),m}]$ because

$$[T_{g(\alpha),m}] \cap \{f_{\gamma'} : \gamma' < g(\alpha)\} = \emptyset.$$

This contradicts that $\delta = \aleph_1$. $\square$

**Remark** In Theorem 3.9 the assumptions *CH* and $cf(\Xi_{\mathcal{V}}) = \aleph_1$ couldn't be eliminated. If one replaces $cf(\Xi_{\mathcal{V}}) = \aleph_1$ by $cf(\Xi_{\mathcal{V}}) = (2^{\aleph_0})^+$ (with or without *CH*), then the result will be just opposite. If one replaces *CH* by $\neg$*CH* (we can even weaken the condition $cf(\Xi_{\mathcal{V}}) = \aleph_1$ to $cf(\Xi_{\mathcal{V}}) = \kappa$ for any uncountable regular $\kappa \leqslant 2^{\aleph_0}$), then the compactness of Loeb probability spaces is undecidable in ZFC.

Let $(\Omega, \mathcal{B}, L_\mu)$ be a Loeb probability space generated by the internal normalized uniform counting measure $(\Omega, \mathcal{A}, \mu)$. We denote $\bar{\mathcal{B}}$ for the Loeb algebra, *i.e.* the Boolean algebra $\mathcal{B}$ modulo the ideal of $L_\mu$-measure zero sets. For each element $B \in \mathcal{B}$ we denote $\bar{B} \in \bar{\mathcal{B}}$ for the equivalence class containing $B$. Note that each $\bar{B} \in \bar{\mathcal{B}}$ contains an internal set in $\mathcal{A}$.

**Definition 3.10** *An automorphism of $\bar{\mathcal{B}}$ is a bijection $\Phi$ from $\bar{\mathcal{B}}$ to $\bar{\mathcal{B}}$ such that $\Phi$ is a Boolean algebra homomorphism and preserves the measure,* i.e. *for any $A, B \in \mathcal{B}$, $\Phi(\bar{A}) = \bar{B}$ implies $L_\mu(A) = L_\mu(B)$.*

**Definition 3.11** *A bijection $T : \Omega \mapsto \Omega$ is called a point-automorphism iff both $T$ and $T^{-1}$ are measurable and for any $B \in \mathcal{B}$ one has $L_\mu(B) = L_\mu(T[B])$.*

It is easy to see that a point-automorphism induces, in a natural way, an automorphism of $\bar{\mathcal{B}}$.

**Theorem 3.12** *(D. Ross [25]) Suppose $\mathcal{V}$ is fully saturated. Suppose $(\Omega, \mathcal{B}, L_\mu)$ is a Loeb probability space generated by an internal normalized uniform counting measure space $(\Omega, \mathcal{A}, \mu)$. Then every automorphism $\Phi$ of $\bar{\mathcal{B}}$ is induced by a point-automorphism $T$.*

**Proof:** Let $\mathcal{A} = \{A_\alpha : \alpha < \Xi_\mathcal{V}\}$. We construct two sequences $\langle B_\alpha : \alpha < \Xi_\mathcal{V}\rangle$ and $\langle C_\alpha : \alpha < \Xi_\mathcal{V}\rangle$ such that

(1) $\{B_\alpha : \alpha < \Xi_\mathcal{V}\} = \{C_\alpha : \alpha < \Xi_\mathcal{V}\} = \mathcal{A}$,

(2) for any $\alpha < \Xi_\mathcal{V}$, $\Phi(\bar{B}_\alpha) = \bar{C}_\alpha$,

(3) for any $\alpha_0 < \alpha_1 < \ldots < \alpha_{n-1} < \Xi_\mathcal{V}$ and for any $h \in 2^n$ one has

$$|\bigcap_{i=0}^{n-1} B_{\alpha_i}^{h(i)}| = |\bigcap_{i=0}^{n-1} C_{\alpha_i}^{h(i)}|,$$

where $A^0 = A$, $A^1 = \Omega \setminus A$ and $|\cdot|$ means internal cardinality.

**Claim** The theorem follows from the construction.

Proof of Claim: For any $x \in \Omega$ there is an $\alpha$ such that $\{x\} = B_\alpha$ by (1). It is easy to see by (3) that $C_\alpha$ is also a singleton $\{y\}$ for some $y \in \Omega$. Let $T(x) = y$. Then it is easy to check again by (3) that $T$ is a well-defined bijection. Also it is not hard to check that $T[B_\alpha] = C_\alpha$. So one has $|A| = |T[A]|$ for every $A \in \mathcal{A}$. This implies that $T$ and $T^{-1}$ are measurable and preserve the measure. So $T$ is a point-automorphism. By (2) and (3) one can easily see that $\Phi$ is induced by $T$. $\square$(Claim)

We now construct $B_\alpha$ and $C_\alpha$ by induction. Let

$$\mathfrak{A} = (\Omega \cup \mathcal{A} \cup {}^*\mathbb{R}; \Omega, \mathcal{A}, {}^*\mathbb{R}, \in, \mu, \cap, \setminus, +, \cdot, <, 0, 1)$$

be the internally presented structure same as in the proof of Theorem 2.7. Suppose we have found $\{B_\beta : \beta < \alpha\}$ and $\{C_\beta : \beta < \alpha\}$ such that (2) and (3) are true up to stage $\alpha$.

Case 1: $\alpha$ is even. We pick $B_\alpha$ first. Let

$$\gamma = \min\{\delta : A_\delta \notin \{B_\beta : \beta < \alpha\}\}$$

and let $B_\alpha = A_\gamma$. This step guarantees (1). Let $\Phi(\bar{B}_\alpha) = \bar{A}$ for some $A \in \mathcal{A}$. We define a set of formulas $,_\alpha(x)$ with only one free variable $x$, which expresses that $x$ is a candidate for $C_\alpha$. The set $,_\alpha(x)$ contains exactly the following:

(a) $\mathcal{A}(x)$, *i.e.* $x$ is an internal subset of $\Omega$,

(b) $\mu(x\Delta A) < \frac{1}{m}$ for every $m \in \mathbb{N}$, *i.e.* the symmetric difference of $x$ and $A$ will have Loeb measure zero,

(c)

$$\mu((\bigcap_{i=0}^{n-1} B_{\alpha_i}^{h(i)}) \cap B_\alpha^j) = \mu((\bigcap_{i=0}^{n-1} C_{\alpha_i}^{h(i)}) \cap x^j)$$

for any $\alpha_0 < \alpha_1 < \ldots < \alpha_{n-1} < \alpha$, for any $h \in 2^n$ and for any $j = 0, 1$. Note that for any internal sets $A, B \subseteq \Omega$ one has $\mu(A) = \mu(B)$ iff $|A| = |B|$.

Since $card(,_\alpha(x)) = card(\alpha) < \Xi_\mathcal{V}$ and $,_\alpha(x)$ is clearly finitely realizable, then, by full saturation, $,_\alpha(x)$ is realized by some $C \in \mathcal{A}$. Let $C_\alpha = C$. Clearly, (2) and (3) are true up to stage $\alpha + 1$.

Case 2: $\alpha$ is odd. We pick $C_\alpha$ first and then $B_\alpha$ by symmetry.

Finally, (1) is true because of the way we choose $B_\alpha$ when $\alpha$ is even, and $C_\alpha$ when $\alpha$ is odd. □

### 3.3. EXERCISES

**Exercise 3.13** *(D. Ross [26]) Show that SMA implies that*

$$cf(^*\mathbb{N}) = coin(^*\mathbb{N} \setminus \mathbb{N}) = cf(\Xi_\mathcal{V}).$$

*By comparing with Exercise 2.17 this exercise witnesses that IP does not imply SMA.*

**Exercise 3.14** *Let SMA$_0$ be SMA restricted only on finite languages in Definition 3.1. Show that SMA is equivalent to SMA$_0$ plus countable saturation.*

**Exercise 3.15** *Assuming CH. Suppose $\mathcal{V}$ is an ultrapower of the standard superstructure modulo an ultrafilter on a countable set. Show that every Loeb probability space generated by an internal normalized uniform counting measure on a hyperfinite set in $\mathcal{V}$ is compact.*

${}^*$**Exercise 3.16** *Show that Theorem 3.12 is still true if $\mathcal{V}$ satisfies SMA instead of full saturation (see [15] for hint).*

**Exercise 3.17** *(D. Ross [25]) Let $\mathcal{V}$ be fully saturated. Let $(\Omega, \mathcal{B}, L_\mu)$ be a Loeb probability space generated by an internal normalized uniform counting measure on a hyperfinite set $\Omega$. Let $\bar{\mathcal{B}}$ be the Loeb algebra. Show the following:*

*(1) There exist automorphisms $\Phi$ of $\bar{\mathcal{B}}$, which are not induced by any internal point-automorphisms,*

*(2) Let $\bar{\mathcal{C}} \subseteq \bar{\mathcal{B}}$ be a subalgebra such that $card(\bar{\mathcal{C}}) < \Xi_{\mathcal{V}}$. Then for any automorphism $\Phi$ of $\bar{\mathcal{C}}$ there exists an internal point-automorphism $T$ such that $\Phi$ is induced by $T$.*

${}^*$**Exercise 3.18** *If $\mathcal{V}$ satisfies SMA but is not fully saturated, then for any Loeb probability space $(\Omega, \mathcal{B}, L_\mu)$ as in Exercise 3.17 there exists a subalgebra $\bar{\mathcal{C}} \subseteq \bar{\mathcal{B}}$ with $card(\bar{\mathcal{C}}) < \Xi_{\mathcal{V}}$ and there exists an automorphism $\Phi$ of $\bar{\mathcal{C}}$ such that $\Phi$ is not induced by any internal point-automorphism. (This exercise and Exercise 3.16 are related, in fact, to a question posed in [26]. This exercise witnesses that SMA does not imply full saturation.)* Hint: *Show first that $\Xi_{\mathcal{V}}$ is a singular strong limit cardinal. Then construct a subalgebra of cardinality $cf(\Xi_{\mathcal{V}})$ together with an automorphism by a diagonal method so that the automorphism avoids to be induced by any internal point-automorphisms (see [15] for more help).*

## 4. The $\lambda$-Bolzano-Weierstrass Property

Let $\mathcal{F} = (F; +, \cdot, <, 0, 1)$ be an ordered field. Let $\lambda$ be an uncountable regular cardinal. By a bounded $\lambda$-sequence in $\mathcal{F}$ we mean a sequence $\langle a_\alpha : \alpha < \lambda \rangle$ in $\mathcal{F}$ such that $\{a_\alpha : \alpha < \lambda\} \subseteq [-r, r]$ for some positive $r \in F$. A $\lambda$-sequence $\langle a_\alpha : \alpha < \lambda \rangle$ in $\mathcal{F}$ is said to converge in $\mathcal{F}$ iff there is an $r \in F$ such that for any positive $\epsilon \in F$ there is an $\beta < \lambda$ such that $\{a_\alpha : \beta < \alpha < \lambda\} \subseteq [r - \epsilon, r + \epsilon]$.

**Definition 4.1** *Given a nonstandard universe $\mathcal{V}$ and let ${}^*\mathcal{R}$ be the hyperreal field in $\mathcal{V}$. Let $\lambda$ be an uncountable regular cardinal less than or equal to $card({}^*\mathbb{R})$. ${}^*\mathcal{R}$ satisfies the $\lambda$-Bolzano-Weierstrass property iff every bounded $\lambda$-sequence in ${}^*\mathcal{R}$ has a convergent $\lambda$-subsequence in ${}^*\mathcal{R}$. $\mathcal{V}$ satisfies the $\lambda$-Bolzano-Weierstrass property iff ${}^*\mathcal{R}$ does.*

Clearly, the $\lambda$-Bolzano-Weierstrass property is a natural generalization of the Bolzano-Weierstrass property for the standard real field.

**Proposition 4.2** *(H. J. Keisler and J. H. Schmerl [21]) There exist countably saturated nonstandard universes satisfying the $\lambda$-Bolzano-Weierstrass property for $\lambda = (2^{\aleph_0})^+$.*[6]

**Definition 4.3** *A nonstandard universe $\mathcal{V}$ is called $\lambda$-Archimedean iff $card({}^*\mathbb{N}) = \lambda$ and $card(\{0, 1, \ldots, H\}) < \lambda$ for every $H \in {}^*\mathbb{N}$.*

**Proposition 4.4** *(J. Cowles and R. LaGrange [6]) If $\mathcal{V}$ satisfies the $\lambda$-Bolzano-Weierstrass property, then $\mathcal{V}$ is $\lambda$-Archimedean.*

**Proof:** Suppose $\mathcal{V}$ is not $\lambda$-Archimedean.

Case 1: $card({}^*\mathbb{N}) < \lambda$. Since the set of all ${}^*$rational numbers has cardinality $card({}^*\mathbb{N}) < \lambda$, then there are no $\lambda$-convergent sequences in ${}^*\mathbb{R}$. But there are $\lambda$-sequences in ${}^*\mathcal{R}$.

Case 2: $card({}^*\mathbb{N}) \geqslant \lambda$. Because $\mathcal{V}$ is not $\lambda$-Archimedean, there exists an $H \in {}^*\mathbb{N}$ and an $S \subseteq \{0, 1, \ldots, H\}$ such that $card(S) = \lambda$. Clearly, the set $S$ could be ordered as a $\lambda$-sequence. That sequence is bounded and has no convergent $\lambda$-subsequence because it is discrete. □

**Remark** Suppose $\mathcal{U}$ is a regular ultrafilter on any $\kappa$. Then the ultrapower of the standard superstructure modulo $\mathcal{U}$ could never be $\lambda$-Archimedean for any $\lambda$. So the existence of an $\lambda$-Archimedean ultrapower would imply the existence of some non-regular ultrafilters, which may imply the consistency of some large cardinals. By assuming the consistency of a measurable cardinal it is consistent that there exists a $\lambda$-Archimedean ultrapower for some $\lambda$ [18]. It is still open whether such kind of ultrapowers could satisfy the $\lambda$-Bolzano-Weierstrass property.

**Proposition 4.5** *(J. Cowles and R. LaGrange [6]) If $\mathcal{V}$ satisfies the $\lambda$-Bolzano-Weierstrass property, then the hyperreal field in $\mathcal{V}$ is Scott complete. (See Exercise 2.13 for the definition of Scott completeness.)*

**Proof:** Suppose ${}^*\mathcal{R}$ is not Scott complete. Then there is an upper bounded regular gap $I$ in ${}^*\mathbb{R}$. It is easy to see that the cofinality of $I$ is same as the cofinality of ${}^*\mathbb{N}$. But $\mathcal{V}$ is $\lambda$-Archimedean. So there exists an increasing $\lambda$-sequence cofinal in $I$. Clearly, the sequence is bounded and has no convergent $\lambda$-subsequence. □

**Remarks:** (1) Since *IP* implies the hyperreal field is not Scott complete, the $\lambda$-Bolzano-Weierstrass property is inconsistent with *IP*. (2) The reader who is interested in doing research on this subject should consult the papers [6], [18], [19], [21], [28], [29], [31], [32] and [33].

[6] The exact result in [21] is that: Suppose $\kappa$ and $\lambda$ are uncountable cardinals, $\kappa$ is regular, $\kappa < \lambda$, and $\eta^\delta < \lambda$ whenever $\delta < \kappa$ and $\eta < \lambda$. Then there exists a $\kappa$-saturated nonstandard universe satisfying the $\lambda$-Bolzano-Weierstrass property.

**Exercise 4.6** *(1) Let $\mathcal{F}$ be an ordered field. Show that the unit interval $[0,1]$ in $\mathcal{F}$ is compact iff $\mathcal{F}$ is isomorphic to the standard real field.*

*(2) Suppose $\aleph_1 = card({}^*\mathbb{N})$ in $\mathcal{V}$ ($\mathcal{V}$ may not be countably saturated). Show that $\mathcal{V}$ satisfies the $\aleph_1$-Bolzano-Weierstrass property iff the unit interval of the hyperreal field is Lindelöf.*

## References

1. Barwise, K. J., (1973) Back and forth through infinitary logic, *Studies in Model Theory*, M. Morley ed., MAA Studies **8**, pp. 5–34.
2. Benedikt, M., (1993) *Nonstandard Analysis and Special Ultrafilters*, Ph.D. Thesis, University of Wisconsin.
3. Benedikt, M., (to appear) Ultrafilters which extend measures, *The Journal of Symbolic Logic.*
4. Benedikt, M., Dong, G., Libkin, L., and Wong, L., (to appear) Model theoretic results on constraint queries, *The Journal of the Association for Computing machinery.*
5. Chang, C.C. and Keisler, H.J., (1990) *Model Theory.* 3rd ed., North-Holland, Amsterdam, (2nd ed., 1977).
6. Cowles, J. and LaGrange, R., (1983) Generalized Archimedean fields, *Notre Dame Journal of Formal Logic*, **24**, pp. 133–140.
7. Henson, C.W., (1974) The isomorphism property in nonstandard analysis and its use in the theory of Banach space, *The Journal of Symbolic Logic* **39**, pp. 717–731.
8. Henson, C.W., (1975) When do two Banach spaces have isometrically isomorphic nonstandard hulls, *Israel Journal of Mathematics* **22**, pp. 57–67.
9. Henson, C.W., (1979) unbounded Loeb measure, *Proceedings of the American Mathematical Society* **74**, pp. 143–150.
10. Henson, C.W. and Keisler, H.J., (1986) On the strength of nonstandard analysis, *The Journal of Symbolic Logic* **51**, pp. 377–386.
11. Henson, C.W. and Wattenberg, F., (1981) Egoroff's theorem and the distribution of standard points in a nonstandard model, *Proceedings of American Mathematical Society* **81**, pp. 455–461.
12. Jin, R., (1992) The isomorphism property versus the special model axiom, *The Journal of Symbolic Logic* **57**, pp. 975–987.
13. Jin, R., (1992) A theorem on the isomorphism property, *The Journal of Symbolic Logic* **57**, pp. 1011–1017.
14. Jin, R., (to appear) Type two cuts, bad cuts and very bad cuts, *The Journal of Symbolic Logic.*
15. Jin, R., (in preparation) Distinguishing strong saturation properties in nonstandard analysis.
16. Jin, R. and Shelah, S., (1994) The strength of the isomorphism property, *The Journal of Symbolic Logic* **59**, pp. 292–301.
17. Jin, R. and Shelah, S., (submitted) Compactness of Loeb spaces, *The Journal of Symbolic Logic.*
18. Jin, R. and Shelah, S., (in preparation) Possible size of an ultrapower of $\omega$.
19. Keisler, H.J., (1974) Models with tree structures, *Proceedings of Symposia in Pure Mathematics* **25**, American Mathematical Society, Providence, Rhode Island, pp. 331–348.
20. Keisler, H.J. and Leth, S.C., (1991) Meager sets on the hyperfinite time line, *The Journal of Symbolic Logic* **56**, pp. 71–102.

21. Keisler, H.J. and Schmerl, J.H., (1991) Making the hyperreal line both saturated and complete, *The Journal of Symbolic Logic* **56**, pp. 1016–1025.
22. Loeb, P., (1975) Conversion from nonstandard to standard measure spaces and applications in probability theory, *Transactions of American Mathematical Society* **211**, pp. 113–122.
23. Luxemburg, W.A.J., (1969) A general theory of monads, *Applications of Model Theory to Algebra, Analysis and Probability*, (W.A.J. Luxemburg, editor), Holt, Rinehart and Winston, New York, pp. 18–86.
24. Ross, D., (1983) *Measurable Transformations in Saturated Models of Analysis*, Ph.D. Thesis, University of Wisconsin.
25. Ross, D., (1987) Automorphisms of the Loeb algebra, *Fundamenta Mathematicae* **128**, pp. 29–36.
26. Ross, D., (1990) The special model axiom in nonstandard analysis, *The Journal of Symbolic Logic* **55**, pp. 1233–1242.
27. Ross, D., (1992) Compact measures have Loeb preimages, *Proceedings of American Mathematical Society* **115**, pp. 365–370.
28. Schmerl, J.H., (1985) Peano arithmetic and a question of Sikorski on ordered field, *Israel Journal of Mathematics* **50**, pp. 145–159.
29. Scott, D., (1969) On completing ordered fields, *Applications of Model Theory to Algebra, Analysis and Probability*, (W.A.J. Luxemburg, editor), Holt, Rinehart and Winston, New York, pp. 274–278.
30. Shelah, S., (1971) Every two elementarily equivalent models have isomorphic ultrapowers, *Israel Journal of Mathematics* **10**, pp. 224–233.
31. Shelah, S., (1978) Models with second order properties, II: Trees with no undefined branches, *Annals of Mathematical Logic* **14**, pp. 73–87.
32. Shelah, S., (1983) Models with second order properties, IV: A general method for eliminating diamonds, *Annals of Pure and Applied Logic* **25**, pp. 183–212.
33. Sikorski, R., (1948) On an ordered algebraic field, *Comptes Rendus des Séances de la Sociétés Sciences et des Lettres de Varsovie, Classe III* **41**, pp. 69–96.