Nonstandard Methods For Upper Banach Density Problems

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Abstract

A general method is developed by using nonstandard analysis for formulating and proving a theorem about upper Banach density parallel to each theorem about Shnirel'man density or lower asymptotic density.

There are many interesting results about Shnirel'man density or lower asymptotic density (see [4, Chapter 1] for example) in additive number theory. There are also a few interesting results about upper Banach density (see [3] or [2]) in combinatorial number theory. However, dealing with upper asymptotic density or upper Banach density in additive number theory is still an uncharted area. One of the major untouched problem in this area is finding the growth and structure of sums of sets of zero lower asymptotic density but positive upper density or upper Banach density. In this paper, we show a general method how, using nonstandard analysis, one can easily derive a parallel result about upper Banach density whenever one has a result about Shnirel'man density or lower asymptotic density in additive number theory. Serving as the testing cases of the idea, four parallel theorems about upper Banach density are formulated and proven in this paper. In §1, these four parallel theorems are stated. In §2, a brief introduction of nonstandard analysis is given. The introduction is intended for the reader without knowledge of nonstandard analysis. The reader who knows nonstandard analysis should ignore this section. In §3, all four theorems stated in §1 are proven using nonstandard analysis developed in §2. In §4, some comments are made.

[0] *Mathematics Subject Classification* Primary 11B05, 11B13, 03H05, 03H15
[0] *Keywords:* upper Banach density, nonstandard analysis
[0] The research was partially supported by Ralph E. Powe Junior Faculty Enhancement Award from Oak Ridge Associated Universities and by Faculty Research and Development Summer Grant from College of Charleston.

1 Results

Some notations are needed before I can state the theorems. We denote by $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{R}$ the set of all natural numbers, the set of all integers and the set of all real numbers, respectively. Let $\mathbf{N}_+$ denote the set of all positive integers. The letters $A, B, C, \ldots$ are always used for subsets of $\mathbf{Z}$ and the letters $k, m, n, \ldots$ are always used for (standard) integers. The Greek letters $\alpha, \beta, \epsilon, \ldots$ are always used for (standard) real numbers. For any integers a, b with $a \leqslant b$, we denote exclusively by $[a, b]$ the interval of integers $\{c : c \text{ is an integer and } a \leqslant c \leqslant b\}$. When $a > b$, let $[a, b]$ be the empty set. For any $A, B \subseteq \mathbf{Z}$, $n \in \mathbf{Z}$ and $h \in \mathbf{N}_+$, let $A \pm n = \{a \pm n : a \in A\}$, $A \pm B = \{a \pm b : a \in A \text{ and } b \in B\}$ and

$$hA = \{a_1 + a_2 + \cdots + a_h : a_1, a_2, \ldots, a_h \in A\}.$$

For any integers a, b and any set A, let $A(a) = |A \cap [1, a]|$ and $A(a, b) = |A \cap [a, b]|$ where $|\cdot|$ means the cardinality. A set $B \subseteq \mathbf{N}$ is called a basis if there exists an $h \in \mathbf{N}_+$, which is called the order of the basis B, such that $hB = \mathbf{N}$. A set $B \subseteq \mathbf{N}$ is called an asymptotic basis if there exists an $h \in \mathbf{N}_+$, which is called the asymptotic order of B, such that $\mathbf{N} \smallsetminus hB$ is a finite set. For a set A, the Shnirel'man density $\sigma(A)$, the lower asymptotic density $\underline{d}(A)$ and the upper Banach density $BD(A)$ are defined as the following:

$$\sigma(A) = \inf_{n \geqslant 1} \frac{A(n)}{n}$$

$$\underline{d}(A) = \liminf_{n \to \infty} \frac{A(n)}{n}$$

$$BD(A) = \lim_{n \to \infty} \sup_{0 \leqslant k \leqslant m, m-k=n} \frac{A(k, m)}{m - k + 1}.$$

Although the set A may contain negative integers, the three definitions above involve only the non-negative part of A. A set A is called thick if $BD(A) = 1$. A set $B \subseteq \mathbf{N}$ is called a Banach basis if there exists an $h \in \mathbf{N}_+$, which is called the Banach order of B, such that hB is thick. Note that a set is thick iff it contains k consecutive natural numbers for any $k \in \mathbf{N}$.

Now we are ready to state four results about the upper Banach density. The first result is a theorem parallel to Shnirel'man's theorem [4, page 8] or [12, page 195]. Shnirel'man's theorem says that for any set $A \subseteq \mathbf{N}$, if $\sigma(A) > 0$ and $0 \in A$, then

A is a basis. Shnirel'man's theorem can also be stated in terms of lower asymptotic density: if $\underline{d}(A) > 0$ and $0, 1 \in A$, then A is a basis. Our parallel theorem is the following.

Theorem 1 *Let $A \subseteq \mathbf{N}$ and let a_0 be the least element in A. If $BD(A) > 0$ and the greatest common divisor of all positive integers in $A - a_0$ is 1, then A is a Banach basis.*

Note that if A is a Banach basis, then the greatest common divisor of all positive integers in $A - a_0$ must be 1. The reason is the following. If the common divisor of all positive numbers in $A - a_0$ is $c > 1$, then hA is a subset of $\{ha_0 + nc : n \in \mathbf{N}\}$ which can never be thick.

The second result is a theorem parallel to Mann's theorem [4, page 5]. Mann's theorem says that for any sets $A, B \subseteq \mathbf{N}$, if $0 \in A \cap B$, then

$$\sigma(A + B) \geqslant \min\{\sigma(A) + \sigma(B), 1\}.$$

Using Mann's theorem one can give a quantitative proof of Shnirel'man's theorem: if $\sigma(A) > 0$ and $0 \in A$, then A is a basis of order at most $\lceil \frac{1}{\sigma(A)} \rceil$, where $\lceil \alpha \rceil$ is the least integer greater than or equal to α. The following is the parallel theorem.

Theorem 2 *Let $A, B \subseteq \mathbf{N}$. Then*

$$BD(A + B + \{0, 1\}) \geqslant \min\{BD(A) + BD(B), 1\}.$$

Note that the term $\{0, 1\}$ can be replaced by $\{c, c + 1\}$ for any $c \in \mathbf{N}$ because $BD(C) = BD(C + c)$ for any $C \subseteq \mathbf{N}$. Note also that $\{0, 1\}$ can't be omitted because, for example, that all even numbers plus all even numbers are all even numbers. Using Theorem 2, one can also give a quantitative proof of a variation of Theorem 1: if $BD(A) > 0$ and A contains two consecutive numbers, then A is a Banach basis of order at most $2\lceil \frac{1}{BD(A)} \rceil - 1$. This result is also optimal. Let c be an integer greater than 1 and let $A = \{2nc : n \in \mathbf{N}\} \cup \{2nc + 1 : n \in \mathbf{N}\}$. Then $BD(A) = \frac{1}{c}$. Hence, A is a Banach basis of order $2\lceil \frac{1}{BD(A)} \rceil - 1 = 2c - 1$. But the set $(2c - 2)A$ is disjoint from the set $\{2nc - 1 : n \in \mathbf{N}\}$. Hence, A is not a Banach basis of order $2c - 2$.

The third result is a theorem parallel to Plünnecke's theorem [13, page 225] about essential components. A set $B \subseteq \mathbf{N}$ is called an essential component if for any $A \subseteq \mathbf{N}$

with $0 < \sigma(A) < 1$, one has $\sigma(A+B) > \sigma(A)$. Plünnecke's theorem says that if B is a basis of order $h \geqslant 2$, then for every $A \subseteq \mathbf{N}$, $\sigma(A+B) \geqslant \sigma(A)^{1-\frac{1}{h}}$. As a corollary, any basis of finite order is an essential component. Let's call a set $B \subseteq \mathbf{N}$ a piecewise basis of order $h_p \in \mathbf{N}_+$ if there exists a sequence of intervals $\langle [a_n, b_n] : n \in \mathbf{N}\rangle$ such that $\lim_{n\to\infty}(b_n - a_n) = \infty$ and $[a_n, b_n] \subseteq h_p((B - a_n) \cap \mathbf{N}) + a_n$ for every $n \in \mathbf{N}$. It is easy to see that a basis of order h is a piecewise basis of order h. Our parallel theorem is the following.

Theorem 3 *Suppose B is a piecewise basis of order h_p. Then for any $A \subseteq \mathbf{N}$,*

$$BD(A+B) \geqslant BD(A)^{1-\frac{1}{h_p}}.$$

The fourth result is a parallel theorem to Erdös-Landau's theorem [4, page 10] and to Rohrbach's theorem [4, page 45]. Let B be a basis of order h. For any $m \in \mathbf{N}$ let $h(m) = \min\{h' : m \in h'B\}$. Clearly, $h(m) \leqslant h$. The number h^* is called the average order of B where

$$h^* = \sup_{n \geqslant 1} \frac{1}{n} \sum_{m=1}^{n} h(m).$$

Clearly, $h^* \leqslant h$. Let B be an asymptotic basis of order h_a such that $\mathbf{N} \setminus h_a B \subseteq [0, b_0]$. The number h_a^* is called the average asymptotic order of B where

$$h_a^* = \limsup_{n\to\infty} \frac{1}{n} \sum_{m=b_0+1}^{n} h(m).$$

Also clearly, $h_a^* \leqslant h_a$. Erdös-Landau's theorem says that if B is a basis of average order h^*, then for any $A \subseteq \mathbf{N}$,

$$\sigma(A+B) \geqslant \sigma(A) + \frac{1}{2h^*}\sigma(A)(1-\sigma(A)).$$

(Erdös-Landau's theorem is a direct consequence of Plünnecke's theorem. See the comments in [4, page 12] and [13, Corollary 7.2, page 226].) Rohrbach's theorem is an asymptotic analogue of Erdös-Landau's theorem, which says that if B is an asymptotic basis of average asymptotic order h_a^*, then for any $A \subseteq \mathbf{N}$,

$$\underline{d}(A+B) \geqslant \underline{d}(A) + \frac{1}{2h_a^*}\underline{d}(A)(1-\underline{d}(A)).$$

Before stating the parallel theorem, we need the definition of a piecewise asymptotic basis which is the "Banach" version of an asymptotic basis. A set $B \subseteq \mathbf{N}$ is called a piecewise asymptotic basis of piecewise asymptotic order h_{pa} if there exists a sequence of intervals $\langle [a_n, b_n] : n \in \mathbf{N} \rangle$ and there exists a number $k \in \mathbf{N}$ such that $\lim_{n\to\infty}(b_n - a_n) = \infty$ and

$$[a_n + k, b_n] \subseteq h_{pa}((B - a_n) \cap \mathbf{N}) + a_n$$

for every $n \in \mathbf{N}$. Note that a piecewise basis is a piecewise asymptotic basis. One needs only to take $k = 0$. Also a piecewise asymptotic basis is a Banach basis. One can easily construct a Banach basis which is not a piecewise asymptotic basis and construct an piecewise asymptotic basis which is not a piecewise basis. In Theorem 1, one can't replace the conclusion "A is a Banach basis" by "A is a piecewise asymptotic basis". Let B be a piecewise asymptotic basis. Suppose $\mathcal{I} = \langle [a_n, b_n] : n \in \mathbf{N} \rangle$, $k \in \mathbf{N}$ and $h_{pa} \in \mathbf{N}_+$ are given such that $\lim_{n\to\infty}(b_n - a_n) = \infty$ and $[a_n + k, b_n] \subseteq h_{pa}((B - a_n) \cap \mathbf{N}) + a_n$ for every $n \in \mathbf{N}$. For each $n \in \mathbf{N}$ and each $m \in [a_n + k, b_n]$, let

$$h_n(m) = \min\{h' \in \mathbf{N}_+ : [a_n + k, b_n] \subseteq h'((B - a_n) \cap \mathbf{N}) + a_n\}.$$

Let

$$h_n^* = \sup_{a_n + k \leqslant m \leqslant b_n} \frac{1}{m - a_n - k + 1} \sum_{i = a_n + k}^{m} h_n(i).$$

and let

$$h_{\mathcal{I},k}^* = \limsup_{n\to\infty} h_n^*.$$

Now the piecewise asymptotic average order h_{pa}^* of a piecewise asymptotic basis is defined as

$$h_{pa}^* = \inf\{h_{\mathcal{I},k}^* : \text{ for all suitable } \mathcal{I} \text{ and } k\}.$$

Now we are ready to state the theorem.

Theorem 4 *If $B \subseteq \mathbf{N}$ is a piecewise asymptotic basis of piecewise asymptotic average order h_{pa}^*, then for every $A \subseteq \mathbf{N}$,*

$$BD(A + B) \geqslant BD(A) + \frac{1}{2h_{pa}^*} BD(A)(1 - BD(A)).$$

Note that the piecewise asymptotic basis in above theorem can't be replaced by a Banach basis. A straightforward construction can produce a set A with $BD(A) = \frac{1}{2}$ and a Banach basis B of order 2 such that $BD(A + B) = \frac{1}{2}$. Note also that, mentioned in [4, page 12-13], Erdös and others have proven some results finer than

Erdös-Landau's theorem. The reader should be able to derive some parallel results about upper Banach density finer than Theorem 4 using the same ideas developed in §2 and §3.

2 Nonstandard Analysis

In this section, we briefly introduce the nonstandard analysis. Although the purpose of this introduction is only supplying enough background for this paper, the introduction itself may give the reader some ideas how the nonstandard analysis works in general. The detailed introduction can be found in [11] or [5]. [5] is written for the reader who has no background in mathematical logic.

Let $(\mathbf{R}; +, \cdot, \leqslant, 0, 1)$ be the (standard) real ordered field. We often write $\mathbf{R}$ for this field as well as its base set. Let $\wp(\mathbf{R})$ be the collection of all subsets of $\mathbf{R}$. We call the structure $\mathbf{V} = (\mathbf{R} \cup \wp(\mathbf{R}); +, \cdot, \leqslant, 0, 1, \in, |\cdot|)$ the standard model, where $\in$ is the membership relation between $\mathbf{R}$ and $\wp(\mathbf{R})$ and $|\cdot|$ is the cardinality function from the collection $Fin(\mathbf{R})$ of all finite subsets of $\mathbf{R}$ to $\mathbf{N}$ such that $|A|$ is the number of elements in A. We will also write $\mathbf{V}$ for the set $\mathbf{R} \cup \wp(\mathbf{R})$. Next we use an ultrapower construction to construct a nonstandard model ${}^*\mathbf{V}$ which is an extension of $\mathbf{V}$ and much more.

Ultrafilter on $\mathbf{N}$*:* A collection $\mathcal{U}$ of some subsets of $\mathbf{N}$ is called a filter if

(i) $\emptyset \notin \mathcal{U}$,

(ii) $A \in \mathcal{U}$ and $A \subseteq B$ imply $B \in \mathcal{U}$,

(iii) $A \in \mathcal{U}$ and $B \in \mathcal{U}$ imply $A \cap B \in \mathcal{U}$.

A filter $\mathcal{U}$ on $\mathbf{N}$ is called an ultrafilter if

(iv) for every $A \subseteq \mathbf{N}$, either $A \in \mathcal{U}$ or $\mathbf{N} \smallsetminus A \in \mathcal{U}$.

An ultrafilter $\mathcal{U}$ is called nonprincipal if

(v) $\mathbf{N} \smallsetminus [0, n] \in \mathcal{U}$ for every $n \in \mathbf{N}$.

Ultrafilter Theorem [6, page 55] *Assuming the axiom of choice, there exists a nonprincipal ultrafilter on* $\mathbf{N}$.

From now on, let's fix a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbf{N}$.

Ultrapower construction: Let $\mathbf{V}^{\mathbf{N}} = \{\langle v_n : n \in \mathbf{N}\rangle : v_n \in \mathbf{V}\}$ be the set of all $\mathbf{V}$-sequences. Using the ultrafilter $\mathcal{U}$, one can define an equivalence relation $\sim_{\mathcal{U}}$ on $\mathbf{V}^{\mathbf{N}}$ by letting $\langle u_n : n \in \mathbf{N}\rangle \sim_{\mathcal{U}} \langle v_n : n \in \mathbf{N}\rangle$ iff $\{n \in \mathbf{N} : u_n = v_n\} \in \mathcal{U}$. We denote by $[\langle v_n : n \in \mathbf{N}\rangle]$ the equivalence class containing $\langle v_n : n \in \mathbf{N}\rangle$. Let ${}^*\mathbf{V} = \mathbf{V}^{\mathbf{N}}/\mathcal{U}$ be the set of all equivalence classes. For each $v \in \mathbf{V}$, let ${}^*v \in {}^*\mathbf{V}$ be the equivalence class containing the constant sequence $\langle v : n \in \mathbf{N}\rangle$. Then the map $* : v \mapsto {}^*v$ is an embedding which embeds $\mathbf{V}$ into ${}^*\mathbf{V}$. Hence one can view $\mathbf{V}$ as a subset of ${}^*\mathbf{V}$. As a convention, we simply write r for ${}^*r$ when $r \in \mathbf{R}$. We can also extend $+, \cdot, \leqslant, \in$ and $|\cdot|$ onto ${}^*\mathbf{V}$ as the following.

For any $[\langle u_n : n \in \mathbf{N}\rangle]$ and $[\langle v_n : n \in \mathbf{N}\rangle]$ in ${}^*\mathbf{R}$ and any $[\langle A_n : n \in \mathbf{N}\rangle]$ in ${}^*\wp(\mathbf{R})$, let

$$[\langle u_n : n \in \mathbf{N}\rangle] + [\langle v_n : n \in \mathbf{N}\rangle] = [\langle u_n + v_n : n \in \mathbf{N}\rangle]$$

$$[\langle v_n : n \in \mathbf{N}\rangle] \cdot [\langle u_n : n \in \mathbf{N}\rangle] = [\langle u_n \cdot v_n : n \in \mathbf{N}\rangle]$$

$$[\langle u_n : n \in \mathbf{N}\rangle] \leqslant [\langle v_n : n \in \mathbf{N}\rangle] \text{ iff } \{n \in \mathbf{N} : u_n \leqslant v_n\} \in \mathcal{U}$$

$$[\langle u_n : n \in \mathbf{N}\rangle] \in [\langle A_n : n \in \mathbf{N}\rangle] \text{ iff } \{n \in \mathbf{N} : u_n \in A_n\} \in \mathcal{U}$$

and

$$|[\langle A_n : n \in \mathbf{N}\rangle]| = [\langle |A_n| : n \in \mathbf{N}\rangle].$$

Note that if every A_n is a finite set, then $[\langle A_n : n \in \mathbf{N}\rangle] \in {}^*Fin(\mathbf{R})$ and $|[\langle A_n : n \in \mathbf{N}\rangle]| \in {}^*\mathbf{N}$. The structure $({}^*\mathbf{V}; +, \cdot, \leqslant, 0, 1, \in, |\cdot|)$ is called a nonstandard model. We also use ${}^*\mathbf{V}$ for the nonstandard model. Obviously, ${}^*\mathbf{V}$ is an extension of $\mathbf{V}$. In fact, ${}^*\mathbf{R}$ is a (nonstandard) real ordered field, ${}^*\mathbf{Z}$ is a (nonstandard) integer ring, ${}^*\mathbf{N}$ is a (nonstandard) model of Peano arithmetic, etc. In ${}^*\mathbf{R}$ there are numbers closer to 0 than any standard non-zero real numbers in $\mathbf{R}$. These numbers are called infinitesimals. For example, $[\langle \frac{1}{n} : n \in \mathbf{N}\rangle]$ is a non-zero infinitesimal. There are also integers in ${}^*\mathbf{N}$ which are greater than any standard integers in $\mathbf{N}$. We call those numbers hyperfinite integers. For example, the number $[\langle n : n \in \mathbf{N}\rangle]$ is a hyperfinite integer. The reader is recommanded to visualize a hyperfinite integer not as a sequence, but as a single number extremely far away. We write $H, K, L, \ldots$ as well as $a, b, c \ldots$ for both finite or hyperfinite integers. We call all elements in ${}^*\mathbf{R}$ the numbers in ${}^*\mathbf{V}$ and call elements in ${}^*\wp(\mathbf{R})$ the sets in ${}^*\mathbf{V}$. There are three kinds of subsets of ${}^*\mathbf{R}$. All the subsets having the form ${}^*A$ for some $A \subseteq \mathbf{R}$ are called standard sets. All the subsets having the form $[\langle A_n : n \in \mathbf{N}\rangle]$ with $A_n \subseteq \mathbf{R}$ are called internal sets. A subset of ${}^*\mathbf{R}$ is called an external set if it is not internal. For example, the set $\mathbf{N}$ is an external subset of ${}^*\mathbf{R}$. In fact, every internal subset of ${}^*\mathbf{N}$ bounded above has a largest element. Let $A = [\langle A_n : n \in \mathbf{N}\rangle] \subseteq [0, H]$ for some hyperfinite integer

H. Then $\max A$, the largest element in A, is $[\langle \max A_n : n \in \mathbf{N}\rangle]$. A standard set is internal. The set $[\langle [0,n] : n \in \mathbf{N}\rangle]$ is internal but not standard. ${}^*\wp(\mathbf{R})$ is the collection of all internal subsets of ${}^*\mathbf{R}$.

The structure ${}^*\mathbf{V}$ is not only an extension of $\mathbf{V}$, but also has many other nice properties.

Logical formulas: Let $x, y, \ldots$ denote variables. Let p, q, r be either the variables or the elements, called constants, in ${}^*\mathbf{V}$. Then the following are called atomic formulas:

$p = q$, $p + q = r$, $pq = r$, $p \leqslant q$, $p \in q$ and $|p| = q$.

The meaning of these atomic formulas in ${}^*\mathbf{V}$ should be self-clear. For example, $p \in q$ means p is a number, q is a set and p is a member of q. When the constants in an atomic formula are all standard elements, the truth of the formula in $\mathbf{V}$ is same as the truth of the formula in ${}^*\mathbf{V}$.

Starting from those atomic formulas, one can form all (logical) formulas according to the following three recursive rules.

(1) If φ is a formula, so is $\neg\varphi$.

(2) If φ and ψ are formulas, so is $\varphi \wedge \psi$.

(3) If φ is a formula, so is $\exists x\varphi$ where x can be any variable.

The symbol $\neg$ stands for "not", $\wedge$ stands for "and" and $\exists x$ stands for "there exists an x such that...". Note that the interpretations of $\exists x$ in $\mathbf{V}$ and in ${}^*\mathbf{V}$ are different. The meaning of $\exists x$ in $\mathbf{V}$ is "there exists an x in $\mathbf{V}$ such that $\ldots$" while the meaning in ${}^*\mathbf{V}$ is "there exists an x in ${}^*\mathbf{V}$ such that $\ldots$". In the formula $\exists x\varphi$, φ is called the scope of the quantifier $\exists x$. An occurrence of a variable y in a formula φ is called free if the occurrence is not within the scope ψ of $\exists y$ for any subformula $\exists y\psi$ in φ. The symbols $\vee$ (stands for "or"), $\rightarrow$ (stands for "imply"), $\leftrightarrow$ (stands for "iff") and $\forall$ (stands for "for every") can also be used in logical formulas. Those symbols can be expressed using $\neg$, $\wedge$ and $\exists$ as the following:

$\varphi \vee \psi$ is equivalent to $\neg(\neg\varphi \wedge \neg\psi)$,

$\varphi \rightarrow \psi$ is equivalent to $\neg\varphi \vee \psi$,

$\varphi \leftrightarrow \psi$ is equivalent to $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$,

$\forall x\varphi$ is equivalent to $\neg\exists x\neg\varphi$.

The symbols $\cap$ (stands for "intersect"), $\cup$ (stands for "union"), $\smallsetminus$ (stands for "set-subtract") can also be used in a formula as the abbreviations. For example, $p \in q \cap r$ is equivalent to $(p \in q) \wedge (p \in r)$. The symbols $-$ (stands for "number subtract") and

/ (stands for "divide") can be expressed as: $p-q=r$ iff $q+r=p$ and $p/q=r$ iff $(\neg q=0)\wedge qr=p$.

Claim *Every mathematical statement in the theorems, propositions and lemmas of this paper can be expressed by a logical formula defined above.*

For example, the statement "$\sigma(A)\geqslant\alpha$" in $\mathbf{V}$ can be expressed as

$$\forall x(x\in\mathbf{N}\wedge\neg x=0\rightarrow\forall y(\forall z(\forall u(1\leqslant u\wedge u\leqslant x\leftrightarrow u\in z)\rightarrow y=|A\cap z|)\rightarrow\alpha\leqslant y/x)).$$

Proposition 1 *Let $v^{(1)},\ldots,v^{(k)}\in{}^*\mathbf{V}$ and let $\varphi=\varphi(v^{(1)},\ldots,v^{(k)})$ be a formula such that it contains no free variables and $v^{(1)},\ldots,v^{(k)}$ are only constants in it. Then φ is true in ${}^*\mathbf{V}$ iff*

$$\{n\in\mathbf{N}:\varphi(v_n^{(1)},\ldots,v_n^{(k)})\textit{ is true in }\mathbf{V}\}\in\mathcal{U}$$

where $v^{(i)}=[\langle v_n^{(i)}:n\in\mathbf{N}\rangle]$.

Note that it makes no sense to speak about the truth of a formula in $\mathbf{V}$ when the formula contains some constants in ${}^*\mathbf{V}\smallsetminus\mathbf{V}$.

Proof: We prove the proposition by induction on the complexity of the formulas. It is easy to check that the proposition is true for any atomic formula. Suppose the proposition is true for ψ and χ.

If $\varphi(v^{(1)},\ldots,v^{(k)})$ is $\psi(v^{(1)},\ldots,v^{(k)})\wedge\chi(v^{(1)},\ldots,v^{(k)})$, then φ is true in ${}^*\mathbf{V}$ iff ψ is true in ${}^*\mathbf{V}$ and χ is true in ${}^*\mathbf{V}$ iff

$$\{n:\varphi(v_n^{(1)},\ldots,v_n^{(k)})\text{ is true in }\mathbf{V}\}=$$

$$\{n:\psi(v_n^{(1)},\ldots,v_n^{(k)})\text{ is true in }\mathbf{V}\}\cap\{n:\chi(v_n^{(1)},\ldots,v_n^{(k)})\text{ is true in }\mathbf{V}\}\in\mathcal{U}.$$

If $\varphi(v^{(1)},\ldots,v^{(k)})$ is $\neg\psi(v^{(1)},\ldots,v^{(k)})$, then φ is true in ${}^*\mathbf{V}$ iff ψ is not true in ${}^*\mathbf{V}$ iff $\{n:\psi(v_n^{(1)},\ldots,v_n^{(k)})\text{ is true in }\mathbf{V}\}\notin\mathcal{U}$ iff

$$\{n:\varphi(v_n^{(1)},\ldots,v_n^{(k)})\text{ is true in }\mathbf{V}\}=\mathbf{N}\smallsetminus\{n:\psi(v_n^{(1)},\ldots,v_n^{(k)})\text{ is true in }\mathbf{V}\}\in\mathcal{U}.$$

Suppose $\varphi(v^{(1)},\ldots,v^{(k)})$ is $\exists x\psi(x,v^{(1)},\ldots,v^{(k)})$. Then φ is true in ${}^*\mathbf{V}$ implies that there is a $v\in{}^*\mathbf{V}$ such that $\psi(v,v^{(1)},\ldots,v^{(k)})$ is true in ${}^*\mathbf{V}$ which implies

$$\{n:\psi(v_n,v_n^{(1)},\ldots,v_n^{(k)})\text{ is true in }\mathbf{V}\}\in\mathcal{U}.$$

Hence

$$\{n : \exists x\psi(x, v_n^{(1)}, \ldots, v_n^{(k)}) \text{ is true in } \mathbf{V}\} \in \mathcal{U}.$$

On the other hand,

$$U = \{n : \exists x\psi(x, v_n^{(1)}, \ldots, v_n^{(k)}) \text{ is true in } \mathbf{V}\} \in \mathcal{U}$$

implies that for each $n \in U$, there exists a $v_n \in \mathbf{V}$ such that $\psi(v_n, v_n^{(1)}, \ldots, v_n^{(k)})$ is true in $\mathbf{V}$. Let $v_n = 0$ for each $n \in \mathbf{N} \setminus U$ and let $v = [\langle v_n : n \in \mathbf{N}\rangle]$. Then $\psi(v, v^{(1)}, \ldots, v^{(k)})$ is true in ${}^*\mathbf{V}$. Hence $\exists x\psi(x, v^{(1)}, \ldots, v^{(k)})$ is true in ${}^*\mathbf{V}$. $\square$

Proposition 2 *Let $\varphi(a_1, \ldots, a_m, A_1, \ldots, A_n)$ be a formula such that $a_1, \ldots, a_m \in \mathbf{R}$ and $A_1, \ldots, A_n \subseteq \mathbf{R}$ are only constants in φ and φ contains no free variables. Then $\varphi(a_1, \ldots, a_m, A_1, \ldots, A_n)$ is true in $\mathbf{V}$ iff $\varphi(a_1, \ldots, a_m, {}^*A_1, \ldots, {}^*A_n)$ is true in ${}^*\mathbf{V}$.*

Proof Again by induction. It is easy to check that the proposition is true for every atomic formula. Suppose the proposition is true for ψ and χ.

If $\varphi(a_1, \ldots, a_m, A_1, \ldots, A_n)$ is $\psi(a_1, \ldots, a_m, A_1, \ldots, A_n) \wedge \chi(a_1, \ldots, a_m, A_1, \ldots, A_n)$, then φ is true in $\mathbf{V}$ iff ψ is true in $\mathbf{V}$ and χ is true in $\mathbf{V}$ iff $\psi(a_1, \ldots, a_m, {}^*A_1, \ldots, {}^*A_n)$ is true in ${}^*\mathbf{V}$ and $\chi(a_1, \ldots, a_m, {}^*A_1, \ldots, {}^*A_n)$ is true in ${}^*\mathbf{V}$ iff

$$\varphi(a_1, \ldots, a_m, {}^*A_1, \ldots, {}^*A_n) = \psi(a_1, \ldots, a_m, {}^*A_1, \ldots, {}^*A_n) \wedge \chi(a_1, \ldots, a_m, {}^*A_1, \ldots, {}^*A_n)$$

is true in ${}^*\mathbf{V}$.

If $\varphi(a_1, \ldots, a_m, A_1, \ldots, A_n)$ is $\neg\psi(a_1, \ldots, a_m, A_1, \ldots, A_n)$, then φ is true in $\mathbf{V}$ iff ψ is not true in $\mathbf{V}$ iff $\psi(a_1, \ldots, a_m, {}^*A_1, \ldots, {}^*A_n)$ is not true in ${}^*\mathbf{V}$ iff

$$\varphi(a_1, \ldots, a_m, {}^*A_1, \ldots, {}^*A_n) = \neg\psi(a_1, \ldots, a_m, {}^*A_1, \ldots, {}^*A_n)$$

is true in ${}^*\mathbf{V}$.

Suppose $\varphi(a_1, \ldots, a_m, A_1, \ldots, A_n)$ is $\exists x\psi(x, a_1, \ldots, a_m, A_1, \ldots, A_n)$. φ is true in $\mathbf{V}$ implies that there is an $a \in \mathbf{V}$ (a could be either a number or a set.) such that $\psi(a, a_1, \ldots, a_m, A_1, \ldots, A_n)$ is true in $\mathbf{V}$ which implies $\psi({}^*a, a_1, \ldots, a_m, {}^*A_1, \ldots, {}^*A_n)$ is true in ${}^*\mathbf{V}$. Hence $\varphi(a_1, \ldots, a_m, {}^*A_1, \ldots, {}^*A_n)$ is true in ${}^*\mathbf{V}$. On the other hand, if $\varphi(a_1, \ldots, a_m, {}^*A_1, \ldots, {}^*A_n)$ is true in ${}^*\mathbf{V}$, then there is a $v \in {}^*\mathbf{V}$ such that $\psi(v, a_1, \ldots, a_m, {}^*A_1, \ldots, {}^*A_n)$ is true in ${}^*\mathbf{V}$. By Proposition 1,

$$U = \{n : \psi(v_n, a_1, \ldots, a_m, A_1, \ldots, A_n) \text{ is true in } \mathbf{V}\} \in \mathcal{U}.$$

Since $U \neq \emptyset$, there is an $v_n \in \mathbf{V}$ such that $\psi(v_n, a_1, \ldots, a_m, A_1, \ldots, A_n)$ is true in $\mathbf{V}$. Hence φ is true in $\mathbf{V}$. $\square$

Proposition 2 is called the transfer principle.

Proposition 3 *Let $\varphi(x, v^{(1)}, \ldots, v^{(k)})$ be a formula such that $v^{(1)}, \ldots, v^{(k)} \in {}^*\mathbf{V}$ and x is the only free variable. Then the set $\{a \in {}^*\mathbf{R} : \varphi(a, v^{(1)}, \ldots, v^{(k)})$ is true in ${}^*\mathbf{V}\}$ is internal.*

Proof For each $n \in \mathbf{N}$, let $A_n = \{\alpha \in \mathbf{R} : \varphi(\alpha, v_n^{(1)}, \ldots, v_n^{(k)})$ is true in $\mathbf{V}\} \subseteq \mathbf{R}$. Let $A = [\langle A_n : n \in \mathbf{N}\rangle]$. Then A is an internal set. We leave to the reader to check that

$$A = \{a \in {}^*\mathbf{R} : \varphi(a, v^{(1)}, \ldots, v^{(k)}) \text{ is true in } {}^*\mathbf{V}\}. \quad \square$$

Proposition 4 *Suppose $\{A^{(k)} : k \in \mathbf{N}\}$ is a collection of non-empty internal sets such that $A^{(1)} \supseteq A^{(2)} \supseteq \ldots \supseteq A^{(k)} \supseteq \ldots$. Then there is an $v \in {}^*\mathbf{V}$ such that $v \in A^{(k)}$ for every $k \in \mathbf{N}$.*

Proof For convenience, let $A^{(0)} = {}^*\mathbf{V}$. For each k, pick a $v^{(k)} \in A^{(k)}$ and let

$$U_k = \{n \in \mathbf{N} : v_n^{(k)} \in A_n^{(k)} \subseteq A_n^{(k-1)} \subseteq \cdots \subseteq A_n^{(0)}\} \smallsetminus [0, k].$$

By Proposition 1, $U_k \in \mathcal{U}$. It is also clear that $\cap_{k=0}^{\infty} U_k = \emptyset$. For each $n \in \mathbf{N}$, let $k_n = \max\{k : n \in U_k\}$ and define $v_n^{(H)} = v_n^{(k_n)}$. Let $v^{(H)} = [\langle v_n^{(H)} : n \in \mathbf{N}\rangle] \in {}^*\mathbf{V}$.

Claim: $v^{(H)} \in A^{(k)}$ for every $k \in \mathbf{N}$.

Given a $k \in \mathbf{N}$, it suffices to show that $\{n : v_n^{(H)} \in A_n^{(k)}\} \supseteq U_k$. For each $n \in U_k$, $k_n \geqslant k$ by the maximality of k_n and $n \in U_{k_n}$. So $v_n^{(H)} = v_n^{(k_n)} \in A_n^{(k_n)} \subseteq A_n^{(k)}$. Hence $U_k \subseteq \{n : v_n^{(H)} \in A_n^{(k)}\}$. $\square$

Proposition 4 is called the countable saturation property.

Loeb spaces: Given a hyperfinite integer H, $\Omega = [0, H-1]$ is a hyperfinite set. Let $A \subseteq \Omega$ be an internal set. Then $|A|$ is an integer between 0 and H. Hence $|A|/H$ is a number in ${}^*\mathbf{R}$ between 0 and 1. By the completeness of $\mathbf{R}$, one can find a unique standard real number α between 0 and 1 such that $|A|/H$ is infinitesimally close to α. Let's call α the standard part of $|A|/H$ denoted by $st(|A|/H) = \alpha$. (In fact, st is defined on every number r in ${}^*\mathbf{R}$ as long as r is between two standard real numbers.) Let Σ_0 be the collection of all internal subsets of Ω and let $\mu(A) = st(|A|/H)$ for every

$A \in \Sigma_0$. Then (Ω, Σ_0, μ) is a finitely-additive probability space from the standard point of view. For any subset S of Ω, internal or external, define

$$\bar{\mu}(S) = \inf\{\mu(A) : A \in \Sigma_0 \text{ and } A \supseteq S\}$$

$$\underline{\mu}(S) = \sup\{\mu(A) : A \in \Sigma_0 \text{ and } A \subseteq S\}$$

and let

$$\Sigma = \{S \subseteq \Omega : \bar{\mu}(S) = \underline{\mu}(S)\}.$$

It is easy to see that $\Sigma_0 \subseteq \Sigma$. For each $S \in \Sigma$, define $\mu_L(S) = \bar{\mu}(S) = \underline{\mu}(S)$. Then (Ω, Σ, μ_L) is a standard, countably-additive, atomless, complete probability space, which is called a hyperfinite Loeb space generated by a normalized uniform counting measure $|\cdot|/H$. Let's call it simply a Loeb space on $[0, H-1]$. Note that the Loeb space construction can be carried out on any hyperfinite set instead of $[0, H-1]$. The reader should notice that the verification of the countable-additivity requires using Proposition 4. Loeb space is a very important tool for applying nonstandard analysis to other fields of mathematics, especially to probability theory (see [1]).

3 Proofs

We introduce Birkhoff ergodic theorem (see, for example, [14, page 30] or [3, page 59]) and prove several lemmas before proving the theorems. Let (Ω, Σ, μ) be a probability space. A bijection T from Ω to Ω is called a measure-preserving transformation if both T and T^{-1} are measurable and $\mu(E) = \mu(T[E])$ for every measurable set $E \in \Sigma$. Let T^0 be the identity function. For any $n \in \mathbf{N}_+$, let $T^n(x) = T(T^{n-1}(x))$.

Birkhoff Ergodic Theorem *Suppose (Ω, Σ, μ) is a probability space and T is a measure-preserving transformation from Ω to Ω. For any function $f \in L^1(\Omega)$, there exists a function $\bar{f} \in L^1(\Omega)$ such that*

$$\mu(\{x \in \Omega : \lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} f(T^m(x)) = \bar{f}(x)\}) = 1.$$

Lemma 1 *Let α be a real number. For any set $A \subseteq \mathbf{N}$, $BD(A) \geqslant \alpha$ iff there is an infinitesimal $\iota \geqslant 0$ and an interval $I = [H, K] \subseteq {}^*\mathbf{N}$ of hyperfinite length such that*

$$\frac{{}^*A(H, K)}{K - H + 1} \geqslant \alpha - \iota.$$

Proof "$\Rightarrow$": Assume $BD(A) \geqslant \alpha$. Let $\varphi(\alpha, \mathbf{N}, A)$ be the statement "for every x in $\mathbf{N}_+$, there exists a, b in $\mathbf{N}$ such that $b - a > x$ and $\frac{A(a,b)}{b-a+1} \geqslant \alpha - \frac{1}{x}$". Then $\varphi(\alpha, \mathbf{N}, A)$ is true in $\mathbf{V}$. By Proposition 2, $\varphi(\alpha, {}^*\mathbf{N}, {}^*A)$ is true in ${}^*\mathbf{V}$. Hence one can choose a hyperfinite integer in ${}^*\mathbf{N}$ for x and the proof is done.

"$\Leftarrow$": Assume $\frac{{}^*A(H,K)}{K-H+1} \geqslant \alpha - \iota$. For each $n \in \mathbf{N}$, let $\varphi(n, \alpha, {}^*\mathbf{N}, {}^*A)$ be the statement "there exists $x, y \in {}^*\mathbf{N}$ such that $y - x > n$ and $\frac{{}^*A(x,y)}{y-x+1} \geqslant \alpha - \frac{1}{n}$". Since $x = H$ and $y = K$ witness the truth of $\varphi(n, \alpha, {}^*\mathbf{N}, {}^*A)$, $\varphi(n, \alpha, \mathbf{N}, A)$ is true in $\mathbf{V}$ for each $n \in \mathbf{N}$ again by Proposition 2. Hence $BD(A) \geqslant \alpha$. $\square$

For each internal set $C \subseteq {}^*\mathbf{Z}$, we define $\underline{d}(C) = \underline{d}(C \cap \mathbf{N})$. We also do the same for σ and BD. Keep in mind that the definitions of $\underline{d}(C)$, $\sigma(C)$ and $BD(C)$ involve only the part of C in $\mathbf{N}$ although C may contain hyperfinite integers or negative integers.

Lemma 2 *Suppose $A \subseteq \mathbf{N}$ and $BD(A) = \alpha$. Then there is an interval of hyperfinite length $[H, K]$ such that for almost all $x \in [H, K]$ in terms of the Loeb measure μ_L on $[H, K]$, $\underline{d}({}^*A - x) = \alpha$. On the other hand, if $A \subseteq \mathbf{N}$ and there is an $a \in {}^*\mathbf{N}$ such that $\underline{d}({}^*A - a) \geqslant \alpha$, then $BD(A) \geqslant \alpha$.*

Proof Suppose $BD(A) = \alpha$. By Lemma 1, there is an interval of hyperfinite length $[H, K]$ such that $\frac{{}^*A(H,K)}{K-H+1}$ is infinitesimally close to α. Hence, the Loeb measure of the set ${}^*A(H, K)$ in $[H, K]$ is α. Let T be the map from $[H, K]$ to $[H, K]$ such that $T(K) = H$ and $T(x) = x + 1$ for every $x \in [H, K - 1]$. Then T is a Loeb measure-preserving transformation. Let f be the characteristic function of the set ${}^*A(H, K)$. By Birkhoff ergodic theorem there is a measurable function $\bar{f}$ such that for almost all $x \in [H, K]$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} f(T^m(x)) = \bar{f}(x).$$

Since the integration over $[H, K]$ of the left-side is α, then $\int_{[H,K]} \bar{f} d\mu_L = \alpha$. We want to show that $\bar{f}(x) = \alpha$ almost surely. Note that the set $\cap_{n=0}^{\infty}[H, K - n]$ has Loeb measure 1.

Suppose there is an $a \in \cap_{n=0}^{\infty}[H, K - n]$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} f(T^m(a)) = \beta > \alpha.$$

The interpretation of the above limit shows that $\frac{{}^*A(a,a+n)}{n+1} > \frac{\beta+\alpha}{2}$ for n large enough. Let D be the set of all those b's in ${}^*\mathbf{N}$ such that $\frac{{}^*A(a,a+b)}{b+1} > \frac{\beta+\alpha}{2}$. By Proposition 3, D

is an internal subset of ${}^*\mathbf{N}$ which contains all large enough n in $\mathbf{N}$. Since $\mathbf{N} \smallsetminus [0,n]$ is not internal in ${}^*\mathbf{V}$ for every $n \in \mathbf{N}$, there is a hyperfinite integer L in D. Hence $\frac{{}^*A(a,a+L)}{L+1} > \frac{\beta+\alpha}{2}$. By Lemma 1, $BD(A) \geqslant \frac{\beta+\alpha}{2} > \alpha$. This contradicts $BD(A) = \alpha$.

Suppose $\mu_L(\{x \in [H,K] : \bar{f}(x) < \alpha\}) > 0$. Then $\int_{[H,K]} \bar{f}(x) d\mu_L = \alpha$ implies that $\mu_L(\{x \in [H,K] : \bar{f}(x) > \alpha\}) > 0$. Hence there exists an $a \in \cap_{n=0}^{\infty}[H, K-n]$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} f(T^m(a)) > \alpha.$$

Now a contradiction can be derived by the same reason in the paragraph above.

The first half of the lemma is proven by the fact that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} f(x+m)) = \alpha$$

implies $\underline{d}({}^*A - x) = \alpha$.

The proof of the second half of the lemma is in fact included in the second paragraph of the proof of the first half. $\square$

Given a set $A \subseteq \mathbf{N}$ and an interval $[a,b] \subseteq \mathbf{N}$. Let

$$\sigma_{[a,b]}(A) = \inf_{a \leqslant m \leqslant b} \frac{A(a,m)}{m-a+1}$$

and let

$$BSD(A) = \lim_{n\to\infty} \sup_{0 \leqslant a \leqslant b, b-a=n} \sigma_{[a,b]}(A).$$

Let's call $BSD(A)$ the Banach-Shnirel'man density of A. Clearly, $BSD(A) \leqslant BD(A)$.

Lemma 3 *Let α be a real number. For any set $A \subseteq \mathbf{N}$, $BSD(A) \geqslant \alpha$ iff there is an infinitesimal $\iota \geqslant 0$ and an interval $[H,K] \subseteq {}^*\mathbf{N}$ of hyperfinite length such that*

$$\inf_{H \leqslant L \leqslant K} \frac{{}^*A(H,L)}{L-H+1} \geqslant \alpha - \iota.$$

Proof The proof is similar to the proof of Lemma 1. I leave the proof to the reader. $\square$

Lemma 4 *Let $A \subseteq \mathbf{N}$. Then $BSD(A) \geqslant \alpha$ iff there exists an $a \in {}^*\mathbf{N}$ such that*

$$\sigma({}^*A - a + 1) \geqslant \alpha.$$

Proof The proof of the lemma is also similar to the proof of Lemma 2.

"$\Rightarrow$": Assume $BSD(A) \geqslant \alpha$. By Proposition 2, there exists an interval $[H, K]$ of hyperfinite length such that

$$\inf_{H \leqslant L \leqslant K} \frac{{}^*\!A(H, L)}{L - H + 1} \geqslant \alpha - \iota$$

for some infinitesimal ι. It is easy to see now that $\sigma({}^*\!A - H + 1) \geqslant \alpha$.

"$\Leftarrow$": Assume $\sigma({}^*\!A - a + 1) \geqslant \alpha$. Then by Proposition 3, there exists a hyperfinite integer H such that

$$\inf_{0 \leqslant L \leqslant H} \frac{{}^*\!A(a, a + L)}{L + 1} \geqslant \alpha.$$

Certainly $\alpha \geqslant \alpha - \iota$ for any $\iota \geqslant 0$. The conclusion follows from Lemma 3. $\square$

Lemma 5 *Let $A \subseteq \mathbf{N}$ be such that $\underline{d}(A) = \alpha$ and $\epsilon > 0$. Then there is an $n_0 \in \mathbf{N}$ such that $\sigma(A - n_0) \geqslant \alpha - \epsilon$.*

Proof Suppose the lemma is not true. Then one can find a strictly increasing sequence of natural numbers $\langle i_n : n \in \mathbf{N} \rangle$ such that $\frac{A(i_n, i_{n+1} - 1)}{i_{n+1} - i_n} \leqslant \alpha - \epsilon$. This implies $\underline{d}(A) \leqslant \alpha - \epsilon$ which contradicts $\underline{d}(A) = \alpha$. $\square$

Lemma 6 *Let $A \subseteq \mathbf{N}$. Then $BD(A) = BSD(A)$.*

Proof Clearly, $BSD(A) \leqslant BD(A)$. Given any $\epsilon > 0$, it suffices to show that $BSD(A) \geqslant BD(A) - \epsilon$.

Let $BD(A) = \alpha$. By Lemma 2, there is an $x \in {}^*\mathbf{N}$ such that $\underline{d}({}^*\!A - x) = \alpha$. By Lemma 5, one can find a $y \in {}^*\mathbf{N}$, $y \geqslant x$ such that $\sigma({}^*\!A - y) \geqslant \alpha - \epsilon$. By Lemma 4, one has $BSD(A) \geqslant \alpha - \epsilon$. $\square$

Now we are ready to prove the theorems stated in §1.

Proof of Theorem 1 Suppose $BD(A) = \alpha > 0$. Since $\gcd(A - a_0) = 1$, there exists an $h_0 \in \mathbf{N}_+$ such that $h_0(A - a_0)$ contains two consecutive numbers. Hence the set $h_0 A$ contains two consecutive numbers $c, c + 1$. By Lemma 6 and Lemma 4, there is an $x \in {}^*\mathbf{N}$, such that $\sigma({}^*\!A - x + 1) = \alpha$. Note that $x \in {}^*\!A$. Now

$$\sigma((1 + h_0){}^*\!A - x - c) \geqslant \sigma({}^*\!A + \{c, c + 1\} - x - c) \geqslant \sigma({}^*\!A - x + 1) = \alpha$$

and

$$0 = x + c - x - c \in (1 + h_0){}^*\!A - x - c.$$

By Shnirel'man's theorem, there is an $h_1 \in \mathbf{N}_+$ such that

$$h_1((1+h_0){}^*\!A - x - c) = h_1(1+h_0){}^*\!A - h_1(x+c) \supseteq \mathbf{N}.$$

By Proposition 3, there is a hyperfinite integer H such that

$$h_1(1+h_0){}^*\!A - h_1(x+c) \supseteq [0, H].$$

Let $h = h_1(1+h_0)$. Then

$${}^*(hA) = h{}^*\!A \supseteq [h_1(x+c), h_1(x+c)+H].$$

By Lemma 1, $BD(hA) = 1$. $\square$

Proof of Theorem 2 The proof of Theorem 2 needs Besicovitch's theorem [4, page 6], which says if $1 \in A$, $0 \in B$ and β is a non-negative real number such that $\inf_{n \geqslant 1} \frac{B(n)}{n+1} \geqslant \beta$, then

$$\sigma(A+B) \geqslant \min\{\sigma(A)+\beta, 1\}.$$

Let $BD(A) = \alpha$ and $BD(B) = \beta$. Without loss of generality, let's assume $\beta \leqslant \alpha$ and $\alpha + \beta \leqslant 1$. Hence $\beta \leqslant \frac{1}{2}$. By Lemma 6 and Lemma 4, there exist $a \in {}^*\!A$ and $b \in {}^*\!B$ such that $\sigma({}^*\!A - a + 1) = \alpha$ and $\sigma({}^*\!B - b + 1) = \beta$. Let $B' = B + \{0, 1\}$. Clearly, $1 \in {}^*\!A - a + 1$ and $0 \in {}^*\!B' - b$. We want to check that

$$\inf_{n \geqslant 1} \frac{({}^*\!B' - b)(n)}{n+1} \geqslant \beta.$$

Let

$$k_0 = \min\{n \in \mathbf{N} : n \notin {}^*\!B' - b\}$$

and let

$$k_1 = \min\{n \in \mathbf{N} : n > k_0 \text{ and } n \in {}^*\!B - b + 1\}.$$

Obviously, $1 < k_0 < k_1 - 1$. Let $n \in \mathbf{N}_+$.

Case 1: $1 \leqslant n < k_0$. Then $({}^*\!B' - b)(n) = n \geqslant \frac{1}{2}(n+1) \geqslant \beta(n+1)$.

Case 2: $k_0 \leqslant n < k_1 - 1$. Then

$$({}^*\!B' - b)(n) \geqslant ({}^*\!B - b + 1)(n) = ({}^*\!B - b + 1)(n+1) \geqslant \beta(n+1).$$

This is because $\sigma({}^*\!B - b + 1) = \beta$.

Case 3: $n \geqslant k_1 - 1$. Then because $k_1 - 1 \in ({}^*\!B' - b) \setminus ({}^*\!B - b + 1)$, one has

$$({}^*\!B' - b)(n) \geqslant ({}^*\!B - b + 1)(n) + 1 \geqslant \beta n + 1 \geqslant \beta(n+1).$$

Following above three cases, we conclude

$$\inf_{n \geqslant 1} \frac{({}^*\!B' - b)(n)}{n+1} \geqslant \beta.$$

Applying Besicovitch's theorem, we have

$$\sigma(({}^*\!A - a + 1) + ({}^*\!B' - b)) = \sigma(({}^*\!A + {}^*\!B') - (a + b - 1)) \geqslant \alpha + \beta.$$

By Lemma 4 and ${}^*\!A + {}^*B' = {}^*(A + B')$, one has

$$BSD(A + B + \{0,1\}) = BSD(A + B') \geqslant \alpha + \beta.$$

And by Lemma 6, one has

$$BD(A + B + \{0,1\}) = BD(A + B') \geqslant \alpha + \beta. \quad \square$$

Proof of Theorem 3 Let B be a piecewise basis of piecewise order h. Let $\langle [a_n, b_n] : n \in \mathbf{N} \rangle$ be a sequence of intervals such that $\lim_{n=0}^{\infty}(b_n - a_n) = \infty$ and

$$h((B - a_n) \cap \mathbf{N}) + a_n \supseteq [a_n, b_n]$$

for $n \in \mathbf{N}$. Let $\varphi(h, B, \mathbf{N})$ be the statement "for every $x \in \mathbf{N}$, there exist y and z in $\mathbf{N}$ such that $z - y > x$ and $h((B - y) \cap \mathbf{N}) + y \supseteq [y, z]$". By Proposition 2, $\varphi(h, {}^*B, {}^*\mathbf{N})$ is true in ${}^*\mathbf{V}$. Let x be a hyperfinite integer. Then there exists an interval $[H, K]$ of hyperfinite length such that $h(({}^*\!B - H) \cap {}^*\mathbf{N}) + H \supseteq [H, K]$ or $h(({}^*\!B - H) \cap {}^*\mathbf{N}) \supseteq [0, K - H]$. This shows $h(({}^*\!B - H) \cap \mathbf{N}) = \mathbf{N}$. Let $A \subseteq \mathbf{N}$ be such that $BD(A) = \alpha$. By Lemma 6 and Lemma 4, there exists an $a \in {}^*\!A$ such that $\sigma({}^*\!A - a + 1) = \alpha$. By Plünnecke's theorem, $\sigma(({}^*\!A - a + 1) + ({}^*\!B - H)) \geqslant \alpha^{1 - \frac{1}{h}}$. Then, by Lemma 6,

$$BD(A + B) \geqslant \sigma(({}^*\!A + {}^*\!B - (a + H - 1))) = \sigma(({}^*\!A - a + 1) + ({}^*\!B - H)).$$

So we conclude that

$$BD(A + B) \geqslant BD(A)^{1 - \frac{1}{h}}. \quad \square$$

Proof of Theorem 4 Let B be a piecewise asymptotic basis of piecewise asymptotic average order h^*_{pa}. Given $\epsilon > 0$, there exists a suitable sequence $\mathcal{I}$ of intervals and a $k \in \mathbf{N}$ such that

$$h^*_{pa} + \frac{\epsilon}{2} \geqslant h^*_{\mathcal{I},k}.$$

By Proposition 2, there exists an interval $[H, K]$ of hyperfinite length such that

$$\sup_{H+k \leqslant L \leqslant K} \frac{1}{L - H - k + 1} \sum_{i=H+k}^{L} h_{[H,K]}(i) \leqslant h^*_{\mathcal{I},k} + \frac{\epsilon}{2}$$

where $h_{[H,K]}(i) = \min\{h' \in {}^*\mathbf{N} : i \in h'(({}^*B - H) \cap {}^*\mathbf{N}) + H)\}$. Obviously, the asymptotic average order $\bar{h}$ of the asymptotic basis $({}^*B - H) \cap \mathbf{N}$ satisfies

$$\bar{h} \leqslant h^*_{\mathcal{I},k} + \frac{\epsilon}{2} \leqslant h^*_{pa} + \epsilon.$$

Let $BD(A) = \alpha$. Then by Lemma 2, there exists an $a \in {}^*\mathbf{N}$ such that $\underline{d}({}^*A - a) = \alpha$. Applying Rohrbach's theorem, we have

$$\underline{d}(({}^*A - a) + ({}^*B - H)) = \underline{d}(({}^*A + {}^*B) - (a + H)) \geqslant \alpha + \frac{1}{2(h^*_{pa} + \epsilon)} \alpha(1 - \alpha).$$

By Lemma 1, we have

$$BD(A + B) \geqslant BD(A) + \frac{1}{2(h^*_{pa} + \epsilon)} BD(A)(1 - BD(A)).$$

Since ϵ can be arbitrarily small, the conclusion follows. $\square$

4 Comments

(1) The main goal of this paper is not just for supplying the proofs to the theorems in §1. In fact, the whole procedure of producing the proofs reveals a general method, using nonstandard analysis, of deriving a result about upper Banach density parallel to each existing result about lower asymptotic density or Shnirel'man density in an extremely efficient way. Given a set A with $BD(A) \geqslant \alpha$, there is a copy of $\mathbf{N}$ in a remote area where ${}^*A$ has lower asymptotic density or Shnirel'man density α. By applying the existing theorem about lower asymptotic density or Shnirel'man density, one can obtain a result on that area. Then one can pull the result down to the standard world to obtain the parallel result.

(2) There are other papers using nonstandard analysis to study the sequences of natural numbers. See [8], [9], [10] and [7] for example. In [7], one of the consequences of the main nonstandard result is in additive number theorem. The consequence, suggested to me by S. Leth, says that for any $A, B \subseteq \mathbf{N}$ with $BD(A) > 0$ and $BD(B) > 0$, the set $A + B$ is piecewise syndetic. A set $S \subseteq \mathbf{N}$ is piecewise syndetic if $S + [0, k]$ is thick for some $k \in \mathbf{N}$. This consequence is interesting because it is a special case of a general phenomenon which says that if A and B are large in terms of "measure", then $A + B$ is not small in terms of "order-topology". The consequence is interesting also because it complements a result which says that if $BD(A) > 0$, then $A - A = \{a - a' : a, a' \in A \text{ and } a' \leqslant a\}$ is syndetic [3, page 75, Proposition 3.19]. A set $S \subseteq \mathbf{N}$ is syndetic if $S + [0, k] \supseteq \mathbf{N} \setminus [0, n]$ for some $k, n \in \mathbf{N}$.

(3) Our development of the nonstandard analysis in §2 is very restrictive. The guideline I followed is that I should provide, as concrete as possible, only necessary background of nonstandard analysis for the proofs of this paper. For broader applications in other mathematical fields, one may need more than that in §2. (a) One can use other ways instead of ultrapower construction to construct a nonstandard model. One can also construct an ultrapower by using a nonprincipal ultrafilter on a larger set I instead of on $\mathbf{N}$. (b) The standard model may contain much more that $\mathbf{V}$ defined in §2. Nonstandard analysts usually take the standard model $\mathbf{V}$ to be the standard superstructure defined as the following. Let $V_0 = \mathbf{R} \cup X$ where X is any needed set, and $V_{n+1} = V_n \cup \wp(V_n)$. Then $\mathbf{V} = (\cup_{n=0}^{\infty} V_n; \in)$. From this $\mathbf{V}$, one can construct the nonstandard model ${}^*\mathbf{V}$ as, for example, an ultrapower of $\mathbf{V}$. Note that the functions such as $+$, $\cdot$ and $|\cdot|$, and relations such as $\leqslant$ and $\in$ can be viewed as elements in $\mathbf{V}$. (c) I deliberately blurred the distinction between syntax and semantics of the logical formulas in order to reduce the hard-ness for the reader with no logic background. We rely heavily on the reader's common-sense understanding of the truth of a formula in a model. The rigorous treatment can be found in any first-year logic course textbook.

References

[1] Arkeryd, L., Cutland, N. J. and Henson, C. W., edited, *Nonstandard Analysis: Theory and Applications*, Kluwer Academic Publishers 1997.

[2] Bergelson, Vitaly, *Ergodic Ramsey Theory*–an Update, Ergodic theory of $\mathbf{Z}^d$ actions (Warwick, 1993-1994) 1-61, London Mathematical Society Lecture Note Ser. 228, Cambridge University Press, Cambridge 1996.

[3] Furstenberg, H., *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, 1981.

[4] Halberstam, H. and Roth, K. F., *Sequences*, Oxford University Press, 1966

[5] Henson, C. W., *Foundations of nonstandard analysis*–A gentle introduction to nonstandard extension, in **Nonstandard Analysis: Theory and Applications**, ed. by N. J. Cutland, C. W. Henson, and L. Arkeryd, Kluwer Academic Publishers 1997.

[6] Jech, T., *Set Theory*, Academic Press, New York, 1978.

[7] Jin, Renling, *Sumset phenomenon*, in preparation.

[8] Leth, Steven C., (1988) *Sequences in countable models of the natural numbers*, Studia Logica, **47**, no. 3, pp. 243—263.

[9] Leth, Steven C., (1988) *Some nonstandard methods in combinatorial number theory*, Studia Logica, **47**, no. 3, pp. 265—278.

[10] Leth, Steven C., (1988) *Application of Nonstandard Models and Lebesgue Measure to Sequences of Natural Numbers*, Transactions of the American Mathematical Society, **307**, No. 2, pp. 457–468.

[11] Lindstrom, T., *An invitation to nonstandard analysis*, in **Nonstandard Analysis and Its Application**, ed. by N. Cutland, Cambridge University Press 1988.

[12] Nathanson, Melvyn B., *Additive Number Theory–the Classical Bases*, Springer, 1996.

[13] Nathanson, Melvyn B., *Additive Number Theory–Inverse Problems and the Geometry of Sumsets*, Springer, 1996.

[14] Petersen, Karl, *Ergodic Theory*, Cambridge University Press, 1983.