Freiman's $3k-3+b$ conjecture for almost bi-arithmetic progression

Renling Jin

College of Charleston
jinr@cofc.edu
July 28, 2010

Abstract

A set $B = I_1 \cup I_2 \subseteq \mathbb{N}$ is called a bi-arithmetic progression of length $|B|$ and difference d if I_1 and I_2 are non-empty arithmetic progressions of the same difference d and I_1+I_1, I_1+I_2, I_2+I_2 are pairwise disjoint. For a non-empty finite set C of integers let $\operatorname{diam}(C) = \max\{|a-b| : a,b \in C\}$. Let $\operatorname{diam}(\emptyset) = -\infty$. A set A of integers is called an ϵ–almost subset of a bi-arithmetic progression B if $\operatorname{diam}(A \smallsetminus B)/|B| < \epsilon$. In this paper we prove the following theorem[1]: There exists an $\epsilon > 0$ such that if A is a sufficiently large set of integers, A is a non-trivial ϵ–almost subset of a bi-arithmetic progression[2], and $|A+A| = 3|A|-3+b$ for $0 \leqslant b < \frac{1}{3}|A|-2$, then A is either a subset of an arithmetic progression of length at most $2|A|-1+2b$ or a subset of a bi-arithmetic progression of length at most $|A|+b$.

---

* *Mathematics Subject Classification 2000* Primary 11B05, 11B13, 11U10, 03H15

* *Keywords:* Freiman's inverse problem, arithmetic progression, bi-arithmetic progression, additive number theory, nonstandard analysis

* The author was supported in part by the NSF grant DMS–RUI#0500671.

[1] We will in fact prove a slightly stronger theorem, see Theorem 1.4.

[2] An ϵ–almost subset A of a bi-arithmetic progression $I_1 \cup I_2$ is non-trivial if $A \cap I_i \neq \emptyset$ for $i = 1, 2$.

Contents

1 Introduction

For $A, B \subseteq \mathbb{N}$ let $A \pm B$ denote the set $\{a \pm b : a \in A \text{ and } b \in B\}$ and $2A$ denote the set $A + A$. A set of the form $\{a + id : i = 0, 1, \ldots, k-1\}$ for integers $a \geqslant 0$ and $d > 0$ is called an *arithmetic progression*, or *a.p.* for short, of length k and difference d. The difference of an *a.p.* of one element can be any positive integer. In the next paragraph we give an equivalent definition of bi-arithmetic progression using Freiman isomorphism as well as the definition of tri-arithmetic progression.

Let A be a subset of an abelian semi-group $\mathcal{G}$ and B be a subset of an abelian semi-group $\mathcal{H}$. A bijection $\phi : A \mapsto B$ is called a Freiman isomorphism of order 2 if for any $a, b, c, d \in A$,

$$\phi(a) + \phi(b) = \phi(c) + \phi(d) \text{ if and only if } a + b = c + d.$$

Notice that if ϕ is a Freiman isomorphism of order 2 from A to B, then $|2A| = |2B|$. Since the order of the Freiman isomorphism used in this paper is always 2, we will omit the word "order 2".

A set $B \subseteq \mathbb{Z}$ is called a *bi-arithmetic progression*, or *b.p.* for short, if B is Freiman isomorphic to $J_1 \cup J_2 \subseteq \mathbb{Z}^2$ with usual vector addition where

$$J_1 = \{(0,0),(1,0),\ldots,(u_1,0)\} \tag{1}$$

$$J_2 = \{(0,1),(1,1),\ldots,(u_2,1)\} \tag{2}$$

for some non-negative integers u_1, u_2.

A set B is called a *tri-arithmetic progression*, or *t.p.* for short, if B is Freiman isomorphic to $J_1 \cup J_2 \cup J_3 \subseteq \mathbb{Z}^2$ where J_1 and J_2 are defined in (1)–(2) and

$$J_3 = \{(l_3,2),(l_3+1,2),\ldots,(u_3,2)\} \tag{3}$$

for some integers l_3, u_3 such that $u_3 - l_3 \geqslant 0$.

If B is a *b.p.* and is Freiman isomorphic via ϕ to $J_1 \cup J_2$, we often write $I_i = \phi^{-1}(J_i)$ for $i = 1,2$. It is easy to see that this definition of *b.p.* is equivalent to the definition of *b.p.* in the abstract. If B is a *t.p.* and is Freiman isomorphic to $J_1 \cup J_2 \cup J_3$, via Freiman isomorphism ϕ, we often write $I_i = \phi^{-1}(J_i)$ for $i = 1,2,3$. Notice that if B is a *t.p.*, then $2I_1, I_1 + I_2, 2I_2, I_2 + I_3, 2I_3$ are pairwise disjoint and $2I_1, I_1 + I_2, I_1 + I_3, I_2 + I_3, 2I_3$ are pairwise disjoint. Because $(I_1 + I_3) \cap (2I_2)$ can be non-empty, we need to allow the left-most element of J_3 in (3) to be $(l_3, 2)$ instead of $(0, 2)$ for flexibility.

When we say that a set A is a subset of a *b.p.* $I_1 \cup I_2$ we implicitly assume that $A \cap I_i \neq \emptyset$ for $i = 1,2$. When we say that A is a subset of a *t.p.* $I_1 \cup I_2 \cup I_3$, we implicitly assume that $A \cap I_i \neq \emptyset$ for $i = 1,2,3$. By calling A a *tight* subset of an *a.p.*, *b.p.*, or *t.p.* B, respectively, we mean that $A \subseteq B$ and for any *a.p.*, *b.p.*, or *t.p.* $B' \subseteq B$, respectively, $A \subseteq B'$ implies $B' = B$. Notice that if A is a subset of a *b.p.* $B = I_1 \cup I_2$ such that $A \cap I_i$ is a tight subset of the *a.p.* I_i for $i = 1,2$, then A is a tight subset of the *b.p.* B. However, if A is a tight subset of a *b.p.* $B = I_1 \cup I_2$, $A \cap I_i$ may not necessarily be a tight subset of the *a.p.* I_i for $i = 1$ or 2. By saying that A is an ϵ*–almost tight subset* of a *b.p.* B we mean that $A \cap B$ is a tight subset of the *b.p.* B and $\mathrm{diam}(A \smallsetminus B)/|B| < \epsilon$.

The study of ϵ–almost bi-arithmetic progression is motivated by Freiman's $3k-3+b$ conjecture stated below. In fact, the results in this paper will be needed in a forthcoming paper, which contains a solution of the conjecture.

Conjecture 1.1 (G. A. Freiman [3, 4]) *There is $k \in \mathbb{N}$ such that if $A \subseteq \mathbb{N}$ is finite, $|A| > k$, and $|2A| = 3|A| - 3 + b$ for $0 \leqslant b < \frac{1}{3}|A| - 2$, then A is either a tight subset of an* a.p. *of length at most $2|A| - 1 + 2b$ or a tight subset of a* b.p. *of length at most $|A| + b$.*

Conjecture 1.1 was posed for an attempt to generalize Theorem A.1 and Theorem A.2. Some efforts have been made on Conjecture 1.1 by Freiman, Lev, and the author. For example, it is shown in [2] that if A is already a subset of a *b.p.*, then A satisfies Conjecture 1.1.

Theorem 1.2 (G. A. Freiman) *Suppose $|A| > 10$ and A is a subset of a* b.p. *If $|A + A| = 3|A| - 3 + b$ for $0 \leqslant b < |A| - 3$, then A is a subset of a* b.p. *of length at most $|A| + b$.*

Theorem 1.2 is a consequence of the proof of Theorem A.3 in [2]. In [2, page 28] the value of b in Theorem A.3 is restricted to be less than $\frac{1}{3}|A| - 2$. However, the restriction is only used for the first half of the theorem. It is not hard to see that the restriction can be weakened to $0 \leqslant b < |A| - 3$ in Theorem 1.2 here when we already know that A is a subset of a *b.p.* (see, for example, [14]).

Freiman also showed that if a tight subset A of an *a.p.* I satisfies the conditions of Conjecture 1.1 such that $|A|/|I|$ is sufficiently small, then A must be a subset of a *b.p.* (cf. [6, the last paragraph in the proof of Theorem 1.4 by Theorem 1.7]) and hence A satisfies the conclusion of Conjecture 1.1 by Theorem 1.2. Because of this we can assume that the set A under our consideration is a tight subset of an *a.p.* I such that $|A|/|I|$ is greater than some fixed positive number. Lev in [9] showed that if $A \subseteq \{0, 1, \ldots, p\}$ for a sufficiently large prime number p and $0, p \in A$ such that $|A| \leqslant p/35$, then A must be a subset of a *b.p.* and hence satisfies Conjecture 1.1. In [6] the author proved a weak version of Conjecture 1.1, which generalized Theorem A.1 and Theorem A.2, without requiring that $|A|/|I|$ be small, by proving the following theorem.

Theorem 1.3 *There is an integer $k > 0$ and a real number $\epsilon > 0$ such that if $|A| > k$ and $|2A| = 3|A| - 3 + b$ for $0 \leqslant b \leqslant \epsilon|A|$, then A is either a tight subset of an* a.p. *of length at most $2|A| - 1 + 2b$ or a tight subset of a* b.p. *of length at most $|A| + b$.*

The proof of Theorem 1.3 in [6] consists of the derivation of some lemmas and verification of a large number of cases. In the process of confirming Conjecture 1.1 we need to strengthen lemmas in [6] and verify cases when the condition $0 \leqslant b \leqslant \epsilon|A|$ for some small positive ϵ is replaced by the condition $0 \leqslant b < \left(\frac{1}{3} + \epsilon\right)|A|$. In this paper we just want to do that when assuming that A is an ϵ–almost bi-arithmetic progression with the help of a key lemma [6, Lemma 2.2]. The results in this paper can be viewed as generalizations of Theorem 1.2 and improvements of [6, Lemma 4.2 and Lemma 4.3].

The following is the main theorem of this paper.

Theorem 1.4 *There exists a positive real number ϵ and a positive integer K such that if A is an ϵ–almost tight subset of a* b.p.*, $|A| > K$, A is not a subset of a* t.p.*, and $|2A| = 3|A| - 3 + b$ for $0 \leqslant b < \left(\frac{1}{3} + \epsilon\right)|A|$, then A is either a tight subset of an* a.p. *of length at most $2|A| - 1 + 2b$ or a tight subset of a* b.p. *of length at most $|A| + b$.*

Remark 1.5 *If A is a sufficiently large tight subset of a* t.p. *but not a subset of any* b.p. *and $|2A| = 3|A| - 3 + b$ for $0 \leqslant b < \left(\frac{1}{3} + \epsilon\right)|A|$ for $\epsilon < \frac{1}{6}$, then, by Theorem A.6, $b \geqslant \frac{1}{3}|A| - 2$ and the length of the* t.p. *is at most $\frac{3}{4}(|A| + b + 2)$.*

By Lemma 3.4 we can easily see that Theorem 1.4 implies the result stated in the abstract. Notice that if A is a subset of a *b.p.*, then A is trivially an ϵ–almost subset of a *b.p.* By Theorem 1.2 we can assume that A is not a subset of a *b.p.* in the proof of Theorem 1.4. In this paper we will assume, without loss of generality, that $\gcd(A - \min A) = 1$.

As we have done before, we would like to use methods and notations from nonstandard analysis. The tools from nonstandard analysis have been shown useful and efficient when dealing with some additive/combinatorial number theoretic problems in author's previous work. The reader is recommended to consult one of [5, 8, 11] for the basic notation, ideas, and principles in nonstandard analysis. Other introductory texts for nonstandard analysis should also be sufficient. When we use nonstandard tools we always work within a fixed countably saturated nonstandard universe.

For each standard set A let ${}^*A$ denote the nonstandard version of A in the nonstandard universe. For example, ${}^*\mathbb{N}$ is the set of all non-negative integers in the nonstandard universe and ${}^*\mathbb{Z}$ is the set of all integers in the nonstandard universe. For any $a, b \in {}^*\mathbb{Z}$ we write $[a, b]$ exclusively for the interval of all integers between a and b including a, b. For any set $A \subseteq {}^*\mathbb{Z}$ and $a, b \in {}^*\mathbb{Z}$ let $A[a, b]$ denote the set $A \cap [a, b]$ and $A(a, b)$ count the number of elements in $A[a, b]$. By a set in this paper we will always mean, unless specified, either a standard set or an internal set of integers. Integers in ${}^*\mathbb{N} \smallsetminus \mathbb{N}$ are called hyperfinite integers and often denoted by H, N, etc. (a hyperfinite integer is greater than every standard positive integer and is still finite in a nonstandard sense.) For the convenience of handling nonstandard arguments, we introduce some notations of comparison (quasi-order), which will be extensively used throughout the whole paper. For real numbers r, s in ${}^*\mathbb{R}$, by $r \approx s$ we mean that $r - s$ is an infinitesimal, i.e., $|r - s| < \frac{1}{n}$ for any standard positive integer $n \in \mathbb{N}$; by $r \ll s$ ($r \gg s$) we mean $r < s$ ($r > s$) and $r \not\approx s$; by $r \lessapprox s$ ($r \gtrapprox s$) we mean $r < s$ ($r > s$) or $r \approx s$. Given a hyperfinite integer H and two real numbers r, s, by $r \sim_H s$ we mean $\frac{s-r}{H} \approx 0$; by $r \prec_H s$ ($r \succ_H s$) we mean $r < s$ ($r > s$) and $r \not\sim_H s$; by $r \preccurlyeq_H s$ ($r \succcurlyeq_H s$) we mean $r \prec_H s$ ($r \succ_H s$) or $r \sim_H s$. The subscript H is often omitted when H is clearly given. The symbols $\succcurlyeq$, $\succ$, etc. offer notational convenience. The next is the definition of a nonstandard version of the ϵ–almost subset of a *b.p.*

Definition 1.6 *A set $A \subseteq {}^*\mathbb{Z}$ is called an almost subset of a* b.p. *B if $diam(A \smallsetminus B)/|B| \approx 0$.*

Notice that if $A \smallsetminus B \neq \emptyset$ in Definition 1.6, then $|B|$ must be hyperfinite.

Next we state a theorem in a nonstandard form, which implies Theorem 1.4.

Theorem 1.7 *Let H be a hyperfinite integer and $A \subseteq [0, H]$ be such that*

$$0 = \min A,\ \gcd(A) = 1,\ \textit{and}\ H = \max A. \tag{4}$$

Suppose A is an almost tight subset of a b.p.*,*

$$A \text{ is neither a subset of a b.p. nor a subset of a t.p.,} \tag{5}$$

and

$$|2A| = 3|A| - 3 + b \text{ for } 0 \leqslant b \precsim \frac{1}{3}|A|. \tag{6}$$

Then

$$H + 1 \leqslant 2|A| - 1 + 2b. \tag{7}$$

The letter b in (6) is reserved for the quantity $|2A| - 3|A| + 3$ throughout this paper. By Theorem 1.3 we can always assume that $b \succ 0$. Notice that if A satisfies (4), then "A is a subset of an *a.p.* of length at most $2|A| - 1 + 2b$" is equivalent to (7). We will prove first that Theorem 1.7 implies Theorem 1.4 in the next section and then focus on proving Theorem 1.7.

For reader's convenience we list some existing theorems in Appendix. We will refer to these theorems frequently in our proofs.

2 Theorem 1.7 implies Theorem 1.4

Proof of Theorem 1.4 assuming Theorem 1.7

Suppose Theorem 1.4 is not true. For each $k \in \mathbb{N}$ with $\epsilon_k = \frac{1}{k}$ and $K = k$ there is a counterexample $A_k \subseteq \mathbb{Z}$ of Theorem 1.4, i.e., $|A_k| > k$, A_k is a ϵ_k–almost tight subset of a *b.p.*, A_k is not a subset of a *t.p.*, $|2A_k| = 3|A_k| - 3 + b$ for $0 \leqslant b < \left(\frac{1}{3} + \epsilon_k\right)|A_k|$, A_k is neither a subset of an *a.p.* of length at most $2|A_k| - 1 + 2b$ nor a subset of a *b.p.* of length at most $|A_k| + b$. Let N be a hyperfinite integer and A_N be the correspondent term in the internal sequence $\{A_k : k \in {}^*\mathbb{N}\}$. Without loss of generality we can assume that A_N satisfies (4). By a result of Freiman mentioned in the second paragraph after Theorem 1.2 we can assume that $|A_N|/H \gg 0$. By the transfer principle we have that, A_N is a $\frac{1}{N}$–almost tight subset of a *b.p.*, A_N is not a subset of a *t.p.*, $|2A_N| = 3|A_N| - 3 + b$ for $0 \leqslant b < \left(\frac{1}{3} + \frac{1}{N}\right)|A_N|$, and A_N is neither a subset of an *a.p.* of length at most $2|A_N| - 1 + 2b$ nor a subset of a *b.p.* of length at most $|A_N| + b$. By Theorem 1.2 we can assume that A_N is not a subset of a *b.p.* Notice that N is hyperfinite and $\frac{1}{N} \approx 0$. Hence $A = A_N$ is an almost subset of a *b.p.* and satisfies (4), (5), and (6), but fails (7). Therefore, A becomes a counterexample of Theorem 1.7. This completes the proof.

□

3 Common lemmas

The first two lemmas are elementary but very useful. See Remark 3.3.

Lemma 3.1 *Suppose that $A \subseteq [0, H]$, $|2A| = 3|A| - 3 + b$, and $H + 1 > 2|A| - 1 + 2b$. Let $A \subseteq A' \subseteq [0, H]$ be such that $|A'| = |A| + 1$ and $|2A'| = 3|A'| - 3 + b' \leqslant |2A| + 2$. then $b - 3 \leqslant b' \leqslant b - 1$ and $H + 1 > 2|A'| - 1 + 2b'$.*

Proof: Since $|2A| = 3|A| - 3 + b = 3|A'| - 3 + (b-3) \leqslant |2A'| = 3|A'| - 3 + b' \leqslant |2A| + 2 = 3|A| - 3 + b + 2 = 3|A'| - 3 + b - 1$, then $b - 3 \leqslant b' \leqslant b - 1$. Also $H + 1 > 2|A| - 1 + 2b = 2|A'| - 2 - 1 + 2b = 2|A'| - 1 + 2(b-1) \geqslant 2|A'| - 1 + 2b'$.

□

Lemma 3.2 *Suppose that $A \subseteq [0, H]$, $|2A| = 3|A| - 3 + b$, and $H + 1 > 2|A| - 1 + 2b$. Let $A' \subseteq A \subseteq [0, H]$ be such that $|A'| = |A| - 1$ and $|2A'| = 3|A'| - 3 + b' \leqslant |2A| - m$ for $m = 2, 3, 4$. Then we have the following.*

1. *If $m = 4$, then $b' \leqslant b - 1$ and $H + 1 > 2|A'| - 1 + 2b'$.*
2. *If $m = 3$, then $b' \leqslant b$ and $H + 1 > 2|A'| - 1 + 2b'$.*
3. *If $m = 2$, then $b' \leqslant b + 1$ and $H + 1 > 2|A'| - 1 + 2b'$.*

Proof Suppose $m = 4$. Then $3|A'| - 3 + b' = |2A'| \leqslant |2A| - 4 = 3|A| - 3 + b - 4 = 3|A'| - 3 + b - 1$, which implies $b' \leqslant b - 1$ and $H + 1 > 2|A| - 1 + 2b = 2|A'| + 2 - 1 + 2b = 2|A'| - 1 + 2(b + 1) > 2|A'| - 1 + 2b'$.

Suppose $m = 3$. Then $3|A'| - 3 + b' = |2A'| \leqslant |2A| - 3 = 3|A| - 3 + b - 3 = 3|A'| - 3 + b$, which implies $b' \leqslant b$ and $H + 1 > 2|A| - 1 + 2b = 2|A'| + 2 - 1 + 2b = 2|A'| - 1 + 2(b + 1) > 2|A'| - 1 + 2b'$.

Suppose $m = 2$. Then $3|A'| - 3 + b' = |2A'| \leqslant |2A| - 2 = 3|A| - 3 + b - 2 = 3|A'| - 3 + b + 1$, which implies $b' \leqslant b + 1$ and $H + 1 > 2|A| - 1 + 2b = 2|A'| + 2 - 1 + 2b = 2|A'| - 1 + 2(b + 1) \geqslant 2|A'| - 1 + 2b'$.

□

Remark 3.3 *Although Lemma 3.1 and Lemma 3.2 are elementary, the repeated use of them becomes very powerful. In the proof of Theorem 1.7 we first assume the theorem is not true and then derive a contradiction. Suppose* A *is a counterexample for Theorem 1.7. If we can manage to add one element in* $[0, H]$ *to* A *to form a new set* A' *so that* $2A'$ *has at most two new elements not in* $2A$*, then* A' *is again a counterexample for Theorem 1.7 by Lemma 3.1. If this step can be repeated for many times, we can come to a stage that the resulting set* A' *has a very nice form and is still a counterexample for Theorem 1.7. On the other hand, when* A' *has a nice form, it can be easily proved that* A' *can't be the counterexample, which leads to a contradiction. By the same idea we sometimes need to delete elements from* A *to form a new set* A'*, which has a nicer form than the original* A*. Lemma 3.2 allows us to do that under certain cases. However, one may need to be careful to make sure that (4), (5), and (6) remain true. Suppose* A *is a counterexample for Theorem 1.7. If deleting one element from* A *results reducing the size of* $2A$ *by at least* 4 *and (4) and (5) remain true, then the resulting set* A' *is still a counterexample for Theorem 1.7. However, if deleting one element from* A *results reducing the size of* $2A$ *by* 2 *or* 3*, then deleting significant amount of them may cause the resulting set* A' *not satisfying (6).*

The next lemma shows why Theorem 1.4 implies the result stated in the abstract.

Lemma 3.4 *If* A *is a tight subset of a* t.p. *then* $|2A| \geqslant \frac{10}{3}|A| - 5$*, i.e.,* $b \geqslant \frac{1}{3}|A| - 2$*.*

Proof: If $0 \leqslant b < \frac{1}{3}|A| - 2$ in Theorem A.5, then the length of the *t.p.* is less than or equal to $\frac{3}{4}(|A| + b + 2)$, which is less than $\frac{3}{4}(|A| + \frac{1}{3}|A| - 2 + 2) = |A|$. This is absurd because A cannot have more terms than the *t.p.* which contains A.

□

Next we prove a lemma, which eliminates the case for $d > 4$. Suppose A is an almost tight subset of a *b.p.* $I_1 \cup I_2$ of difference d satisfying (4). Let

$$A_i = A \cap I_i \text{ for } i = 1, 2 \text{ and } A_0 = A \setminus (A_1 \cup A_2), \tag{8}$$

$$l_i = \min(A \cap I_i) \text{ and } u_i = \max(A \cap I_i) \text{ for } i = 0, 1, 2. \tag{9}$$

Lemma 3.5 *Suppose $A \subseteq [0, H]$ is an almost tight subset of a* b.p. $I_1 \cup I_2$ *of difference d and satisfies (4), (5), and (6), but fails (7). Then $d \leqslant 4$.*

Proof Suppose $d > 4$. Without loss of generality, let $|A_1| \geqslant |A_2|$. If $A_1 \cup A_2$ is a subset of an *a.p.* of difference $d' > 1$, then there is $a \in A_0$ such that $a \not\equiv l_1 \pmod{d'}$. Hence $|2A| \geqslant |a + (A_1 \cup A_2)| + |2(A_1 \cup A_2)| \succcurlyeq 4|A|$, which contradicts (6). Thus we can assume that $A_1 \cup A_2$ is not a subset of an *a.p.* of difference > 1.

If for every $a \in A_0$ we have either $a \equiv l_1 \pmod{d}$ or $a \equiv l_2 \pmod{d}$, then A is a subset of a *b.p.*, which contradicts (5). Hence we can assume that there is $a \in A_0$ such that $a \not\equiv l_1 \pmod{d}$ and $a \not\equiv l_2 \pmod{d}$.

We can assume that either $a + A_1$ or $a + A_2$ is disjoint from $2(A_1 \cup A_2)$ by the following argument: Assume the contrary. Then $a + l_1 \equiv 2l_2 \pmod{d}$ and $a + l_2 \equiv 2l_1 \pmod{d}$, which implies $3(l_2 - l_1) \equiv 0 \pmod{d}$. Since $d > 4$, there is a factor $d' > 1$ of d such that $d' | (l_2 - l_1)$, which implies that $A_1 \cup A_2$ is a subset of an *a.p.* of difference $\geqslant d' > 1$.

If $a + A_1$ is disjoint from $2(A_1 \cup A_2)$, then $|2A| \succcurlyeq 3|A| + |A_1| \succ 3|A| + \frac{1}{2}|A|$, which contradicts (6). Hence we can assume that $a + A_2$ is disjoint from $2(A_1 \cup A_2)$, which implies that $|A_2| \preccurlyeq \frac{1}{3}|A|$, $|A_1| \succcurlyeq \frac{2}{3}|A|$, and $a + l_1 \equiv 2l_2 \pmod{d}$. If $|A_1| \succ \frac{2}{3}|A|$, then

$$\begin{aligned} |2A| &\succcurlyeq |2A_1| + |A_2 + A_1| + |a + A_1| + |a + A_2| \\ &\succcurlyeq 3|A| + |A_1| - |A_2| \succ 3|A| + \frac{2}{3}|A| - \frac{1}{3}|A| \succcurlyeq \frac{10}{3}|A|, \end{aligned}$$

which contradicts (6). Hence we can assume that $|A_1| \sim 2|A_2| \sim \frac{2}{3}|A|$.

We can assume that $\{a\} \cup A_1 \cup A_2$ is a subset of a *t.p.* by the following argument: Assume the contrary. Then $2a$ must be congruent to $2l_1$, $2l_2$, or $l_1 + l_2$ modulo d. Since $a + l_1 \equiv 2l_2 \pmod{d}$ and $a \not\equiv l_1 \pmod{d}$, we have $2a \equiv 2l_1 \pmod{d}$ or $2a \equiv l_1 + l_2 \pmod{d}$. If $2a \equiv 2l_1 \pmod{d}$, then $4l_2 \equiv 2l_1 + 2a \equiv 4l_1 \pmod{d}$, which implies $4(l_2 - l_1) \equiv 0 \pmod{d}$. Since $d > 4$, there is a factor $d' > 1$ of d such that $d'|(l_2 - l_1)$, which contradicts that $A_1 \cup A_2$ is not an *a.p.* of difference > 1. If $2a \equiv l_1 + l_2 \pmod{d}$, then $4l_2 \equiv 2l_1 + 2a \equiv 3l_1 + l_2 \pmod{d}$, which implies $3(l_2 - l_1) \equiv 0 \pmod{d}$. We again have a contradiction.

Since A is not a subset of a *t.p.*, there is $a' \in A_0$ such that a' is not congruent to a, l_1, or l_2 modulo d. By the same argument as above we can assume that $a' + A_2$ is disjoint from $2(A_1 \cup A_2)$. If $(a + A_2) \cap (a' + A_2) = \emptyset$, then $|2A| \succcurlyeq 3|A| + 2|A_2| \succ \frac{10}{3}|A|$. If $(a + A_2) \cap (a' + A_2) \neq \emptyset$, then $a + l_2 \equiv a' + l_2 \pmod{d}$, which implies that $a \equiv a' \pmod{d}$, a contradiction. This completes the proof of the lemma.

$\square$

By Lemma 3.5 we can assume that A is an almost subset of a *b.p.* of difference $d = 1, 2, 3, 4$ in Theorem 1.7.

Lemma 3.6 *Suppose $A \subseteq [0, H]$ is an almost tight subset of a* b.p. $I_1 \cup I_2$ *of difference $d \leqslant 4$ and satisfies (4), (5), and (6), but fails (7). Then $A_1 \cup A_2$ is not a subset of an* a.p. *of difference > 1. Furthermore, we can choose the* b.p. $I_1 \cup I_2$ *so that $A_0 = A \smallsetminus (I_1 \cup I_2)$ is a subset of an* a.p. *I_0 of difference d with $l_0 = \min A_0 = \min I_0$, $u_0 = \max A_0 = \max I_0$, I_0 is disjoint from $I_1 \cup I_2$, and if $d = 3, 4$, then $l_0 \not\equiv l_i \pmod{d}$ for $i = 1, 2$.*

Proof Suppose $A_1 \cup A_2$ is a subset of an *a.p.* of difference $d' > 1$. By (4) there is $a \in A$ such that $\{a\} \cup A_1 \cup A_2$ is not a subset of an *a.p.* of difference > 1. Hence $|2A| \succcurlyeq |a + (A_1 \cup A_2)| + |2(A_1 \cup A_2)| \succcurlyeq 4|A|$, which contradicts (6).

For $d = 1, 2$ we can easily re-arrange the elements in A to form A'_0, A'_1, and A'_2 so that A is an almost tight subset of a *b.p.* $I'_1 \cup I'_2$ and $A'_0 = A \smallsetminus (I'_1 \cup I'_2)$ is a subset of an *a.p.* I'_0 of difference d, which is disjoint from $I'_1 \cup I'_2$.

Suppose $d = 3$. Let $A'_i = A_i \cup \{a \in A_0 : a \equiv l_i \pmod{3}\}$ for $i = 1, 2$ and $A'_0 = A_0 \smallsetminus (A'_1 \cup A'_2)$. Then $A'_1 \cup A'_2$ is a subset of a *b.p.* $I'_1 \cup I'_2$ of difference 3 and A'_0 is a subset of an *a.p.* I'_0 of difference 3 disjoint from $I'_1 \cup I'_2$. Clearly, $l_0 \not\equiv l_i \pmod{d}$ for $i = 1, 2$.

Suppose $d = 4$. By the same argument as above we can assume that $a \not\equiv l_i \pmod 4$ for $i = 1, 2$ for every $a \in A_0$. Without loss of generality, we can assume that $|A_1| \geqslant |A_2|$. If there are $a, a' \in A_0$ such that $a \not\equiv a' \pmod 4$, then either $a + A_1$ or $a' + A_1$ is disjoint from $2(A_1 \cup A_2)$ because otherwise we have $a + l_1 \equiv a' + l_1 \pmod 4$, which contradicts $a \not\equiv a' \pmod 4$. Hence $|2A| \succcurlyeq 3|A| + \frac{1}{2}|A| \succ \frac{10}{3}|A|$, which contradicts (6). Thus we can assume that A_0 is a subset of an *a.p.* I_0 of difference 4 disjoint from $I_1 \cup I_2$ and $l_0 \not\equiv l_i \pmod d$ for $i = 1, 2$.

□

Suppose that $A \subseteq [0, H]$ is an almost a tight subset of *b.p.* $I_1 \cup I_2$ of difference d. Let A_i, l_i, and u_i be defined by (8) and (9) satisfying the conclusion of Lemma 3.6.

Define sets S_i and numbers s_i by

$$S_i = I_i \smallsetminus A_i \text{ and } s_i = |S_i| \tag{10}$$

for $i = 1, 2$. For $i = 1, 2$ define

$$b_i = |2A_i| - 2|A_i| + 1. \tag{11}$$

For $0 \leqslant i < j \leqslant 2$ let

$$b_{i,j} = |A_i + A_j| - |A_i| - |A_i| + 1. \tag{12}$$

By Theorem A.1 we have $s_i \leqslant b_i$ if $|2A_i| < 3|A_i| - 3$ for $i = 1, 2$.

When $|A_i|/|I_i| > \frac{1}{2}$ for $i = 1$ or 2, we define

$$L_i = \max\left(\{l_i\} \cup \left\{x \in I_i : A_i(l_i, x - d) \leqslant \frac{1}{2d}(x - l_i)\right\}\right) \text{ and } \tag{13}$$

$$R_i = \min\left(\{u_i\} \cup \left\{x \in I_i : A_i(x + d, u_i) \leqslant \frac{1}{2d}(u_i - x)\right\}\right). \tag{14}$$

Notice that $L_i, R_i \in A_i$. Since $A_i(l_i, L_i - d) \leqslant \frac{1}{2}|I_i \cap [l_i, L_i - d]|$ when $l_i < L_i$ and $A_i(R_i + d, u_i) \leqslant \frac{1}{2}|I_i \cap [R_i + d, u_i]|$ when $R_i < u_i$, we have the following inequality for s_i when $s_i > 0$:

$$s_i \geqslant \max\left\{\frac{1}{2d}(L_i - l_i),\ \frac{1}{2d}(u_i - R_i)\right\}. \tag{15}$$

Furthermore, if $L_i \leqslant R_i$, then

$$s_i + A_i(l_i, L_i - d) + A_i(R_i + d, u_i) \geqslant \frac{1}{d}(L_i - l_i) + \frac{1}{d}(u_i - R_i).$$

Hence

$$s_i \geqslant \frac{1}{2d}(L_i - l_i) + \frac{1}{2d}(u_i - R_i). \tag{16}$$

If we do not assume $L_i > l_i$ and $R_i < u_i$, then (15) is still true. If we assume only "$L_i \preccurlyeq R_i$" instead of "$L_i \leqslant R_i$", then (16) is still true when $\leqslant$ and $\geqslant$ are replaced by $\preccurlyeq$ and $\succcurlyeq$, respectively. Notice that $A_i(l_i, L_i - d) \sim \frac{1}{2d}(L_i - l_i)$ and $A_i(R_i + d, u_i) \sim \frac{1}{2d}(u_i - R_i)$.

Suppose $L_i \prec R_i$ for $i = 1$ or 2. We define

$$p_i = |(I_i \setminus A_i) \cap [L_i, R_i]|. \tag{17}$$

Lemma 3.7 *Suppose $L_i \prec R_i$ for $i = 1$ or 2. If $|2A_i| = 2|A_i| - 1 + b_i < 3|A_i| - 3$, then $s_i \leqslant b_i - p_i$ and $|I_i| = |A_i| + s_i \leqslant |A_i| + b_i - p_i$.*

Proof Let $A_i' = A_i \cup I_i[L_i, R_i]$. If $x \in A_i' \setminus A_i$, then $L_i < x < R_i$. Hence $x + A_i \subseteq 2A_i$. This shows that $2A_i' = 2A_i$. Therefore,

$$|2A_i'| = |2A_i| = 2|A_i| - 1 + b_i = 2|A_i'| - 1 + b_i - 2p_i < 3|A_i| - 3 \leqslant 3|A_i'| - 3.$$

By Theorem A.1 we have $|A_i| + s_i = |I_i| \leqslant |A_i'| + b_i - 2p_i = |A_i| + b_i - p_i$, which implies that $s_i \leqslant b_i - p_i$ and $|I_i| \leqslant |A_i| + b_i - p_i$.

□

Lemma 3.8 *Suppose $|I_2| > |I_1|$, $L_i \prec R_i$ for $i = 1, 2$, $L_1' = \max\{L_1, l_1 + (L_2 - l_2)\} \prec R_1' = \min\{R_1, u_1 - (u_2 - R_2)\}$, and $q = |(I_1 \setminus A_1) \cap [L_1', R_1']|$. If $s_2 < |A_1| + q - 1$, then $|A_1 + A_2| \geqslant |I_2| + |A_1| - 1 + q$.*

Proof Let $A_1' = A_1 \cup (I_1[L_1', R_1'])$. Then $|A_1'| = |A_1| + q$ and $A_1' + A_2 = A_1 + A_2$. Hence $|A_1 + A_2| = |A_1' + A_2| \geqslant |I_2| + |A_1'| - 1 = |I_2| + |A_1| - 1 + q$ by Theorem A.4.

□

Let $C \subseteq [0, H]$ and I be an *a.p.* We say that C is *full* in I if $C \subseteq I$ and $|C| \sim |I|$. We call a set C being full if C is full in some *a.p.*

Remark 3.9 *For proving Theorem 1.7 we intend to find the good lower bound of $|2A|$ in various cases. We hope to eliminate the cases by showing that $|2A| \succ \frac{10}{3}|A|$, which contradicts (6). If this cannot be achieved, then we will try to show that (7) is true. Since finding a good lower bound of $|2A|$ depends on the configuration of A, we need to divide the proof into many different cases according to various configurations of A. Sometimes, a subtle difference between two configurations of A requires very different treatments.*

4 Proof of Theorem 1.7 when $d = 4$

In this section we prove Theorem 1.7 when $d = 4$.

Lemma 4.1 *Suppose $A \subseteq [0, H]$ is an almost tight subset of a* b.p. $I_1 \cup I_2$ *of difference d $= 4$ satisfying (4), (5), and (6) but failing (7). Suppose A_0 is a subset of an* a.p. I_0 *of difference* 4 *such that $l_0 \not\equiv l_i \pmod 4$ for $i = 1, 2$. If $(l_0 + I_2) \cap (2I_1) \neq \emptyset$. Then A_1 is full in I_1, A_2 is full in I_2, $2|A_1| \sim |A_2|$, $l_0 + l_2 \sim 2l_1$, and $l_0 + u_2 \sim 2u_1$.*

Proof It is easy to see that I_0+I_1 is disjoint from $2I_0$, I_0+I_2, $2I_1$, and I_1+I_2. Also I_0+I_2 is disjoint from $2I_0$, I_0+I_1, $2I_2$, and I_1+I_2. If $(I_0+I_1)\cap(2I_2) \neq \emptyset$, then $l_0 + l_1 \equiv 2l_2 \pmod 4$ and $l_0 + l_2 \equiv 2l_1 \pmod 4$. Subtracting the first equation from the second we have $l_2 - l_1 \equiv 2(l_1 - l_2) \pmod 4$, which implies $l_1 \equiv l_2 \pmod 4$. Therefore, we can assume that $(I_0 + I_1) \cap (2I_2) = \emptyset$. If $|A_1| \succ \frac{1}{3}|A|$, then

$$|2A| \succcurlyeq |2A_2| + |A_2 + A_1| + |2A_1| + |A_0 + A_1| \tag{18}$$
$$\succcurlyeq 4|A_1| + 3|A_2| \sim 3|A| + |A_1| \succ \frac{10}{3}|A|, \tag{19}$$

which contradicts (6). If $|A_1| \prec \frac{1}{3}|A|$, then $|A_2| \succ \frac{2}{3}|A|$. Hence

$$|2A| \succcurlyeq |2A_2| + |A_2 + A_1| + |A_2 + A_0| + |A_0 + A_1| \tag{20}$$
$$\succcurlyeq 4|A_2| + 2|A_1| \sim 3|A| + |A_2| - |A_1| \succ \frac{10}{3}|A|, \tag{21}$$

which again contradicts (6). Therefore we have that $|A_1| \sim \frac{1}{3}|A|$ and $|A_2| \sim \frac{2}{3}|A|$, which imply $|A_2| \sim 2|A_1|$.

Notice that in order to prevent $|2A| \succ \frac{10}{3}|A|$ in the inequalities above, we must have $|2A_2| \sim 2|A_2|$, $|2A_1| \sim 2|A_1|$ $|A_1 + A_2| \sim |A_1| + |A_2|$, $|A_0 + A_1| \sim |A_1|$, and $|A_0 + A_2| \sim |A_2|$. Now clearly we have $|2A| \sim \frac{10}{3}|A|$. By Lemma A.1 we have that A_i is full in I_i for $i = 1, 2$ and $|I_2| \sim 2|I_1|$. If $l_0 + l_2 \not\sim 2l_1$ or $u_0 + u_2 \not\sim 2u_1$, then we can replace $|A_0 + A_2|$ by $|(A_0 + A_2) \cup (2A_1)|$ in (18) or replace $|2A_1|$ by $|(A_0 + A_2) \cup (2A_1)|$ in (20) to conclude $|2A| \succ \frac{10}{3}|A|$. This completes the proof.

□

We are now ready to prove the following part of Theorem 1.7.

Theorem 4.2 *Suppose $A \subseteq [0, H]$ is an almost tight subset of a* b.p. $I_1 \cup I_2$ *of difference $d = 4$ satisfying (4), (5), and (6). Then A satisfies (7).*

Proof Assume the contrary. By Lemma 3.6 we can assume that $A_0 = A \setminus (I_1 \cup I_2)$ is a subset of an *a.p.* I_0 of difference 4 with $l_0 = \min I_0$, $u_0 = \max I_0$, and $l_0 \not\equiv l_i \pmod 4$ for $i = 1, 2$. We can also assume, without loss of generality, that $(I_0 + I_2) \cap (2I_1) \neq \emptyset$ because if $(I_0 + I_2) \cap (2I_1) = \emptyset$ and $(I_0 + I_1) \cap (2I_2) = \emptyset$, then $|2A| \succcurlyeq 4|A|$. By Lemma 4.1 we have that A_1 is full in I_1, A_2 is full in I_2, $2|A_1| \sim |A_2|$, $l_0 + l_2 \sim 2l_1$, and $l_0 + u_2 \sim 2u_1$. It is easy to see that $2A_1, A_1 + A_2, 2A_2, A_0 + A_1$ are pairwise disjoint.

If $u_0 < l_2$, then $(2A_0) \cap (2A_2) = \emptyset$, which implies that A is a subset of a *t.p.* Hence we can assume that $u_0 \geqslant l_2$. By a symmetric argument we can also assume that $l_0 \leqslant u_2$. Without loss of generality we can assume that $u_0 \prec u_2$. This implies that $u_2 = H$. We now divide the rest of the proof into three cases for $l_0 = 0$, $l_1 = 0$, or $l_2 = 0$.

Case 4.2.1 $l_0 = 0$.

Notice that $|A_2 + A_1| \geqslant |I_2| + |A_1| - 1 + q$ by Lemma 3.8, where q is defined in that lemma. Hence we have

$$|2A| \geqslant |2A_2| + |A_2 + A_1| + |A_0 + A_2| + |A_0 + A_1| \tag{22}$$

$$
\begin{aligned}
&+2A_0(0, l_2-1) - 1 && (23)\\
&\geqslant 2|A_2| - 1 + b_2 + |I_2| + |A_1| - 1 + q && (24)\\
&+A_2(L_2, L_2 + (u_0 - l_0)) + |A_2| - 1 && (25)\\
&+A_1(L_1, L_1 + (u_0 - l_0)) + |A_1| - 1 + 2A_0(0, l_2 - 1) - 1 && (26)\\
&\geqslant 3|A| - 3 + b_2 + |I_2| - |A_1| - 3|A_0| && (27)\\
&+A_2(L_2, L_2 + (u_0 - l_0)) + |I_0| + 2A_0(0, l_2 - 1) - 2, && (28)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
b \geqslant\ & b_2 + |I_2| - |A_1| - 3|A_0| \\
& +A_2(L_2, L_2 + (u_0 - l_0)) + |I_0| + 2A_0(0, l_2 - 1) - 2
\end{aligned}
$$

by the definition of b in (6). Hence

$$
\begin{aligned}
H + 1 &= u_2 - l_2 + l_2 + 1 \leqslant 4|I_2| - 4 + l_2 + 1 \\
&\leqslant 2(|A_2| + s_2) + 2|I_2| + l_2 - 3 \\
&\leqslant 2|A| - 2|A_1| - 2|A_0| + 2b_2 - 2p_2 + 2|I_2| + l_2 - 3 \\
&\leqslant 2|A| - 2|A_0| - 2p_2 + l_2 - 3 \\
&\quad +2(b + 3|A_0| - A_2(L_2, L_2 + (u_0 - l_0)) - |I_0| - 2A_0(0, l_2 - 1) + 2) \\
&\leqslant 2|A| - 1 + 2b + 4|A_0| - 4|I_0| - 4A_0(0, l_2 - 1) + 2 + l_2 \\
&\leqslant 2|A| - 1 + 2b
\end{aligned}
$$

because $A_0(l_2, u_0) = A_0(l_2 + 2, u_0)$ by the fact that $l_0 + 2 \equiv l_2 \pmod 4$ and

$$
\begin{aligned}
& 4|A_0| - 4|I_0| - 4A_0(0, l_2 - 1) + 2 + l_2 \\
&\quad \leqslant 4A_0(l_2 + 2, u_0) - u_0 - 2 + l_2 \leqslant u_0 - l_2 + 2 - u_0 - 2 + l_2 = 0.
\end{aligned}
$$

Hence (7) is true.

□ (Case 4.2.1)

Case 4.2.2 $l_1 = 0$.

Since $l_0 + l_2 \sim 2l_1 = 0$, we have $l_0 \sim l_2 \sim 0$. Notice that $u_0 \geqslant l_2 + 2$. Without loss of generality we can assume that $l_0 \geqslant l_2 + 2$ by the following argument: Suppose $l_0 < l_2 + 2$. Then $l_0 < l_2$ by the fact that $l_0 \equiv l_2 + 2 \pmod 4$. Let $A' = A \setminus \{l_0\}$. Then $2l_0, l_0 + 0 \notin (2A')$. By Lemma 3.2 for $m = 2$ we can replace A by A'. We now repeat this step until $l_0 \geqslant l_2 + 2$. Notice that (6) remains true because $|A_0| \sim 0$.

We can assume that either $|A_0| = 1$ or $u_0 + u_2 \leqslant 2u_1$ by the following argument: Suppose $l_0 < u_0$ and $u_0 + u_2 > 2u_1$. Let $A' = A \setminus \{u_0\}$. Then $u_0 + u_1, u_0 + u_2 \notin (2A')$. By Lemma 3.2 for $m = 2$ again we can replace A by A'. Now repeat this step until either $|A_0| = 1$ or $u_0 + u_2 \leqslant 2u_1$. Notice that we cannot remove u_0 from A if $A_0 = \{u_0\}$ because if we do, the resulting set would be a subset of a *b.p.*

Suppose $A_0 = \{l_0\} = I_0$ and $l_0 + u_2 > 2u_1$. We now want to expand A in $I_0 \cup I_1 \cup I_2$ by Lemma 3.1. Let $b_0 = b$ be defined in (6). Without loss of generality we can assume that $|A|$ is maximal for $A \subseteq I_0 \cup I_1 \cup I_2$ being a counterexample of Theorem 4.2 and $0 \leqslant |2A| - 3|A| + 3 \leqslant b_0$[3].

Claim 4.2.1 $A_1 = I_1$ and $A_2 = I_2$.

Proof of Claim 4.2.1 Step 1: If $x \in I_2$ and $l_2 \prec x \prec u_2$, then $x \in A$. Suppose $x \notin A$. Let $A' = A \cup \{x\}$. By the fact that A_i is full in I_i for $i = 1, 2$ we have that $x + A_2 \subseteq 2A_2$ and $x + A_1 \subseteq A_1 + A_2$. Hence $2A' \subseteq (2A) \cup \{x + l_0\}$. This contradicts the maximality of $|A|$ by Lemma 3.1.

Step 2: If $x \in I_1$ and $l_1 \prec x \prec u_1$, then $x \in A$. Suppose $x \notin A$. Let $A' = A \cup \{x\}$. By the fact that $l_0 + l_2 \sim 0$, $l_0 + u_2 \sim 2u_1$, and Step 1 above we have that $x + A_2 \subseteq A_1 + A_2$ and $x + A_1 \subseteq A_0 + A_2$. Hence $2A' \subseteq (2A) \cup \{x + l_0\}$. This contradicts the maximality of $|A|$ by Lemma 3.1.

For $i = 1, 2$ let $m_i = \frac{l_i + u_i}{2}$ and

$$x_{i,L} = \max\left(\{l_i\} \cup \{x \in I_i \setminus A_i : x \leqslant m_i\}\right) \text{ and } \tag{29}$$

$$x_{i,R} = \min\left(\{u_i\} \cup \{x \in I_i \setminus A_i : x \geqslant m_i\}\right). \tag{30}$$

Notice that $x_{i,L} \sim l_i$ and $x_{i,R} \sim u_i$ by Step 1 and Step 2 above.

Step 3: $x_{1,L} - l_1 = x_{2,L} - l_2$ and $u_1 - x_{1,R} = u_2 - x_{2,R}$. Suppose $x_{1,L} - l_1 < x_{2,L} - l_2$. Then $x_{2,L} \notin A$. Let $A' = A \cup \{x_{2,L}\}$. Then $2A' \subseteq (2A) \cup \{x_{2,L} + l_2, x_{2,L} + l_0\}$. This again contradicts the maximality of $|A|$ by Lemma

[3] This condition is internal, which is needed to replace the external condition (6)

3.1. By the same reason we can derive a contradiction by assuming that $x_{1,L} - l_1 > x_{2,L} - l_2$. We can prove $u_1 - x_{1,R} = u_2 - x_{2,R}$ by a similar argument.

Step 4: $x_{2,L} = l_2$ and hence $x_{1,L} = l_1$ by Step 3. Suppose $x_{2,L} > l_2$. Let $A' = A \cup \{x_{2,L}\}$. Since $l_1 = 0$, $x_{2,L} + l_0 \geqslant l_1 + x_{1,L} + (x_{2,L} + l_0 - x_{1,L}) \in (2A_1)$. Hence $2A' \subseteq (2A) \cup \{x_{2,L} + l_1, x_{2,L} + l_2\}$. This again contradicts the maximality of $|A|$ by Lemma 3.1.

Step 5: $x_{1,R} = u_1$ and hence $x_{2,R} = u_2$ by Step 3. Suppose $x_{1,R} < u_1$. Then $x_{1,R} \notin A$. Let $A' = A \cup \{x_{1,R}\}$. Since $l_0 + u_2 > 2u_1$ and $u_1 - x_{1,R} = u_2 - x_{2,R}$, we have $l_0 + x_{2,R} > u_1 + x_{1,R}$. Hence $u_1 + x_{1,R} \in A_0 + A_2$. This implies that $2A' \subseteq (2A) \cup \{l_0 + x_{1,R}, u_2 + x_{1,R}\}$, which again contradicts the maximality of $|A|$.

□ (Claim 4.2.1)

Following (22) with $|A_0 + A_2|$ being replaced by $|(2A_1) \cup (A_0 + A_2)|$ we have that

$$\begin{aligned} |2A| &\geqslant 2|A_2| - 1 + |A_1| + |A_2| - 1 \\ &\quad + \frac{1}{4}(l_0 + u_2) + 1 + |A_1| \\ &= 3|A| - 5 + \frac{1}{4}(l_0 + u_2 - u_1). \end{aligned}$$

Hence $b \geqslant \frac{1}{4}(l_0 + u_2 - u_1) - 2$. Therefore,

$$\begin{aligned} H + 1 &= u_2 - l_2 + l_2 - u_1 + u_1 + 1 \\ &= 2|A| - 2|A_0| - 4 + \frac{1}{2}(u_2 - l_2) + \frac{1}{2}u_1 + l_2 - u_1 + 1 \\ &= 2|A| - 1 + \frac{1}{2}(u_2 + l_2 - u_1) - 4 \\ &< 2|A| - 1 + \frac{1}{2}(u_2 + l_0 - u_1) - 2 \leqslant 2|A| - 1 + 2b, \end{aligned}$$

which contradicts our assumption that (7) is false.

Remark 4.2.1 Notice that when we add one element x to A in the proof above to form a set A', b' is less than b. Is it possible that b is non-negative but

b' becomes negative, which makes a part of (6) false? In fact it is impossible when A is assumed to fail (7) because if it is possible, then $b \leqslant b' + 3 \sim 0$, which implies $H + 1 \leqslant 2|A| - 1 + 2b$ by Theorem 1.3.

Suppose now $u_0 + u_2 \leqslant 2u_1$ is true. Then by (22) with $|A_0 + A_2|$ being replaced by $|2A_1|$ we have $|2A| \geqslant 2|A_2| - 1 + b_1 + |A_2| + |I_1| - 1 + 2|A_1| - 1 + b_1 + |A_1| + |A_0| - 1 = 3|A| - 3 - 2|A_0| + |I_1| + b_1 + b_2 - 1$. Hence $b \geqslant -2|A_0| + |I_1| + b_1 + b_2 - 1$. Therefore,

$$\begin{aligned} H + 1 &= u_2 - l_2 + l_2 - u_1 + u_1 + 1 \\ &\leqslant 2|A_2| + 2|A_1| + 2(b_1 + b_2) + 2|I_2| + 2|I_1| + l_2 - u_1 - 7 \\ &\leqslant 2|A| - 2|A_0| + 2(b + 2|A_0| - |I_1| + 1) + 2|I_2| + 2|I_1| + l_2 - u_1 - 7 \\ &\leqslant 2|A| + 2b + \frac{1}{2}(u_0 - l_0) + 2 + \frac{1}{2}(u_2 - l_2) + 2 + l_2 - u_1 - 5 \\ &\leqslant 2|A| - 1 + 2b + \frac{1}{2}(u_0 + u_2 - 2u_1 + l_2 - l_0) \\ &< 2|A| - 1 + 2b, \end{aligned}$$

which contradicts the failure of (7).

□ (Case 4.2.2)

Case 4.2.3 $l_2 = 0$.

Following (22) we have

$$\begin{aligned} |2A| \geqslant\ & 2|A_2| - 1 + b_2 + |I_2| + |A_1| - 1 + |A_2| + |A_0| - 1 \\ & + |A_1| + |A_0| - 1 \geqslant 3|A| + |I_2| - |A_1| - |A_0| + b_2 - 4, \end{aligned}$$

then $b \geqslant |I_2| - |A_1| - |A_0| + b_2 - 1$.

Hence $\frac{1}{4}H + 1 = |I_2|$ implies that

$$H + 1 = 4|I_2| - 3 \leqslant 2|A| - 1 + 2(|I_2| - |A_1| - |A_0| + b_2 - 1) \leqslant 2|A| - 1 + 2b,$$

which contradicts the assumption that A fails (7). This completes the proof of Theorem 4.2.

□

5 Proof of Theorem 1.7 when $d = 3$

In this section we prove Theorem 1.7 when $d = 3$. First we prove a couple of lemmas.

Lemma 5.1 *Suppose that* $A \subseteq [0, H]$ *is an almost tight subset of a* b.p. $I_1 \cup I_2$ *of difference* $d = 3$ *and satisfies (4), (5), and (6), but fails (7). Then* $|A_1 + A_2| \prec |A_1| + |A_2| + \min\{|A_1|, |A_2|\}$.

Proof: Notice that l_0, l_1, l_2 are pairwise non-congruent modulo 3 by Lemma 3.6. Suppose this lemma is not true. Without loss of generality we assume that $\min\{|A_1|, |A_2|\} = |A_1|$. Then we have $|A_1 + A_2| \succcurlyeq 2|A_1| + |A_2|$. If $|A_1| \sim 0$, then $|2A| \succcurlyeq 4|A_2| \sim 4|A|$, which contradicts (6). Hence we can assume $|A_1| \succ 0$.

Suppose $|A_1| \succcurlyeq \frac{1}{3}|A|$. Then

$$|2A| \succcurlyeq |(2A_1) \cup (A_0 + A_2)| \tag{31}$$

$$+|(A_1 + A_2) \cup (2A_0)| + |(2A_2) \cup (A_0 + A_1)| \tag{32}$$

$$\succcurlyeq 2|A_1| + b_1 + 2|A_1| + |A_2| + 2|A_2| + b_2 \tag{33}$$

$$\sim 3|A| + |A_1| + b_1 + b_2 \succcurlyeq \frac{10}{3}|A| + b_1 + b_2. \tag{34}$$

By (6) we have $b_1 + b_2 \sim 0$. Hence by Theorem A.1 we have that A_i is full in I_i for $i = 1, 2$. Let $d_i = \gcd(A_i - l_i)$ for $i = 1, 2$. If $d_1 = d_2$, then $|A_1 + A_2| \sim |A_1| + |A_2|$, which contradicts the assumption that $|A_1 + A_2| \succcurlyeq 2|A_1| + |A_2|$. If $d_1 \neq d_2$, then either $|(A_0 + A_2) \setminus (2A_1)| \succ 0$ or $|(A_0 + A_1) \setminus (2A_2)| \succ 0$. Hence by (31) and (32) we have $|2A| \succ 3|A| + |A_1| \succcurlyeq \frac{10}{3}|A|$, which contradicts (6).

Suppose $|A_1| \prec \frac{1}{3}|A|$. Then $|A_2| \succ \frac{2}{3}|A|$. By (31) and (32) we now have

$$|2A| \succcurlyeq 3|A| + |A_2| - |A_1| \succ 3|A| + \frac{2}{3}|A| - \frac{1}{3}|A| = \frac{10}{3}|A|,$$

which again contradicts (6).

□

Lemma 5.2 *Suppose that $A \subseteq [0, H]$ is an almost tight subset of a* b.p. $I_1 \cup I_2$ *of difference $d = 3$ and satisfies (4), (5), and (6), but fails (7). Let $g_i = \gcd(A_i - l_i)$ for $i = 1, 2$. Then $g_1 = g_2 = 3$.*

Proof Notice that $g = \gcd(g_1, g_2) = 3$. If $g_1 \neq g_2$, then $|A_1 + A_2| \succcurlyeq |A_1| + |A_2| + \min\{|A_1|, |A_2|\}$, which contradicts Lemma 5.1. Hence $g = g_1 = g_2 = 3$.

□

Lemma 5.3 *Suppose that $A \subseteq [0, H]$ is an almost tight subset of a* b.p. $I_1 \cup I_2$ *of difference* 3 *and satisfies (4), (5), and (6), but fails (7). Then $s_i \prec \frac{1}{2}|I_i|$ for $i = 1, 2$, where s_i is defined by (10).*

Proof Suppose $s_1 \succcurlyeq \frac{1}{2}|I_1|$. Then $|A_1| \preccurlyeq \frac{1}{2}|I_1|$. If $|I_2| \preccurlyeq |I_1|$, then

$$|A_1 + A_2| \succcurlyeq \min\{|I_1| + |A_2|, |A_1| + 2|A_2|\} \succcurlyeq |A_1| + |A_2| + \min\{|A_1|, |A_2|\}$$

by Theorem A.4. But this contradicts Lemma 5.1. Hence we can assume that $|I_1| \prec |I_2|$. If $s_2 \succcurlyeq \frac{1}{2}|I_1|$, then

$$|A_1 + A_2| \succcurlyeq \min\{2|A_1| + |A_2|, |A_1| + |A_2| + s_2\} \succcurlyeq 2|A_1| + |A_2|,$$

which again contradicts Lemma 5.1. Thus we can assume $s_2 \prec \frac{1}{2}|I_1|$, which implies that $A_2(l_2, l_2 + u_1 - l_1 - 3) \succ \frac{1}{2}|I_1| \succcurlyeq |A_1|$. Now by Theorem A.4 we have

$$\begin{aligned} &|A_1 + A_2| && (35)\\ &\quad \succcurlyeq |A_1 + A_2[l_2, l_2 + u_1 - l_1 - 3]| + |u_1 + A_2[l_2 + u_1 - l_1, u_2]| && (36)\\ &\quad \succcurlyeq 2|A_1| + A_2(l_2, l_2 + u_1 - l_1 - 3) + A_2(l_2 + u_1 - l_1, u_2) && (37)\\ &\quad = 2|A_1| + |A_2|, && (38) \end{aligned}$$

which again contradicts Lemma 5.1. Therefore, we conclude that $s_1 \prec \frac{1}{2}|I_1|$. We can prove that $s_2 \prec \frac{1}{2}|I_2|$ by an symmetric argument.

□

Remark 5.4 *Lemma 5.3 shows that if A satisfies (4), (5), and (6), but fails (7), then A_i for $i = 1, 2$ occupies significantly more than half of the space inside its hosting* a.p. I_i.

Lemma 5.5 *Suppose that $A \subseteq [0, H]$ is an almost tight subset of a* b.p. $I_1 \cup I_2$ *of difference* 3 *and satisfies (4), (5), and (6), but fails (7). Then* $|A_1 + A_2| \succcurlyeq |A_1| + |A_2| + \max\{s_1, s_2\}$.

Proof Without loss of generality let $|I_2| \succcurlyeq |I_1|$. When $s_2 \succcurlyeq s_1$, we have $|A_1 + A_2| \succcurlyeq |A_1| + |I_2| = |A_1| + |A_2| + s_2$ by Theorem A.4. When $s_2 \prec s_1$, then by (36) we have

$$|A_1+A_2| \succcurlyeq |I_1|+A_2(l_2, l_2+u_1-l_1-3)+A_2(l_2+u_1-l_1, u_2) = |A_1|+s_1+|A_2|.$$

□

We now prove the second part of Theorem 1.7 for $d = 3$.

Theorem 5.6 *Suppose $A \subseteq [0, H]$ is an almost tight subset of a* b.p. $I_1 \cup I_2$ *of difference $d = 3$ satisfying (4), (5), and (6). Then A satisfies (7).*

Proof Assume that the theorem is not true, i.e., A satisfies (4), (5), and (6), but fails (7). We assume that $|A|$ is maximal. Without loss of generality let $u_2 = H$. By symmetry we can assume $l_0 \sim u_0 \prec H$. We will derive a contradiction in each of the three cases for $l_2 \sim 0$, $l_1 \sim 0$, or $l_0 \sim 0$.

Case 5.6.1 $l_2 \sim 0$.

We will prove two Claims first. Then we derive a contradiction.

Claim 5.6.1.1 $2(u_1 - l_1) \preccurlyeq u_2 - l_2 \sim H$.

Proof of Claim 5.6.1.1 By the failure of (7) we have $|A|+b < \frac{1}{2}(H+2)$. By (31) and (32) we have $|2A| \succcurlyeq 3|A| + s_1 + s_2$, which implies $b \succcurlyeq s_1 + s_2$. Hence $\frac{1}{3}(u_1 - l_1) + \frac{1}{3}(u_2 - l_2) \sim |A_1| + s_1 + |A_2| + s_2 \preccurlyeq |A| + b \preccurlyeq \frac{1}{2}H$, which implies $(u_1 - l_1) + (u_2 - l_2) \sim u_1 - l_1 + H \preccurlyeq \frac{3}{2}H$. Hence $u_1 - l_1 \preccurlyeq \frac{1}{2}H$, which implies $2(u_1 - l_1) \preccurlyeq H - l_2$.

□ (Claim 5.6.1.1)

Claim 5.6.1.2 $2l_1 \sim l_2 + l_0$ and $2u_1 \sim u_2 + u_0$, hence $2(u_1 - l_1) \sim H$.

Proof of Claim 5.6.1.2 Suppose the claim is not true. Then we have either $2l_1 \succ l_2 + l_0$ or $2u_1 \prec u_0 + H$ by Claim 5.6.1.1. Suppose $2l_1 \succ l_2 + l_0$.

By Lemma 5.1 we have $|A_1 + A_2| \prec 2|A_1| + |A_2|$. By Theorem A.4 we have that $|A_1 + A_2| \succcurlyeq |A_1| + |I_2| = |A_1| + |A_2| + s_2$ and $|2A_2| \succcurlyeq 2|A_2| + s_2$. Let $w = A_2(2u_1 - l_0, H)$ if $2u_1 \leqslant H + l_0$ and $w = 0$ otherwise. Then by adding the terms $A_2(l_2, 2l_1 - l_0)$ and w to (31), (32), and by Lemma 5.5 we have

$$|2A| \succcurlyeq 3|A| + s_1 + 2s_2 + A_2(l_2, 2l_1 - l_0) + w.$$

This shows $b \succcurlyeq s_1 + 2s_2 + A_2(l_2, 2l_1 - l_0) + w$. Since $\frac{1}{3}H \sim |A_2| + s_2$, we have

$$\begin{aligned} H + 1 &\sim 3|A_2| + 3s_2 \sim 2|A| + |I_2| - 2|A_1| + 2s_2 \\ &\preccurlyeq 2|A| + |I_2| - 2|A_1| + 2(b - s_1 - s_2 - A_2(l_2, 2l_1 - l_0) - w) \\ &= 2|A| + 2b + |I_2| - 2|I_1| - 2(s_2 + A_2(l_2, 2l_1 - l_0) + w). \end{aligned}$$

If $2u_1 \leqslant H + l_0$, then

$$s_2 + A_2(l_2, 2l_1 - l_0) + w \succcurlyeq \frac{1}{3}(2l_1 - l_0 - l_2 + H - 2u_1 + l_0) \sim \frac{1}{3}(2l_1 + H - 2u_1).$$

Hence

$$\begin{aligned} H + 1 &\preccurlyeq 2|A| + 2b + \frac{1}{3}(H - 2u_1 + 2l_1) - \frac{2}{3}(2l_1 + H - 2u_1) \\ &= 2|A| + 2b - \frac{1}{3}(2l_1 - l_0 + H - 2u_1 + l_0) \prec 2|A| + 2b \end{aligned}$$

because $2l_1 - l_0 \succ l_2 \geqslant 0$ and $H - 2u_1 + l_0 \geqslant 0$. If $2u_1 > H + l_0$, then $w = 0$ and $s_2 + A_2(l_2, 2l_1 - l_0) + w \succcurlyeq \frac{1}{3}(2l_1 - l_0 - l_2)$. Hence

$$\begin{aligned} H + 1 &\preccurlyeq 2|A| + 2b + \frac{1}{3}(H - 2u_1 + 2l_1) - \frac{2}{3}(2l_1 - l_0) \\ &\preccurlyeq 2|A| + 2b + \frac{1}{3}(H - 2u_1 + l_0) - \frac{1}{3}(2l_1 - l_0) \prec 2|A| + 2b \end{aligned}$$

because $H - 2u_1 + l_0 < 0$ and $2l_1 - l_0 \succ 0$. Therefore, we have $2|A| + 2b \succ H$, which contradicts the failure of (7). If we assume $2u_1 \prec H + u_0$ first, we can prove that $2|A| + 2b \succ H$ by symmetry.

□ (Claim 5.6.1.2)

We now prove Theorem 5.6 under the case of $l_2 \sim 0$. Suppose $|A_1+A_2| \sim |A_1|+|A_2|+b_{1,2} \succ |A_1|+|A_2|$. Then $|2A| \succcurlyeq 3|A|+s_1+s_2+b_{1,2}$. By (6) we have $b \succcurlyeq s_1+s_2+b_{1,2}$. Hence $\frac{1}{3}H \sim |I_2| \sim |A_2|+s_2$ implies that

$$\begin{aligned} H+1 &\sim 3|A_2|+3s_2 \preccurlyeq 2|A|-2|A_1|+|I_2|+2(b-s_1-b_{1,2}) \\ &\sim 2|A|+2b+|I_2|-2|I_1|-2b_{1,2} \sim 2|A|+2b-2b_{1,2} \prec 2|A|+2b, \end{aligned}$$

which contradicts the failure of (7). Suppose $|A_1+A_2| \sim |A_1|+|A_2|$, then $s_1 \sim s_2 \sim 0$ by Lemma 5.5. Hence $|2A_1| \sim 2|A_1|$ and $|2A_2| \sim 2|A_2|$. By Claim 5.6.1.2 we have that $|2A| \sim 3|A|$ or $b \sim 0$. Since A is not a subset of a $b.p.$ of length $|A|+b$, then (7) is true by Theorem 1.3. This completes the proof of Case 5.6.1.

□ (Case 5.6.1)

Case 5.6.2 $l_1 \sim 0$.

By Case 5.6.1 we can assume $l_2 \succ 0$. If $u_1 \sim H$, then by the arguments in Claim 5.6.1.1 and Claim 5.6.1.2 we can show that $u_0+u_1 \sim 2u_2$, which contradicts our assumption that $u_0 \prec H$. Hence we can assume $l_1 \sim 0$, $u_1 \prec H$, $l_2 \succ 0$, and $u_2 = H$. Without loss of generality we can assume $|I_2| \succcurlyeq |I_1|$.

Claim 5.6.2.1 $2(u_1-l_2) \preccurlyeq u_2-l_1$.

Proof of Claim 5.6.2.1 Since we assumed that A fails (7), we have $|A|+b \preccurlyeq \frac{1}{2}H$. This implies $\frac{1}{3}(u_1-l_1)+\frac{1}{3}(u_2-l_2) \sim |A_1|+s_1+|A_2|+s_2 \preccurlyeq |A|+b \preccurlyeq \frac{1}{2}H$. Hence $(u_1-l_1)+(u_2-l_2) \preccurlyeq \frac{3}{2}H \sim \frac{3}{2}(u_2-l_1)$. This implies $2(u_1-l_2) \preccurlyeq u_2-l_1 \sim H$.

□ (Claim 5.6.2.1)

Claim 5.6.2.2 $2l_2 \sim l_1+l_0$ and $2u_1 \sim u_2+u_0$, which implies $2(u_1-l_2) \sim u_2-l_1 \sim H$.

Proof of Claim 5.6.2.2 Suppose the claim is not true. By Claim 5.6.2.1 we have that either $2l_2 \succ l_1+l_0$ or $u_0+u_2 \succ 2u_1$. Suppose $2l_2 \succ l_1+l_0$. Then

$$|2A| \succcurlyeq 3|A|+2s_1+s_2+A_1(l_1, 2l_2-l_0) \succcurlyeq 3|A|+s_1+s_2+\frac{1}{3}(2l_2-l_0-l_1),$$

which implies $b \succcurlyeq s_1 + s_2 + \frac{1}{3}(2l_2 - l_0 - l_1)$. Hence $\frac{1}{3}(u_1 - l_1) + \frac{1}{3}(H - l_2) \sim |A_1| + s_1 + |A_2| + s_2$ implies

$$\begin{aligned} H + 1 &\sim 2|A| + |I_1| + |I_2| + 2(s_1 + s_2) - u_1 + l_1 + l_2 \\ &\preccurlyeq 2|A| + |I_1| + |I_2| + 2(b - \frac{1}{3}(2l_2 - l_0 - l_1)) - u_1 + l_1 + l_2 \\ &\sim 2|A| + 2b + \frac{1}{3}(u_1 - l_1) + \frac{1}{3}(u_2 - l_2) - \frac{2}{3}(2l_2 - l_0 - l_1) - u_1 + l_1 + l_2 \\ &= 2|A| + 2b + \frac{1}{3}(-2u_1 + 4l_1 + u_2 - 2l_2 + 2l_0). \end{aligned}$$

By the failure of (7), we have $-2u_1 + 4l_1 + u_2 - 2l_2 + 2l_0 \succcurlyeq 0$, which implies $2l_2 - l_0 - l_1 \sim 2l_2 - l_0 \preccurlyeq u_2 + l_0 - 2u_1 + 4l_1 \sim u_2 + u_0 - 2u_1$. Since

$$\begin{aligned} |2A| &\succcurlyeq 3|A| + s_1 + 2s_2 + A_2(2u_1 - u_0, u_2) \\ &\succcurlyeq 3|A| + s_1 + s_2 + \frac{1}{3}(u_2 + u_0 - 2u_1), \end{aligned}$$

then $b \succcurlyeq s_1 + s_2 + \frac{1}{3}(u_2 + u_0 - 2u_1)$. Hence $\frac{1}{3}(u_1 - l_1) + \frac{1}{3}(H - l_2) \sim |A_1| + s_1 + |A_2| + s_2$ implies

$$\begin{aligned} H &\sim 2|A| + |I_1| + |I_2| + 2(s_1 + s_2) - u_1 + l_1 + l_2 \\ &\preccurlyeq 2|A| + |I_1| + |I_2| + 2(b - \frac{1}{3}(u_2 + u_0 - 2u_1)) - u_1 + l_1 + l_2 \\ &\sim 2|A| + 2b + \frac{1}{3}(u_1 - l_1) + \frac{1}{3}(u_2 - l_2) - \frac{2}{3}(u_2 + u_0 - 2u_1) - u_1 + l_1 + l_2 \\ &= 2|A| + 2b + \frac{1}{3}(2u_1 + 2l_1 - u_2 + 2l_2 - 2u_0). \end{aligned}$$

Again by the failure of (7) we have $2u_1 + 2l_1 - u_2 + 2l_2 - 2u_0 \succcurlyeq 0$, which implies $2l_2 - l_0 - l_1 \sim 2l_2 - l_0 + 2l_1 \succcurlyeq u_2 + 2u_0 - l_0 - 2u_1 \sim u_2 + u_0 - 2u_1$. Therefore, we have showed that $2l_2 - l_0 - l_1 \sim u_2 + u_0 - 2u_1 \succ 0$, which implies that $u_2 - l_1 \succ 2(u_1 - l_2)$.

Suppose $A_1(l_1, 2l_2 - l_0) \succ 0$. Since

$$\begin{aligned} |2A| &\succcurlyeq 3|A| + s_1 + 2s_2 + A_1(l_1, 2l_2 - l_0) + A_2(2u_1 - u_0, u_2) \\ &\succcurlyeq 3|A| + s_1 + s_2 + A_1(l_1, 2l_2 - l_0) + \frac{1}{3}(u_2 + u_0 - 2u_1), \end{aligned}$$

we have $b \succcurlyeq s_1 + s_2 + A_1(l_1, 2l_2 - l_0) + \frac{1}{3}(u_2 + u_0 - 2u_1)$. Hence $\frac{1}{3}(u_1 - l_1) + \frac{1}{3}(H - l_2) \sim |A_1| + s_1 + |A_2| + s_2$ implies

$$\begin{aligned} H &\preccurlyeq 2|A| + |I_1| + |I_2| \\ &\quad + 2(b - A_1(l_1, 2l_2 - l_0) - \frac{1}{3}(u_2 + u_0 - 2u_1)) - u_1 + l_1 + l_2 \\ &\sim 2|A| + 2b + \frac{1}{3}(u_1 - l_1) + \frac{1}{3}(u_2 - l_2) - \frac{2}{3}(u_2 + u_0 - 2u_1) \\ &\quad - 2A_1(l_1, 2l_2 - l_0) - u_1 + l_1 + l_2 \\ &\sim 2|A| + 2b + \frac{1}{3}(2u_1 + 2l_1 - u_2 + 2l_2 - 2u_0) - 2A_1(l_1, 2l_2 - l_0) \\ &\sim 2|A| + 2b - 2A_1(l_1, 2l_2 - l_0) \prec 2|A| + 2b \end{aligned}$$

because $2u_1 + 2l_1 - u_2 + 2l_2 - 2u_0 \sim (2l_2 - l_0 - l_1) + 3l_1 - (u_2 + u_0 - 2u_1) \sim 0$, which contradicts the failure of (7).

Suppose $A_1(l_1, 2l_2 - l_0) \sim 0$. By symmetry we can also assume $A_2(2u_1 - u_0, u_2) \sim 0$. By Lemma 5.3 we have that $\frac{1}{3}(2l_2 - l_0 - l_1) \preccurlyeq s_1 \prec \frac{1}{2}|I_1|$ and $\frac{1}{3}(u_2 + u_0 - 2u_1) \preccurlyeq s_2 \prec \frac{1}{2}|I_2|$. Notice that we have assumed that $|I_2| \succcurlyeq |I_1|$. Then $l_2 + (2l_2 - l_0 - l_1) \prec u_2 - (u_2 + u_0 - 2u_1)$. Because $A_1(2l_2 - l_0, u_1) \sim |A_1|$, we have

$$\begin{aligned} |A_1 + A_2| &\succcurlyeq |l_1 + A_2[l_2, 3l_2 - l_0 - l_1]| + |A_1[2l_2 - l_0, u_1] + A_2| \\ &\succcurlyeq A_2(l_2, 3l_2 - l_0 - l_1) + A_1(2l_2 - l_0, u_1) + |A_2| + s_2 \\ &\succcurlyeq A_2(l_2, 3l_2 - l_0 - l_1) + |A_1| + |A_2| + s_2. \end{aligned}$$

Hence we have

$$\begin{aligned} |2A| &\succcurlyeq 3|A| + s_1 + 2s_2 + A_2(l_2, 3l_2 - l_0 - l_1) + A_2(2u_1 - u_0, u_2) \\ &\succcurlyeq 3|A| + s_1 + s_2 + \frac{1}{3}(2l_2 - l_0 - l_1 + u_2 + u_0 - 2u_1), \end{aligned}$$

which implies $b \succcurlyeq s_1 + s_2 + \frac{1}{3}(2l_2 - l_0 - l_1 + u_2 + u_0 - 2u_1)$. Hence

$$\begin{aligned} H &\sim 3|A| + 3(s_1 + s_2) - u_1 + l_1 + l_2 \\ &\preccurlyeq 2|A| + |I_1| + |I_2| + 2\left(b - \frac{1}{3}(2l_2 - l_0 - l_1 + u_2 + u_0 - 2u_1)\right) \\ &\quad - u_1 + l_1 + l_2 \end{aligned}$$

$$= 2|A| + 2b + \frac{1}{3}(2u_1 + 4l_1 - u_2 - 2l_2 + 2l_0 - 2u_0)$$
$$\sim 2|A| + 2b + \frac{1}{3}((l_0 + l_1 - 2l_2) + (2u_1 - u_2 - u_0)) \prec 2|A| + 2b$$

because $u_0 \sim l_0$, $l_1 \sim 0$, and $l_0+l_1-2l_2 \sim 2u_1-u_2-u_0 \prec 0$. This contradicts the failure of (7).

$\square$ (Claim 5.6.2.2)

We now prove Theorem 5.6 under the case $l_1 \sim 0$. Suppose $|A_1 + A_2| \sim |A_1|+|A_2|+b_{1,2} \succ |A_1|+|A_2|$. Then $|2A| \succcurlyeq 3|A|+s_1+s_2+b_{1,2}$, which implies $b \succcurlyeq s_1+s_2+b_{1,2}$. Hence $\frac{1}{3}(H-l_2)+\frac{1}{3}(u_1-l_1) = \frac{1}{3}(H-l_1)+\frac{1}{3}(u_1-l_2) \sim \frac{1}{2}H \sim |A_1|+s_1+|A_2|+s_2$ implies $H \sim 2|A|+2(s_1+s_2) \preccurlyeq 2|A|+2b-2b_{1,2} \prec 2|A|+2b$ by Claim 5.6.2.2. This implies (7). Suppose $|A_1 + A_2| \sim |A_1| + |A_2|$, then $b_{1,2} \sim 0$. This implies $s_1 \sim s_2 \sim 0$. Hence $|2A_1| \sim 2|A_1|$ and $|2A_2| \sim 2|A_2|$. By Claim 5.6.2.2 again we have that $|2A| \sim 3|A|$ or $b \sim 0$. By Theorem 1.3 (7) is true. This completes the proof of Case 5.6.2.

$\square$ (Case 5.6.2)

Case 5.6.3 $l_0 \sim 0$. In fact we can assume that $l_0 = 0$.

By Case 5.6.1 and Case 5.6.2 we can assume that $l_1 \succ 0$ and $l_2 \succ 0$, which imply $l_0 = 0$. Since A is not a subset of a *t.p.*, we can assume that $2l_1 \leqslant u_2 + u_0$ and $2l_2 \leqslant u_1 + u_0$.

We can assume that either $A_0 = \{0\}$ or $A \setminus \{u_0\}$ is a subset of a *t.p.* by the following argument: Suppose $|A_0| > 1$ and $A' = A \setminus \{u_0\}$ is not a subset of a *t.p.* Let $u'_0 = \max(A_0 \setminus \{u_0\})$. Since $2u_0, u_0 + u'_0 \notin (2A')$, by Lemma 3.2 with $m = 2$, A' is still a counterexample of Theorem 5.6. Now repeat this step until either $|A_0| = 1$ or A is a subset of a *t.p.* Notice that since $|A_0| \sim 0$, (6) remains true after removing some elements from A_0.

We now need to remove some elements from the left-end of A_1 when $2l_1 < l_2 - 3 - u_0$ and assume that $2l_1 \geqslant l_2 - 3 - u_0$. By symmetry we can also assume that $2l_2 \geqslant l_1 - 3 - u_0$. However, we have to make sure that two things do not occur. The first is that $A'_1 = A_1 \setminus \{l_1\}$ does not become a subset of a *t.p.* The second is that b defined in (6) does not increase. We need this because otherwise we might violate (6) after removing significant number of elements from A_1. This requires that $m \geqslant 4$ when we apply Lemma 3.2.

Claim 5.6.3.1 (i) Suppose $2l_1 < l_2 - 3 - u_0$. Let $A' = A \setminus \{l_1\}$. Then $|2A| - |2A'| \geqslant 4$ and A' is not a subset of a *t.p.* (ii) Suppose $2l_2 < l_1 - 3 - u_0$. Let $A' = A \setminus \{l_2\}$. Then $|2A| - |2A'| \geqslant 4$ and A' is not a subset of a *t.p.*

Proof of Claim 5.6.3.1 We prove (i) only. The proof of (ii) is symmetric.

Clearly, $\{l_1, 2l_1, l_1 + l_2\} \subseteq (2A) \setminus (2A')$. Let $l_1' = \min(A_1 \setminus \{l_1\})$. If $l_1' < l_2 - l_1$, then $l_1 + l_1' \notin (2A')$. Hence $\{l_1, 2l_1, l_1 + l_2, l_1 + l_1'\} \subseteq (2A) \setminus (2A')$, which implies $|2A| - |2A'| \geqslant 4$. Thus we can assume that $l_1' \geqslant l_2 - l_1$.

Notice that we have $l_1' + l_1 \geqslant l_2 > 2l_1 + 3 + u_0$, which implies $l_1 + u_0 < l_1' - 3$. Hence $l_1 + u_0 \notin (2A')$. If $u_0 > 0$, then $\{l_1, 2l_1, l_1 + l_2, l_1 + u_0\} \subseteq (2A) \setminus (2A')$, which implies $|2A| - |2A'| \geqslant 4$. Hence we can assume $u_0 = 0 = l_0$.

Notice that $l_1' - l_1 \geqslant l_2 - 2l_1 > 3$. Let $g = l_1' - l_1 > 3$, $a_1 = \min\{x \in A_1 : x - l_1 \not\equiv 0 \,(\mathrm{mod}\ g)\}$, and $a_2 = \min\{x \in A_2 : x - l_2 \not\equiv 0 \,(\mathrm{mod}\ g)\}$. Notice that a_1 and a_2 exist by Lemma 5.2. If $a_2 > l_1 + a_1$, then $l_1 + a_1 \notin (2A')$. If $a_2 \leqslant l_1 + a_1$, then $l_1 + a_2 \notin (2A')$ by the following argument:

Suppose $l_1 + a_2 = c + e$ for some $c \in A_1 \setminus \{l_1\}$ and $e \in A_2$. Then $c + e = l_1 + a_2 \leqslant 2l_1 + a_1 < l_2 - 3 + a_1$. Since $e \geqslant l_2$, then $l_1 < c < a_1 - 3$, which implies $e = l_1 - c + a_2 < a_2$. Hence $c = l_1 + kg$ and $e = l_2 + k'g$ by the minimality of a_1 and a_2. This shows $a_2 = l_1 + kg + l_2 + k'g - l_1 = l_2 + (k + k')g$, which contradicts the definition of a_2.

Suppose $A' = A \setminus \{l_1\}$ is a subset of a *t.p.* and $l_1' = \min(A_1 \setminus \{l_1\}$. Then $2l_1' > u_0 + u_2$. Since we want the proof to be symmetric for (ii), we do not assume $u_2 = H$ temporarily in this Claim. Since $2l_1 < l_2 - 3 + u_0$, we have $2(l_1' - l_1) \succcurlyeq u_2 - l_2$. This implies that $s_1 \succcurlyeq I_1(l_1, l_1') \succcurlyeq \frac{1}{2}|I_2|$. By (31), (32), and Lemma 5.5 we have $|2A| \succcurlyeq 3|A| + 2s_1 + s_2 \succcurlyeq 3|A| + |A_2| + 2s_2$, which implies $|A_2| \preccurlyeq \frac{1}{3}|A|$ by (6) and $|A_1| \succcurlyeq \frac{2}{3}|A|$. Notice that if $|A_2| \prec \frac{1}{3}|A|$, then $|A_1| \succ \frac{2}{3}|A|$ and, by (31) and (32) with $|(A_1 + A_0) \cup (2A_2)| \succcurlyeq |A_1|$,

$$|2A| \succcurlyeq 3|A| + 2s_1 + |A_1| - 2|A_2| \succcurlyeq 3|A| + |A_1| - |A_2| \succ \frac{10}{3}|A|,$$

which contradicts (6). Hence we can assume that $|A_2| \sim \frac{1}{3}|A|$, which implies that $s_2 \sim 0$, $|A_1| \sim 2|A_2|$, and $s_1 \sim \frac{1}{2}|I_2| \sim I_1(l_1, l_1')$. We now have that A_2

is full in I_2 and $A_1[l_1', u_1]$ is full in $I_1[l_1', u_1]$.

If $l_2 \prec l_1 + l_1'$, then $|(A_2 + A_0) \cup (2A_1)| \succcurlyeq A_2(l_2, l_1 + l_1') + |2A_1|$. By (31) and (32) we have $|2A| \succcurlyeq \frac{10}{3}|A| + A_2(l_2, l_1 + l_1') \succ \frac{10}{3}|A|$, a contradiction to (6). If $l_2 \succcurlyeq l_1 + l_1'$, then $2l_2 \succ l_1'$. Hence $|2A| \succcurlyeq \frac{10}{3}|A| + A_1(l_1', 2l_2) \succ \frac{10}{3}|A|$, a contradiction again.

The proof of (ii) is symmetric. This completes the proof of the claim.

□ (Claim 5.6.3.1)

Notice that b will not become negative after removing l_1 from A in (i) of Claim 5.6.3.1 because otherwise by Theorem A.1 we must have $\gcd(A') > 1$. But $\gcd(A_2 - l_2) = 3$ and $l_2 \not\equiv 0 \pmod 3$ imply $\gcd(A') = 1$. Same for (ii).

By Claim 5.6.3.1 we can now assume that $2l_1 \geqslant l_2 - 3 - u_0$ and $2l_2 \geqslant l_1 - 3 - u_0$ by Lemma 3.2 with $m = 4$.

We now divide the remaining part of the proof into three subcases that (1) $2l_2 \prec u_1$ and $2u_1 \preccurlyeq u_2$, (2) $2l_2 \sim u_1$, and (3) $2l_2 \prec u_1$ and $2u_1 \succ u_2$.

Subcase 5.6.3.1 $2l_2 \prec u_1$ and $2u_1 \preccurlyeq u_2$.

Notice that $2l_1 \geqslant l_2 - 3 - u_0$ and $2u_1 \preccurlyeq u_2$ imply $|I_2| \succcurlyeq 2|I_1|$. Since $2l_2 \leqslant u_1$ and $2l_1 \prec 2u_1 \preccurlyeq u_2$, we can assume after removing elements from the right side of A_0 that $u_0 = l_0$ by Lemma 3.2. We assume that $|A|$ is maximal such that $A \subseteq I_1 \cup I_2 \cup \{0\}$, A satisfies (4), (5), (6) without increasing b, but fails (7). We will show that $A_i = I_i$ for $i = 1, 2$ by Lemma 3.1 and then derive a contradiction by showing that (7) is true.

Let $w = A_2(2u_1 + 3, u_2)$ if $2u_1 < u_2$ and $w = 0$ otherwise. Under this subcase we have

$$\begin{aligned}
3|A| - 3 + b = |2A| &\geqslant |2A_2| + |A_1 + A_2| + |2A_1| + |A_0 + A_0| && (39)\\
&\quad + A_1(l_1, 2l_2 - 3) + A_2(l_2, 2l_1 - 3) + w && (40)\\
&\geqslant 2|A_2| - 1 + s_2 + |A_1| + |A_2| - 1 + b_{1,2} + 2|A_1| - 1 + s_1 + 1 && (41)\\
&\quad + A_1(l_1, 2l_2 - 3) + A_2(l_2, 2l_1 - 3) + w && (42)\\
&= 3|A| - 3 + s_1 + s_2 + b_{1,2} && (43)\\
&\quad + A_1(l_1, 2l_2 - 3) + A_2(l_2, 2l_1 - 3) + w - 2, && (44)
\end{aligned}$$

which implies

$$\frac{1}{3}|A| \succcurlyeq b \geqslant s_1 + s_2 + b_{1,2} \tag{45}$$

$$+A_1(l_1, 2l_2 - 3) + A_2(l_2, 2l_1 - 3) + w - 2. \tag{46}$$

where s_i's are defined in (10) and $b_{1,2}$ is defined in (12).

Claim 5.6.3.2 $s_1+s_2 \preccurlyeq \frac{1}{5}(u_1-l_1)$ and $s_2 \preccurlyeq \frac{1}{6}(u_1-l_1)$. If $s_2 \sim \frac{1}{6}(u_1-l_1)$, then $s_1 \sim 0$, $l_1 \sim 2l_2$, and $A_2(l_2, 2l_1) + w \sim 0$.

Proof of Claim 5.6.3.2 By (45) and (46) we have

$$s_1 + s_2 + b_{1,2} + A_2(l_2, 2l_1) + w \preccurlyeq b \preccurlyeq \frac{1}{3}|A|.$$

Since $b_{1,2} \succcurlyeq \frac{1}{3}s_2 + \frac{1}{3}(s_1 + s_2)$ by Lemma 5.5 and $2u_1 \preccurlyeq u_2$, we have that

$$\begin{aligned}
& s_1 + s_2 + \frac{1}{3}(s_1 + s_2) \\
& \quad + \frac{1}{9}(2l_1 - l_2 + u_2 - 2u_1) + \frac{2}{3}(A_2(l_2, 2l_1) + A_2(2u_1, u_2)) \\
& \preccurlyeq s_1 + s_2 + \frac{1}{3}(s_1 + s_2) + \frac{1}{3}s_2 + A_2(l_2, 2l_1) + w \\
& \preccurlyeq \frac{1}{3}|A| \sim \frac{1}{3}(|I_1| + |I_2|) - \frac{1}{3}(s_1 + s_2).
\end{aligned}$$

This shows that

$$\frac{5}{3}(s_1 + s_2) \preccurlyeq \frac{1}{9}(u_1 - l_1 + u_2 - l_2 - 2l_1 + l_2 - u_2 + 2u_1) = \frac{1}{3}(u_1 - l_1),$$

which implies that $s_1 + s_2 \preccurlyeq \frac{1}{5}(u_1 - l_1)$.

By (45) and (46) again we have

$$\begin{aligned}
2s_2 + \frac{1}{9}(2l_1 - l_2 + u_2 - 2u_1) & \preccurlyeq \frac{7}{3}s_2 + \frac{1}{3}A_2(l_2, 2l_1) + \frac{1}{3}w \\
& \preccurlyeq \frac{4}{3}s_1 + \frac{4}{3}s_2 + b_{1,2} + A_2(l_2, 2l_1) + w + A_1(l_1, 2l_2) \\
& \preccurlyeq \frac{1}{3}(|I_1| + |I_2|) \preccurlyeq \frac{1}{9}(u_2 - l_2 + u_1 - l_1),
\end{aligned}$$

which implies $2s_2 \preccurlyeq \frac{1}{3}(u_1-l_1)$. Hence $s_2 \preccurlyeq \frac{1}{6}(u_1-l_1)$. If $s_2 \sim \frac{1}{6}(u_1-l_1)$, then the inequalities above force that $A_2(l_2, 2l_1)+A_2(2u_1, u_2) \sim 0$, $A_1(l_1, 2l_2) \sim 0$, and $s_1 \sim 0$. Hence $l_1 \sim 2l_2$ because $s_1 + A_1(l_1, 2l_2) \succcurlyeq \frac{1}{3}(2l_2 - l_1) \succcurlyeq 0$.

□ (Claim 5.6.3.2)

Claim 5.6.3.3 $\max\left\{L_2, l_2 + \frac{3}{5}(u_1 - l_1)\right\} \prec \min\left\{R_2, u_2 - \frac{3}{5}(u_1 - l_1)\right\}$. Furthermore, if

$$\max\left\{L_2, l_2 + \frac{3}{5}(u_1 - l_1)\right\} \prec x \prec \min\left\{R_2, u_2 - \frac{3}{5}(u_1 - l_1)\right\}$$

for some $x \in I_2$, then $x \in A_2$.

Proof of Claim 5.6.3.3 First we show that $L_2 \prec u_1 + l_1$ by the following argument: By Claim 5.6.3.2 we have $L_2 - l_2 \preccurlyeq 6s_2 \preccurlyeq u_1 - l_1$. If $L_2 - l_2 \prec u_1 - l_1$, then $L_2 \prec l_2 + u_1 - l_1 \preccurlyeq 2l_1 + u_1 - l_1 = u_1 + l_1$. If $L_2 - l_2 \sim u_1 - l_1$, then $s_2 \sim \frac{1}{6}(u_1 - l_1)$, which implies $2l_1 \sim 4l_2 \succ l_2$ by Claim 5.6.3.2. Hence $L_2 = l_2 + (L_2 - l_2) \prec 2l_1 + (u_1 - l_1) = l_1 + u_1$.

Second we show that $L_2 \prec R_2$ by the following argument: Since $u_2 - R_2 \preccurlyeq 6s_2 \preccurlyeq u_1 - l_1$, we have that $R_2 \succcurlyeq u_2 - u_1 + l_1 \succcurlyeq 2u_1 - u_1 + l_1 = u_1 + l_1$.

Clearly, we have $L_2 \preccurlyeq u_1 + l_1 = 2u_1 - (u_1 - l_1) \prec u_2 - \frac{3}{5}(u_1 - l_1)$ and $R_2 \succcurlyeq u_1 + l_1 = 2l_1 + (u_1 - l_1) \succ l_2 + \frac{3}{5}(u_1 - l_1)$.

Finally, we have

$$\begin{aligned} l_2 + \frac{3}{5}(u_1 - l_1) &\preccurlyeq 2l_1 + \frac{3}{5}u_1 - \frac{3}{5}l_1 = \frac{3}{5}u_1 + \frac{7}{5}l_1 \\ &\prec \frac{7}{5}u_1 + \frac{3}{5}l_1 = 2u_1 - \frac{3}{5}(u_1 - l_1) \preccurlyeq u_2 - \frac{3}{5}(u_1 - l_1). \end{aligned}$$

Above steps imply $\max\left\{L_2, l_2 + \frac{3}{5}(u_1 - l_1)\right\} \prec \min\left\{R_2, u_2 - \frac{3}{5}(u_1 - l_1)\right\}$.

Let $\max\{L_2, l_2 + \frac{3}{5}(u_1 - l_1)\} \prec x \prec \min\{R_2, u_2 - \frac{3}{5}(u_1 - l_1)\}$ for some $x \in I_2$. We want to show that $x \in A_2$. Suppose $x \notin A_2$.

Let $A' = A \cup \{x\}$. By the fact that $L_2 \prec x \prec R_2$ we have $x + (A_2 \cup \{x\}) \subseteq (2A_2)$. Let $y \in A_1$. Choose z, z' in I_2 such that $z < x < z'$, $z' - z \succ \frac{3}{5}(u_1 - l_1)$, $z' - x \leqslant y - l_1$, and $x - z \leqslant u_1 - y$. Notice that $l_1 \leqslant x + y - z' < x + y - z \leqslant u_1$. Then

$$\begin{aligned} A_1(x + y - z', x + y - z) + A_2(z, z') &\succcurlyeq \frac{2}{3}(z' - z) - (s_1 + s_2) \\ &\succcurlyeq \frac{2}{3}(z' - z) - \frac{1}{5}(u_1 - l_1) \succ \frac{2}{3}(z' - z) - \frac{1}{5}\left(\frac{5}{3}(z' - z)\right) = \frac{1}{3}(z' - z), \end{aligned}$$

which implies that $A_1[x+y-z', x+y-z] \cap (x+y-A_2[z,z']) \neq \emptyset$. Hence $y + x \in A_1[x+y-z', x+y-z] + A_2[z,z'] \subseteq A_1 + A_2$. This shows that $x + A_1 \subseteq (2A)$. Now we have $(2A') \subseteq (2A) \cup \{x+0\}$ where $0 = l_0 = u_0$. This contradicts the maximality of $|A|$ by Lemma 3.1.

□ (Claim 5.6.3.3)

Claim 5.6.3.4 If $x \in I_2$ such that

$$\min\left\{L_2, l_2 + \frac{3}{5}(u_1 - l_1)\right\} \prec x \prec \max\left\{R_2, u_2 - \frac{3}{5}(u_1 - l_1)\right\},$$

then $x \in A$. Notice that max and min in this claim and in Claim 5.6.3.3 switch places.

Proof of Claim 5.6.3.4 Let $x_{2,L}$ and $x_{2,R}$ be defined by (29) and (30) with m_2 being replaced by some z between $\max\left\{L_2, l_2 + \frac{3}{5}(u_1 - l_1)\right\}$ and $\min\left\{R_2, u_2 - \frac{3}{5}(u_1 - l_1)\right\}$. We want to show that $x_{2,L} \preccurlyeq L_2$, $x_{2,L} \preccurlyeq l_2 + \frac{3}{5}(u_1 - l_1)$, $x_{2,R} \succcurlyeq R_2$, and $x_{2,R} \succcurlyeq u_2 - \frac{3}{5}(u_1 - l_1)$.

Suppose $x_{2,L} \succ L_2$. Let $A' = A \cup \{x_{2,L}\}$. Then $x_{2,L} + (A_2 \cup \{x_{2,L}\}) \subseteq (2A_2)$ because of $x_{2,L} \prec R_2$. Notice also that $x_{2,L} + l_1$ is the only element of $x_{2,L} + A_1$ possibly not in $A_1 + A_2$. Hence $(2A') \setminus (2A) \subseteq \{x_{2,L} + 0, x_{2,L} + l_1\}$, which contradicts the maximality of $|A|$ by Lemma 3.1.

Suppose $x_{2,L} \succ l_2 + \frac{3}{5}(u_1 - l_1)$. Let $A' = A \cup \{x_{2,L}\}$. Then $x_{2,L} + A_1 \subseteq A_1 + A_2$ because of $x_{2,L} \succ l_2 + \frac{3}{5}(u_1 - l_1)$ by the same argument as in Claim 5.6.3.3. Hence $x_{2,L} + l_2$ is the only elements possibly not in $(2A_2)$. Hence $(2A') \setminus (2A) \subseteq \{x_{2,L} + 0, x_{2,L} + l_2\}$. This contradicts the maximality of $|A|$ by Lemma 3.1.

The arguments are symmetric for the proof of $x_{2,R} \succcurlyeq R_2$ and $x_{2,R} \succcurlyeq u_2 - \frac{3}{5}(u_1 - l_1)$.

□ (Claim 5.6.3.4)

Claim 5.6.3.5 $x_{2,R} - x_{2,L} \succcurlyeq (u_2 - l_2) - \frac{3}{5}(u_1 - l_1) \succcurlyeq \frac{7}{5}(u_1 - l_1)$ and $x_{2,R} - x_{2,L} \succ \frac{7}{5}(u_1 - l_1)$.

Proof of Claim 5.6.3.5 We first prove that $x_{2,R} - x_{2,L} \succcurlyeq (u_2 - l_2) - \frac{3}{5}(u_1 - l_1) \succcurlyeq \frac{7}{5}(u_1 - l_1)$. Then show that $x_{2,R} - x_{2,L} \sim \frac{7}{5}(u_1 - l_1)$ is impossible.

Clearly, we have $u_2 - l_2 - \frac{3}{5}(u_1 - l_1) \succcurlyeq 2(u_1 - l_1) - \frac{3}{5}(u_1 - l_1) = \frac{7}{5}(u_1 - l_1)$. Suppose $x_{2,R} - x_{2,L} \prec (u_2 - l_2) - \frac{3}{5}(u_1 - l_1)$. Then we have $x_{2,L} - l_2 + u_2 - x_{2,R} \succ \frac{3}{5}(u_1 - l_1)$. Notice also that $u_2 - x_{2,R} + x_{2,L} - l_2 \preccurlyeq u_2 - R_2 + L_2 - l_2 \preccurlyeq 6s_2 \preccurlyeq u_1 - l_1$.

Suppose $A_1(l_1, l_1 + (x_{2,L} - l_2)) + A_2(l_2, x_{2,L}) \succ \frac{1}{3}(x_{2,L} - l_2)$. Then $x_{2,L} \notin A$ by the definition of $x_{2,L}$ in (29). Let $A' = A \cup \{x_{2,L}\}$. Since

$$A_2[l_2, x_{2,L}] \cap (x_{2,L} + l_1 - A_1[l_1, l_1 + (x_{2,L} - l_2)]) \neq \emptyset,$$

we have $x_{2,L} + l_1 \in A_1 + A_2$. This shows that $(2A') \setminus (2A) \subseteq \{x_{2,L}, x_{2,L} + l_2\}$, which contradicts that $x_{2,L} \notin A$ by Lemma 3.1 and the maximality of $|A|$. Thus we can assume that $A_1(l_1, l_1 + (x_{2,L} - l_2)) + A_2(l_2, x_{2,L}) \preccurlyeq \frac{1}{3}(x_{2,L} - l_2)$. By a symmetric argument we can assume that $A_1(u_1 - (u_2 - x_{2,R}), u_1) + A_2(x_{2,R}, u_2) \preccurlyeq \frac{1}{3}(u_2 - x_{2,R})$. Now we have

$$\begin{aligned} &\frac{1}{3}(x_{2,L} - l_2) + \frac{1}{3}(u_2 - x_{2,R}) + \frac{1}{5}(u_1 - l_1) \\ &\quad \succcurlyeq A_1(l_1, l_1 + (x_{2,L} - l_2)) + A_2(l_2, x_{2,L}) \\ &\qquad + A_1(u_1 - (u_2 - x_{2,R}), u_1) + A_2(x_{2,R}, u_2) + s_1 + s_2 \\ &\quad \succcurlyeq \frac{2}{3}(x_{2,L} - l_2) + \frac{2}{3}(u_2 - x_{2,R}), \end{aligned}$$

which implies $\frac{1}{5}(u_1 - l_1) \succcurlyeq \frac{1}{3}(x_{2,L} - l_2) + \frac{1}{3}(u_2 - x_{2,R})$. This contradicts the assumption that $x_{2,L} - l_2 + u_2 - x_{2,R} \succ \frac{3}{5}(u_1 - l_1)$.

Suppose $x_{2,R} - x_{2,L} \sim \frac{7}{5}(u_1 - l_1)$. Then we have $u_2 - l_2 \sim 2(u_l - l_1)$, which implies $l_2 \sim 2l_1$ and $u_2 \sim 2u_1$ because $l_2 \preccurlyeq 2l_1$ and $2u_1 \preccurlyeq u_2$. We now have $\frac{1}{3}(|I_1| + |I_2|) \sim \frac{1}{3}(u_1 - l_1)$. Hence by (45) we have $\frac{3}{2}(s_1 + s_2) \preccurlyeq s_1 + s_2 + b_{1,2} \preccurlyeq b \preccurlyeq \frac{1}{3}|A| \sim \frac{1}{3}(|I_1| + |I_2|) - \frac{1}{3}(s_1 + s_2)$, which implies $\frac{11}{6}(s_1 + s_2) \preccurlyeq \frac{1}{3}(u_1 - l_1)$ or $s_1 + s_2 \preccurlyeq \frac{2}{11}(u_1 - l_1)$. Notice that we have showed that

$$A_1(l_1, l_1 + (x_{2,L} - l_2)) + A_2(l_2, x_{2,L}) \preccurlyeq \frac{1}{3}(x_{2,L} - l_2) \text{ and}$$

$$A_1(u_1 - (u_2 - x_{2,R}), u_1) + A_2(x_{2,R}, u_2) \preccurlyeq \frac{1}{3}(u_2 - x_{2,R}).$$

We also have that

$$\frac{1}{3}(x_{2,L} - l_2) + \frac{1}{3}(u_2 - x_{2,R}) + \frac{2}{11}(u_1 - l_1)$$

$$\begin{aligned}&\succcurlyeq A_1(l_1, l_1 + (x_{2,L} - l_2)) + A_2(l_2, x_{2,L})\\&\quad +A_1(u_1 - (u_2 - x_{2,R}), u_1) + A_2(x_{2,R}, u_2) + s_1 + s_2\\&\succcurlyeq \frac{2}{3}(x_{2,L} - l_2) + \frac{2}{3}(u_2 - x_{2,R}).\end{aligned}$$

Hence

$$x_{2,R} - x_{2,L} \succcurlyeq u_2 - l_2 - \frac{6}{11}(u_1 - l_1) \sim \frac{16}{11}(u_1 - l_1) \succ \frac{7}{5}(u_1 - l_1),$$

which contradicts $x_{2,R} - x_{2,L} \sim \frac{7}{5}(u_1 - l_1)$.

□ (Claim 5.6.3.5)

Claim 5.6.3.6 $l_1 + (x_{2,L} - l_2) \prec u_1 - (u_2 - x_{2,R})$. Furthermore, if $x \in I_1$ with $l_1 + (x_{2,L} - l_2) \prec x \prec u_1 - (u_2 - x_{2,R})$, then $x \in A_1$.

Proof of Claim 5.6.3.6 Since $x_{2,R} - x_{2,L} \succcurlyeq (u_2 - l_2) - \frac{3}{5}(u_1 - l_1)$ by Claim 5.6.3.5, we have $(u_2 - x_{2,R}) + (x_{2,L} - l_2) \preccurlyeq \frac{3}{5}(u_1 - l_1)$, which implies $l_1 + (x_{2,L} - l_2) \prec u_1 - (u_2 - x_{2,R})$.

Suppose the second half of the claim is not true. Let $A' = A \cup \{x\}$ and let $y \in A'$.

If $y \in A_1 \cup \{x\}$, then

$$\begin{aligned}x_{2,L} &\preccurlyeq 2l_1 - l_2 + x_{2,L} = l_1 + l_1 + (x_{2,L} - l_2)\\&\prec y + x \prec 2u_1 - (u_2 - x_{2,R}) \preccurlyeq x_{2,R}.\end{aligned}$$

Hence $y + x \in A_2 = A_0 + A_2$.

If $y \in A_2$ and $y \prec x_{2,R} - (x - l_1)$, then

$$\begin{aligned}l_1 + x_{2,L} &= l_2 + (l_1 + x_{2,L} - l_2)\\&\prec y + x \prec x_{2,R} - (x - l_1) + x = l_1 + x_{2,R}.\end{aligned}$$

Hence $y + x \in l_1 + A_2 \subseteq (2A)$. If $y \in A_2$ and $y \succcurlyeq x_{2,R} - (x - l_1)$, then $y \succ x_{2,L} + (u_1 - x)$ because

$$x_{2,R} - (x - l_1) - x_{2,L} - (u_1 - x) = x_{2,R} - x_{2,L} - (u_1 - l_1) \succ \frac{2}{5}(u_1 - l_1).$$

Hence

$$u_1 + x_{2,L} = x_{2,L} + (u_1 - x) + x \prec y + x \prec u_2 + u_1 - (u_2 - x_{2,R}) = u_1 + x_{2,R},$$

which implies that $y + x \in u_1 + A_2 \subseteq (2A)$. Thus we have $(2A') \setminus (2A) \subseteq \{l_0 + x\}$, which leads to a contradiction to the maximality of $|A|$ by Lemma 3.1.

$\square$ (Claim 5.6.3.6)

Let $x_{1,L}$ and $x_{1,R}$ be defined by (29) and (30) with m_1 being replaced by some z between $l_1+(x_{2,L}-l_2)$ and $u_1-(u_2-x_{2,R})$. Then $x_{1,L} \preccurlyeq l_1+(x_{2,L}-l_2)$ and $x_{1,R} \succcurlyeq u_1 - (u_2 - x_{2,R})$ by Claim 5.6.3.6. Hence

$$x_{1,R} - x_{1,L} \succcurlyeq (u_1 - l_1) - (u_2 - x_{2,R} + x_{2,L} - l_2) \succcurlyeq \frac{2}{5}(u_1 - l_1)$$

by Claim 5.6.3.5.

Claim 5.6.3.7 $x_{1,L} - l_1 = x_{2,L} - l_2$ and $u_1 - x_{1,R} = u_2 - x_{2,R}$.

Proof of Claim 5.6.3.7 First we assume that $x_{1,L} - l_1 \geqslant u_1 - x_{1,R}$. Under this assumption we prove $x_{1,L}-l_1 = x_{2,L}-l_2$ and then prove $u_1-x_{1,R} = u_2 - x_{2,R}$. By symmetry we can prove $u_1 - x_{1,R} = u_2 - x_{2,R}$ and then prove $x_{1,L} - l_1 = x_{2,L} - l_2$ if $x_{1,L} - l_1 \leqslant u_1 - x_{1,R}$ is true.

Suppose $x_{1,L} - l_1 < x_{2,L} - l_2$. Then $x_{2,L} \notin A_2$. Let $A' = A \cup \{x_{2,L}\}$ and let $y \in A'$.

If $y = l_1$, then $y + x_{2,L} = y + (x_{2,L} - l_2) + l_2 > x_{1,L} + l_2$. If $y + (x_{2,L} - l_2) \geqslant x_{1,R}$, then $l_1 + \frac{3}{5}(u_1 - l_1) \succcurlyeq l_1 + (x_{2,L} - l_2) > x_{1,R}$, which implies $x_{1,L}-l_1 \geqslant u_1-x_{1,R} \succcurlyeq \frac{2}{5}(u_1-l_1)$. Therefore, we have $x_{1,R}-x_{1,L} \preccurlyeq \frac{1}{5}(u_1-l_1)$, which contradicts $x_{1,R} - x_{1,L} \succcurlyeq \frac{2}{5}(u_1 - l_2)$. This shows $y + (x_{2,L} - l_2) < x_{1,R}$, which implies $y+x_{2,L} = y+(x_{2,L}-l_2)+l_2 \in A_1+l_2 \subseteq (2A)$. If $y \in A_1 \setminus \{l_1\}$, then $y + x_{2,L} = l_1 + x_{2,L} + (y - l_1) \in l_1 + A_2 \subseteq (2A)$. It is easy to see that if $y \in (A_2 \cup \{x_{2,L}\}) \setminus \{l_2\}$, then $y + x_{2,L} \in (2A_2)$. Therefore, we have $(2A') \setminus (2A) \subseteq \{x_{2,L}, x_{2,L} + l_2\}$, which contradicts the maximality of $|A|$ by Lemma 3.1.

Suppose $x_{1,L} - l_1 > x_{2,L} - l_2$. Then $x_{1,L} \notin A_1$. Let $A' = A \cup \{x_{1,L}\}$ and let $y \in A'$.

If $y \in A_2$ and $y < x_{2,R} - (x_{1,L} - l_1)$, then $x_{2,L} < y + (x_{1,L} - l_1) < x_{2,R}$, which implies $y + x_{1,L} = l_1 + y + (x_{1,L} - l_1) \in l_1 + A_2 \subseteq (2A)$.

Suppose $y \in A_2$ and $y \geqslant x_{2,R}-(x_{1,L}-l_1)$. Notice that we have $u_2-x_{2,R} \prec x_{1,R}-x_{1,L}$ because

$$\begin{aligned}2(u_2-x_{2,R}) &\preccurlyeq (u_2-x_{2,R})+(x_{2,L}-l_2)\\ &\preccurlyeq \frac{3}{5}(u_1-l_1) \preccurlyeq \frac{3}{2}(x_{1,R}-x_{1,L}) \prec 2(x_{1,R}-x_{1,L}).\end{aligned}$$

Then

$$\begin{aligned}x_{2,L} &\prec x_{2,R}-(u_1-l_1) \preccurlyeq x_{2,R}-(x_{1,L}-l_1)-(x_{1,R}-x_{1,L})\\ &\preccurlyeq y+3-(x_{1,R}-x_{1,L}) \preccurlyeq u_2-(x_{1,R}-x_{1,L}) \prec u_2-(u_2-x_{2,R})=x_{2,R}.\end{aligned}$$

Hence $y+x_{1,L}=y+3-(x_{1,R}-x_{1,L})+(x_{1,R}-3) \in A_2+(x_{1,R}-3) \subseteq (2A)$. Therefore, we have $A_2+x_{1,L} \subseteq (2A)$.

If $y \in (A_1 \cup \{x_{1,L}\}) \setminus \{l_1\}$ and $y \prec l_1+(x_{1,R}-x_{1,L})$, then

$$l_1+x_{1,L} < y+x_{1,L} = l_1+x_{1,L}+(y-l_1) < l_1+x_{1,R}.$$

Hence $y+x_{1,L} \in l_1+A_1 \subseteq (2A)$.

If $y \in (A_1 \cup \{x_{1,L}\}) \setminus \{l_1\}$ and $y \succcurlyeq l_1+(x_{1,R}-x_{1,L})$, then

$$\begin{aligned}x_{2,L} &\preccurlyeq 2l_1-l_2+x_{2,L}=l_1+(l_1+(x_{2,L}-l_2))\\ &\prec l_1+(u_1-(u_2-x_{2,R})) \preccurlyeq l_1+x_{1,R} \prec y+x_{1,L} \preccurlyeq u_1+l_1+(x_{2,L}-l_2)\\ &\prec u_1+(u_1-(u_2-x_{2,R}))=(2u_1-u_2)+x_{2,R} \preccurlyeq x_{2,R}.\end{aligned}$$

Hence $y+x_{1,L} \in A_0+A_2 \subseteq (2A)$.

Now we have $(2A') \setminus (2A) \subseteq \{x_{1,L}, x_{1,L}+l_1\}$, which contradicts the maximality of $|A|$ by Lemma 3.1. Combine the arguments above we conclude that $x_{1,L}-l_1=x_{2,L}-l_2$.

Suppose $u_1-x_{1,R} > u_2-x_{2,R}$. Let $A'=A \cup \{x_{1,R}\}$ and let $y \in A'$.

If $y \in A_2$ and $y \succ x_{2,L}+(u_1-x_{1,R})$, then

$$\begin{aligned}x_{2,L}+u_1 &\prec y+x_{1,R}\\ &= y-(u_1-x_{1,R})+u_1 < u_2-(u_2-x_{2,R})+u_1=x_{2,R}+u_1.\end{aligned}$$

Hence $y+x_{1,R} \in u_1+A_2 \subseteq (2A)$. If $y \in A_2$ and $y \preccurlyeq x_{2,L}+(u_1-x_{1,R})$, then by $x_{1,L}-l_1=x_{2,L}-l_2$, we have

$$\begin{aligned}l_1+x_{2,L} &= x_{1,L}+l_2 \prec x_{1,R}+y = l_1+y+(x_{1,R}-l_1)\\ &\preccurlyeq l_1+x_{2,L}+(u_1-x_{1,R})+(x_{1,R}-l_1)=l_1+x_{2,L}+(u_1-l_1) \prec l_1+x_{2,R}.\end{aligned}$$

Hence $y + x_{1,R} \in l_1 + A_2 \subseteq (2A)$. If $y \in (A_1 \cup \{x_{1,R}\}) \smallsetminus \{u_1\}$ and $y > u_1 - (x_{1,R} - x_{1,L})$, then

$$\begin{aligned} u_1 + x_{1,R} > y + x_{1,R} &= u_1 + x_{1,R} - (u_1 - y) \\ &> u_1 + x_{1,R} - (x_{1,R} - x_{1,L}) = u_1 + x_{1,L}. \end{aligned}$$

Hence $y + x_{1,R} \in u_1 + A_1 \subseteq (2A)$. If $y \in (A_1 \cup \{x_{1,R}\}) \smallsetminus \{u_1\}$ and $y \leqslant u_1 - (x_{1,R} - x_{1,L})$, then

$$\begin{aligned} x_{2,L} &\preccurlyeq 2l_1 - l_2 + x_{2,L} \\ &= l_1 + (l_1 + (x_{2,L} - l_2)) \prec l_1 + (u_1 - (u_2 - x_{2,R})) \preccurlyeq y + x_{1,R} \\ &\preccurlyeq u_1 - (x_{1,R} - x_{1,L}) + x_{1,R} = u_1 + x_{1,L} \preccurlyeq u_1 + l_1 + (x_{2,L} - l_2) \\ &\prec u_1 + (u_1 - (u_2 - x_{2,R}) = 2u_1 - u_2 + x_{2,R} \preccurlyeq x_{2,R}. \end{aligned}$$

Hence $y + x_{1,R} \in A_0 + A_2$. Now we have $(2A') \smallsetminus (2A) \subseteq \{x_{1,R}, x_{1,R} + u_1\}$, which contradicts the maximality of $|A|$ by Lemma 3.1.

Suppose $u_1 - x_{1,R} < u_2 - x_{2,R}$. Let $A' = A \cup \{x_{2,R}\}$ and $y \in A'$.

If $y \in (A_2 \cup \{x_{2,R}\}) \smallsetminus \{u_2\}$, then $y + x_{2,R} \in (2A_2)$ by Claim 5.6.3.5. If $y \in A_1$ and $y < u_1$, then $y + x_{2,R} = u_1 + x_{2,R} - (u_1 - y) \in u_1 + A_2 \subseteq (2A)$. If $y = u_1$, then

$$\begin{aligned} x_{1,L} + u_2 &= l_1 + (x_{1,L} - l_1) + u_2 = l_1 + (x_{2,L} - l_2) + u_2 \\ &\prec u_1 - (u_2 - x_{2,R}) + u_2 = y + x_{2,R} < u_1 + u_2 - (u_1 - x_{1,R}) = x_{1,R} + u_2. \end{aligned}$$

Hence $y + x_{2,R} \in A_1 + u_2 \subseteq (2A)$. We have showed that $(2A') \smallsetminus (2A) \subseteq \{x_{2,R}, x_{2,R} + u_2\}$, which contradicts the maximality of $|A|$ by Lemma 3.1. Now we can conclude that $u_1 - x_{1,R} = u_2 - x_{2,R}$.

If we assume that $x_{1,L} - l_1 < u_1 - x_{1,R}$ in the beginning, then we can show that $u_1 - x_{1,R} = u_2 - x_{2,R}$ first and $x_{1,L} - l_1 = x_{2,L} - l_2$ second by symmetric arguments as above. This completes the proof of the claim.

□ (Claim 5.6.3.7)

Claim 5.6.3.8 If $x_{1,L} - l_1 > 0$, then $l_1 + x_{1,L} = x_{2,L}$.

Proof of Claim 5.6.3.8 By the comments after Claim 5.6.3.1 we can assume that $2l_1 \geqslant l_2 - 3$. Hence

$$l_1 + x_{1,L} = 2l_1 + (x_{1,L} - l_1) \geqslant l_2 - 3 + (x_{2,L} - l_2) = x_{2,L} - 3.$$

Suppose $l_1 + x_{1,L} > x_{2,L}$. Let $A' = A \cup \{x_{1,L}\}$ and let $y \in A'$.

We can first show that $x_{1,L} + (A_2 \smallsetminus \{l_2\}) \subseteq (2A)$ and $x_{1,L} + ((A_1 \cup \{x_{1,L}\}) \smallsetminus \{l_1\}) \subseteq (2A)$ by the arguments similar to the proof of Claim 5.6.3.7. Then we have $x_{1,L} + l_1 = 0 + y$ for some $y \in I_2$ strictly between $x_{2,L}$ and $x_{2,R}$. Hence $(2A') \smallsetminus (2A) \subseteq \{x_{1,L}, x_{1,L} + l_2\}$, which contradicts the maximality of $|A|$ by Lemma 3.1.

Thus we can assume that $l_1 + x_{1,L} \leqslant x_{2,L}$, which implies that $l_1 + x_{1,L} = x_{2,L}$ or $l_1 + x_{1,L} = x_{2,L} - 3$. Suppose $l_1 + x_{1,L} = x_{2,L} - 3$. Let $A' = A \cup \{x_{2,L}\}$. Since $0 + x_{2,L} = l_1 + (x_{1,L} + 3) \in (2A_1)$, it is easy to see that $(2A') \smallsetminus (2A) \subseteq \{x_{2,L} + l_1, x_{2,L} + l_2\}$, which contradicts the maximality of $|A|$ by Lemma 3.1. Hence the only possible value for $l_1 + x_{1,L}$ is $x_{2,L}$. This completes the proof of the claim.

□ (Claim 5.6.3.8)

Claim 5.6.3.9 If $u_1 - x_{1,R} > 0$, then $x_{1,R} \leqslant x_{2,L} + l_2$.

Proof of Claim 5.6.3.9 Suppose $u_1 - x_{1,R} > 0$ and $x_{1,R} > x_{2,L} + l_2$. Let $A' = A \cup \{x_{1,R}\}$. Since $l_2 + x_{2,L} + 3 \leqslant 0 + x_{1,R} \prec x_{2,R} + u_2$, we have $0 + x_{1,R} \in (2A_2) \subseteq (2A)$. Hence by the arguments similar to the proof of Claim 5.6.3.7 we can shows that $(2A') \smallsetminus (2A) \subseteq \{x_{1,R} + u_1, x_{1,R} + u_2\}$, which contradicts the maximality of $|A|$ by Lemma 3.1.

□ (Claim 5.6.3.9)

Claim 5.6.3.10 $x_{1,R} > x_{2,L} + l_2$. Hence $x_{1,R} = u_1$ by Claim 5.6.3.9.

Proof of Claim 5.6.3.10 Suppose $x_{1,R} \leqslant x_{2,L} + l_2$. Notice that if $x_{2,L} > l_2$, then $A_2(l_2, x_{2,L}) \preccurlyeq \frac{1}{6}(x_{2,L} - l_2)$ because otherwise $A_2[l_2, x_{2,L}] \cap (l_2 + x_{2,L} - A_2[l_2, x_{2,L}]) \neq \emptyset$, which implies $l_2 + x_{2,L} \in (2A_2)$ and hence $(2A') \smallsetminus (2A) \subseteq \{x_{2,L}, x_{2,L} + l_1\}$. By the same reason we have $A_i(l_i, x_{i,L}) \preccurlyeq \frac{1}{6}(x_{i,L} - l_i)$ and $A_i(x_{i,R}, u_i) \preccurlyeq \frac{1}{6}(u_i - x_{i,L})$, which implies that

$$\begin{aligned} &A_i(l_i, x_{i,L}) + A_i(x_{i,R}, u_i) \\ &\quad \preccurlyeq \frac{1}{3}(x_{i,L} - l_i + u_i - x_{i,R}) - A_i(l_i, x_{i,L}) - A_i(x_{i,R}, u_i) \preccurlyeq s_i. \end{aligned}$$

Since

$$
\begin{aligned}
|2A| \succcurlyeq\ & |A_2 + A_2[x_{2,L}, u_2]| + |A_1 + A_2| + |2A_1| \\
& + A_1(l_1, x_{1,R}) + A_2(l_2, 2l_1) + A_2(2u_1, u_2) \\
\succcurlyeq\ & 2|A_2| + s_2 - A_2(l_2, x_{2,L}) + |A_1| + |A_2| + b_{1,2} + 2|A_1| + s_1 \\
& + |A_1| - A_1(x_{1,R}, u_1) + A_2(l_2, 2l_1) + A_2(2u_1, u_2),
\end{aligned}
$$

we have

$$
\begin{aligned}
\frac{1}{3}|A| & \succcurlyeq b \succcurlyeq b_{1,2} + |A_1| + A_2(l_2, 2l_1) + A_2(2u_1, u_2) \\
& \succcurlyeq \frac{2}{3}s_1 + \frac{2}{3}|A_1| + \frac{1}{3}|A_1| + \frac{1}{3}s_2 + \frac{1}{3}(A_2(l_2, 2l_1) + A_2(2u_1, u_2)) \\
& \quad + \frac{2}{3}(A_2(l_2, 2l_1) + A_2(2u_1, u_2)) \\
& \succcurlyeq \frac{2}{3}|I_1| + \frac{1}{3}|A_1| + \frac{1}{9}(u_2 - 2u_1 + 2l_1 - l_2) + \frac{2}{3}(A_2(l_2, 2l_1) + A_2(2u_1, u_2)),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& \frac{1}{3}|I_2| - \frac{1}{3}s_2 \\
& \quad \succcurlyeq \frac{2}{3}|I_1| + \frac{1}{9}(u_2 - l_2) - \frac{2}{9}(u_1 - l_1) + \frac{2}{3}(A_2(l_2, 2l_1) + A_2(2u_1, u_2)) \\
& \quad \sim \frac{2}{3}|I_1| + \frac{1}{3}|I_2| - \frac{2}{3}|I_1| + \frac{2}{3}(A_2(l_2, 2l_1) + A_2(2u_1, u_2)).
\end{aligned}
$$

Hence $-\frac{1}{3}s_2 \succcurlyeq \frac{2}{3}(A_2(l_2, 2l_1) + A_2(2u_1, u_2))$. This shows $s_2 \sim 0$, which implies $x_{2,L} \sim l_2$ and $x_{2,R} \sim u_2$. By Claim 5.6.3.7 we also have $x_{1,L} \sim l_1$ and $x_{1,R} \sim u_1$. Hence under the condition of Subcase 5.6.3.1 we have $u_1 \succ 2l_2 \sim l_2 + x_{2,L} \geqslant x_{1,R} \sim u_1$, which is absurd.

□ (Claim 5.6.3.10)

Combining Claim 5.6.3.9 and Claim 5.6.3.10 we can assume that $x_{i,R} = u_i$ for $i = 1, 2$.

Claim 5.6.3.11 $x_{i,L} = l_i$ for $i = 1, 2$.

Proof of Claim 5.6.3.11 Suppose $x_{2,L} > l_2$. Let $A' = A \cup \{x_{2,L}\}$. By Claim 5.6.3.10 we have that $x_{2,L} + l_2 \leqslant 0 + x_{1,R} - 3$. By Claim 5.6.3.8 we have $0 + x_{1,L} + 3 \leqslant x_{2,L} + l_2$. Hence $(2A') \setminus (2A) \subseteq \{x_{2,L} + l_1, x_{2,L}\}$, which contradicts the maximality of $|A|$ by Lemma 3.1.

□ (Claim 5.6.3.11)

Claim 5.6.3.12 Suppose $2u_1 < u_2$. If $A' = A \cup \{u_1 + 3\}$, then A' is an almost *b.p.* satisfying (4), (5), (6) without increasing b, but failing (7).

Proof of Claim 5.6.3.12 Since $2l_2 < 0 + u_1 + 3 \leqslant 2u_2$, we have $u_1 + 3 \in (2A)$. Since $0 + l_2 < 2u_1 + 3 \leqslant 0 + u_2$, we have $u_1 + (u_1 + 3) \in (2A)$. Hence we have $(2A') \setminus (2A) \subseteq \{2(u_1 + 3), (u_1 + 3) + u_2\}$, which implies the claim by Lemma 3.1.

□ (Claim 5.6.3.12)

We are now ready to prove the theorem under Subcase 5.6.3.1. Recall that we have $2l_1 \geqslant l_2 - 3 \succ 0$, $u_1 \geqslant 2l_2 \geqslant l_1 - 3 \succ 0$, $2u_1 \geqslant u_2$ by Claim 5.6.3.12, $A_0 = \{0\}$, $A_1 = I_1$, and $A_2 = I_2$.

By (39) and (40) without w we have

$$\begin{aligned}|2A| &\geqslant 2|A_2| - 1 + |A_1| + |A_2| - 1 + 2|A_1| - 1 + 1 \\ &\quad + A_1(l_1, 2l_2 - 3) + A_2(l_2, 2l_1 - 3) \\ &\geqslant 3|A| - 3 + \frac{1}{3}(l_2 + l_1) - 2,\end{aligned}$$

which implies $b \geqslant \frac{1}{3}(l_1 + l_2) - 2$. Since $u_2 - 2u_1 \leqslant 0$, we have

$$\begin{aligned}H + 1 &= u_2 - l_2 + 1 + l_2 \\ &= 2|A_2| - 1 + \frac{1}{3}(u_2 + 2l_2) \\ &= 2|A| - 1 - 2|A_1| - 2 + \frac{1}{3}(u_2 + 2l_2) \\ &= 2|A| - 1 - \frac{2}{3}(u_1 - l_1) - 4 + \frac{1}{3}(u_2 + 2l_2) \\ &= 2|A| - 1 + \frac{1}{3}(-2u_1 + 2l_1 + u_2 + 2l_2) - 4 \\ &\leqslant 2|A| - 1 + 2\left(\frac{1}{3}(l_1 + l_2) - 2\right) \leqslant 2|A| - 1 + 2b,\end{aligned}$$

which contradicts that A fails (7). This ends the proof of the theorem under Subcase 5.6.3.1.

□ (Subcase 5.6.3.1.)

Subcase 5.6.3.2 $2l_2 \sim u_1$.

Recall that we must have $2l_2 \leqslant u_0 + u_1$ because otherwise A would be a subset of a *t.p.* Under this subcase we may not assume $|A_0| = 1$. If we delete u_0 from A, the resulting set may become a subset of a *t.p.*

Claim 5.6.3.13 $|A_1| \sim \frac{1}{3}|A|$, $s_1 \sim s_2 \sim 0$, $2l_1 \sim l_2$, and $2u_1 \sim u_2$.

Proof of Claim 5.6.3.13 If $|A_1| \succ \frac{1}{3}|A|$, then

$$|2A| \succcurlyeq |2A_2| + |A_1 + A_2| + |2A_1| + A_1(l_1, 2l_2) \succcurlyeq 3|A| + |A_1| \succ \frac{10}{3}|A|,$$

which contradicts (6). Let $w = (2A_1)(u_2 + u_0 + 3, 2u_1)$ if $u_2 + u_0 < 2u_1$ and 0 otherwise. If $|A_1| \prec \frac{1}{3}|A|$, then $|A_2| \succ \frac{2}{3}|A|$ and

$$\begin{aligned}|2A| &\succcurlyeq |2A_2| + |A_1 + A_2| + |A_0 + A_2| + w + A_1(l_1, 2l_2)\\ &\succcurlyeq 3|A| + |A_2| - |A_1| + w \succ \frac{10}{3}|A|,\end{aligned}$$

which again contradicts (6). So we can assume that $|A_1| \sim \frac{1}{3}|A|$. Then

$$|2A| \succcurlyeq 3|A| + s_1 + s_2 + +b_{1,2} + |A_1| + w \sim \frac{10}{3}|A| + s_1 + s_2 + b_{1,2} + w$$

implies that $s_1 \sim s_2 \sim w \sim 0$. Hence $2u_1 \preccurlyeq u_2$ by the fact that $w \sim 0$ and $s_1 \sim 0$. Since $2u_1 \preccurlyeq u_2$, $2l_1 \succcurlyeq l_2$, and $|I_2| \sim |A_2| \sim 2|A_1| \sim 2|I_1|$, we have $2u_1 \sim u_2$ and $2l_1 \sim l_2$.

□ (Claim 5.6.3.13)

By Claim 5.6.3.13 we have that A_i is full in I_i for $i = 1, 2$, which also implies that $L_i \sim l_i$ and $R_i \sim u_i$ for $i = 1, 2$ where L_i and R_i are defined by (13) and (14). We now prove the theorem under Subcase 5.6.3.2 by deriving a contradiction to the failure of (7). Let $b_0 = |2A| - 3|A| + 3$. Again we assume that $|A|$ is maximal among all subsets A' tightly contained in $I_0 \cup I_1 \cup I_2$ satisfying (4), (5), (6) with $0 \leqslant |2A'| - 3|A'| + 3 \leqslant b_0$, but failing (7).

For each $x \in I_2$ with $l_2 \prec x \prec u_2$, we have $x + A_2 \subseteq (2A_2)$, $x + A_1 \subseteq (A_1 + A_2)$, and $x + A_0 \subseteq (2A_1)$. Hence $x \in A$ by the maximality of $|A|$.

Suppose $2u_1 \leqslant u_0 + u_2$. Then by Lemma 3.8

$$\begin{aligned}
|2A| &\geqslant |2A_2| + |A_1 + A_2| + |A_0 + A_2| && (47)\\
&\quad +|A_0 + A_1[l_1, 2l_2 - u_0 - 3]| + |2A_0| && (48)\\
&\geqslant 2|A_2| - 1 + s_2 + |A_1| + |I_2| - 1 + q + \frac{1}{3}u_0 + |A_2| && (49)\\
&\quad +A_1(x, x + u_0 - 3) + A_1(l_1, 2l_2 - u_0 - 3) + 2|A_0| - 1 && (50)\\
&\geqslant 3|A| - 3 - |A_1| - A_1(2l_2 - u_0, u_1) - |A_0| + s_2 && (51)\\
&\quad +|I_2| + q + \frac{1}{3}u_0 + A_1(x, x + u_0 - 3) && (52)\\
&\geqslant 3|A| - 3 - |A_1| - |A_0| + s_2 - \frac{1}{3}(u_1 - 2l_2) + \frac{1}{3}(u_2 - l_2) + \frac{1}{3}u_0 && (53)\\
&\geqslant 3|A| - 3 - |A_1| - |A_0| + s_2 + \frac{1}{3}(u_2 - u_1 + l_2 + u_0) && (54)
\end{aligned}$$

where $x = 2l_2 - u_0$ and q is defined in Lemma 3.8. In fact, the inequalities above are true for any $x \in I_1$ with $l_1 \leqslant x \leqslant u_1 - u_0 + 3$ and q is not needed. Hence $b \geqslant -|A_1| - |A_0| + s_2 + \frac{1}{3}(u_2 - u_1 + l_2 + u_0)$. Now we have

$$\begin{aligned}
H + 1 &= u_2 - l_2 + l_2 + 1 && (55)\\
&= 2|A_2| - 1 + 2s_2 + \frac{1}{3}(u_2 - l_2) + l_2 && (56)\\
&\leqslant 2|A| - 1 - 2|A_1| - 2|A_0| + 2s_2 + \frac{1}{3}(u_2 + 2l_2) && (57)\\
&\leqslant 2|A| - 1 + 2\left(b - \frac{1}{3}(u_2 - u_1 + l_2 + u_0)\right) + \frac{1}{3}(u_2 + 2l_2) && (58)\\
&\leqslant 2|A| - 1 + 2b + \frac{1}{3}(-2u_2 + 2u_1 - 2l_2 - 2u_0 + u_2 + 2l_2) && (59)\\
&= 2|A| - 1 + 2b + \frac{1}{3}(-u_2 + 2u_1 - 2u_0) \leqslant 2|A| - 1 + 2b, && (60)
\end{aligned}$$

which contradicts the assumption that (7) is false.

Suppose $2u_1 > u_0 + u_2$. We first derive a new inequality for $|A_1 + A_2|$. Since $2u_1 \sim u_2$ we can choose $x \in A_1$ with $l_1 \prec x \prec u_1$ such that $x + u_1 \prec u_2$. We also choose $y \in A_2$ such that $x - l_1 \leqslant y - l_2$ and $u_2 - y \succ u_1 - x$. Let $s'_2 = |\{z \in I_2 \setminus A_2 : z < y\}|$, $s'_1 = |\{z \in I_1 \setminus A_1 : z > x\}|$, and q be the number defined in Lemma 3.8 with A_1 and A_2 being replaced by $A_1[l_1, x]$

and $A_2[l_2, y]$. Then we have

$$\begin{aligned}
|A_1 + A_2| &\geqslant |A_1[l_1, x] + A_2[l_2, y]| + |x + A_2[y + 3, u_2 - (u_1 - x) - 3]| \\
&\quad + |A_1[x, u_1] + A_2[u_2 - (u_1 - x), u_2]| \\
&\geqslant A_1(l_1, x) + A_2(l_2, y) - 1 + s_2' + q + A_2(y + 3, u_2 - (u_1 - x) - 3) \\
&\quad + A_1(x, u_1) + A_2(u_2 - (u_1 - x), u_2) - 1 + s_1' \\
&= |A_1| + |A_2| - 1 + s_2' + q + s_1'.
\end{aligned}$$

Hence by (47) and (48) with $|A_0 + A_2|$ replaced by $|(A_0 + A_2) \cup (2A_1)|$, which is greater than or equal to

$$|0 + A_2[l_2, u_1 + x]| + |u_1 + A_1[x + 3, u_1]| = A_2(l_2, u_1 + x) + A_1(x + 3, u_1),$$

we have that

$$\begin{aligned}
|2A| &\geqslant 2|A_2| - 1 + s_2 + |A_1| + |A_2| - 1 + s_2' + q + s_1' + A_2(l_2, u_1 + x) \\
&\quad + A_1(x + 3, u_1) + A_1(x', x' + u_0 - 3) + A_1(l_1, 2l_2 - u_0 - 3) + 2|A_0| - 1 \\
&\geqslant 3|A| - 3 - |A_1| - |A_0| + s_2 - A_1(2l_2 - u_0, u_1) \\
&\quad + \frac{1}{3}(u_1 + x - l_2) + 1 + \frac{1}{3}(u_1 - x) + \frac{1}{3}u_0 \\
&\geqslant 3|A| - 3 - |A_1| - |A_0| + s_2 - \frac{1}{3}(u_1 - 2l_2) + \frac{1}{3}(u_1 + x - l_2) + \frac{1}{3}(u_1 - x) \\
&\geqslant 3|A| - 3 - |A_1| - |A_0| + s_2 + \frac{1}{3}(u_1 + l_2)
\end{aligned}$$

where $x' \in A_1$ such that $l_1 \prec x' \prec x$. By the same argument as in (55), (56), and (57) we have that

$$\begin{aligned}
H + 1 &\leqslant 2|A| - 1 + 2\left(b - \frac{1}{3}(u_1 + l_2)\right) + \frac{1}{3}(u_2 + 2l_2) \\
&\leqslant 2|A| - 1 + 2b + \frac{1}{3}(-2u_1 - 2l_2 + u_2 + 2l_2) \leqslant 2|A| - 1 + 2b,
\end{aligned}$$

which contradicts the assumption that (7) is false. We now complete the proof of Subcase 5.6.3.2.

□ (Subcase 5.6.3.2)

Subcase 5.6.3.3 $2l_2 \prec u_1$ and $2u_1 \succ u_2$.

Subclaim 5.6.3.14 Either $A_1(2l_2, u_1) \succ s_1$ or $A_2(2l_1, u_2) \succ s_2$.

Proof of Subclaim 5.6.3.14 Suppose the subclaim is not true. Then $|A_1| \sim A_1(l_1, 2l_2)+A_1(2l_2, u_1) \preccurlyeq A_1(l_1, 2l_2)+s_1$. Similarly $|A_2| \preccurlyeq A_2(l_2, 2l_1)+s_2$. Hence $|2A| \succcurlyeq 3|A| + s_1 + s_2 + b_{1,2} + A_1(l_1, 2l_2) + A_2(l_2, 2l_1) \succcurlyeq 4|A|$, which contradicts (6).

□ (Subclaim 5.6.3.14)

Suppose $A_1(2l_2, u_1) \succ s_1$. Then we have $|A_1 + A_1[2l_2, u_1]| \succcurlyeq |A_1| + A_1(2l_2, u_1) + s_1$ by Theorem A.4. Since

$$\begin{aligned}
|2A| &\succcurlyeq |2A_2| + |A_1 + A_2| + |A_1 + A_1[2l_2, u_1]| \\
&\quad + A_1(l_1, 2l_2) + A_2(l_2, l_1 + 2l_2) \\
&\succcurlyeq 2|A_2| + s_2 + |A_2| + |A_1| + b_{1,2} + |A_1| + A_1(2l_2, u_1) + s_1 \\
&\quad + A_1(l_1, 2l_2) + A_2(l_2, l_1 + 2l_2) \\
&\succcurlyeq 3|A| + s_1 + s_2 + \frac{1}{3}(l_1 + l_2),
\end{aligned}$$

then $b \succcurlyeq s_1 + s_2 + \frac{1}{3}(l_1 + l_2)$. Since $\frac{1}{3}(u_1 - l_1) + \frac{1}{3}(u_2 - l_2) \sim |A_1| + |A_2| + s_1 + s_2$, we have that

$$\begin{aligned}
H + 1 &\preccurlyeq 2|A| + |I_1| + |I_2| + 2(s_1 + s_2) - u_1 + l_1 + l_2 \\
&\preccurlyeq 2|A| + \frac{1}{3}(u_1 - l_1) + \frac{1}{3}(u_2 - l_2) + 2(b - \frac{1}{3}(l_1 + l_2)) - u_1 + l_1 + l_2 \\
&\preccurlyeq 2|A| + 2b + \frac{1}{3}(u_1 - l_1 + u_2 - l_2 - 2l_1 - 2l_2 - 3u_1 + 3l_1 + 3l_2) \\
&= 2|A| + 2b + \frac{1}{3}(u_2 - 2u_1) \prec 2|A| + 2b
\end{aligned}$$

because of the assumption of $2u_1 \succ u_2$ in this case.

By Claim 5.6.3.14 we can assume $A_2(2l_1, u_2) \succ s_2$. Now we can derive that $H + 1 \prec 2|A| - 1 + 2b$ by switching A_1 and A_2 in the proof of the case above. This completes the proof of Subcase 5.6.3.3, Case 5.6.3, as well as Theorem 5.6.

□

6 Proof of Theorem 1.7 when $d = 2$

In this section we prove Theorem 1.7 when $d = 2$.

Theorem 6.1 *Suppose $A \subseteq [0, H]$ is an almost tight subset of a* b.p. $I_1 \cup I_2$ *of difference $d = 2$ satisfying (4), (5), and (6). Then A satisfies (7).*

Proof Suppose Theorem 6.1 is not true. We can assume that $A_0 = A \setminus (I_1 \cup I_2)$ is a subset of an *a.p.* $|I_0|$ of difference 2 with $l_0 = \min I_0 = \min A_0$ and $u_0 = \max I_0 = \max A_0$. Without loss of generality we can assume the following:

1. I_0 is disjoint from $I_1 \cup I_2$,
2. $l_0 \equiv l_1 \equiv 0 \pmod 2$,
3. $l_2 \equiv 1 \pmod 2$,
4. $u_1 < l_2$ because $I_1 \cup I_2$ is a *b.p.*,
5. $u_1 < l_0$ because otherwise we have $u_0 < l_1$, which implies that A is subset of a *b.p.* $J \cup I_2$ of difference 2,
6. $l_2 < u_0$ because again A is not a subset of a *b.p.*, and
7. $\min\{l_2, l_0\} \leqslant 2u_1$ because otherwise A is a subset of the *b.p.* $[0, u_1] \cup [\min\{l_2, l_0\}, u_2]$ of difference 1.

We will prove the theorem under two cases: $u_0 = H$ and $u_2 = H$ (note that $l_1 = 0$).

Case 6.1.1 $u_0 = H$.

Under this case we have that $u_0 - u_1 > u_2 - l_2$. Let J be an *a.p.* of difference 2, which tightly contains $I_1 \cup I_0$. Hence $\min J = 0$ and $\max J = u_0$. Let $A_J = A_1 \cup A_0$. Let s_J and b_J be defined by (10) and (11) with I_i and i being replaced by J, respectively. We will use these notations from now on.

Claim 6.1.1 $|A_2| \preccurlyeq \frac{1}{3}|A|$, $\gcd(A_1) = 2$, and $s_J \prec |A_1|$.

Proof of Claim 6.1.1 If $|A_2| \succ \frac{1}{3}|A|$, then

$$|2A| \succcurlyeq |2A_1| + |A_1 + A_2| + |2A_2| + |u_0 + A_2| \succcurlyeq 3|A| + |A_2| \succ \frac{10}{3}|A|,$$

which contradicts the assumption that A satisfies (6). Hence we have $|A_2| \preccurlyeq \frac{1}{3}|A|$. Notice that this implies that $|A_1| \succcurlyeq \frac{2}{3}|A|$.

Suppose $\gcd(A_1 - l_1) = g_1 > 2$. Notice that $\gcd((A_1 - l_1) \cup (A_2 - l_2)) = 2$ by the assumption that A is an almost tight subset of a *b.p.* of difference 2. Hence $|A_1 + A_2| \succcurlyeq 2|A_1| + |A_2|$, which implies $|2A| \succcurlyeq 3|A| + |A_1| \succ \frac{10}{3}$, a contradiction to (6).

Suppose $s_J \succcurlyeq |A_1|$. Then $|2A_J| \succcurlyeq 3|A_1|$ by Theorem A.4. Hence

$$\begin{aligned} |2A| &\succcurlyeq |(2A_J) \cup (2A_2)| + |A_J + A_2| \\ &\succcurlyeq |2A_J| + |A_1| + 2|A_2| + b_{1,2} \succcurlyeq 3|A| + |A_1| - |A_2| + b_{1,2} \succcurlyeq \frac{10}{3}|A| + b_{1,2}. \end{aligned}$$

By (6) we have $b_{1,2} \sim 0$, A_1 is full in I_1, A_2 is full in I_2, $|I_1| \sim |A_1| \sim \frac{2}{3}|A|$, and $|I_2| \sim |A_2| \sim \frac{1}{3}|A|$. We also have that $|(2A_J) \cup (2A_2)| \sim |2A_J| \sim 3|A_1|$, which implies that $l_0 \leqslant 2u_1 \preccurlyeq l_0$, $u_0 + u_1 \sim 2u_2$, and $u_1 \sim l_2$. Let $c = \max\{2u_1 - u_0 + 2, 0\}$. By Lemma 3.8 with A_1 and A_2 being switched, we have $|A_1 + A_2| \geqslant |A_1| + |A_2| - 1 + s_1 + q$. Then for some $x \in I_2$ with $l_2 \prec x \prec u_2$ we have

$$\begin{aligned} |2A| &\geqslant |2A_1| + |u_0 + A_1[c, u_1]| + |2A_0| + |A_1 + A_2| + |A_0 + A_2| \\ &\geqslant 2|A_1| - 1 + s_1 + A_1(c, u_1) + 2|A_0| - 1 + |A_1| + |A_2| - 1 \\ &\quad + s_1 + q + |A_2| + A_2(x, x + (u_0 - l_0)) - 1 \\ &\geqslant 3|A| - 3 + \frac{1}{2}(u_1 - c) + 1 - |A_0| - |A_2| + s_1 + \frac{1}{2}(u_0 - l_0). \end{aligned}$$

Hence

$$\begin{aligned} H + 1 &= u_0 - u_1 + 2\left(\frac{1}{2}u_1 + 1\right) - 1 \\ &\leqslant 2|A_1| - 1 + 2s_1 + u_0 - u_1 \\ &= 2|A| - 1 - 2|A_2| - 2|A_0| + 2s_1 + u_0 - u_1 \\ &\leqslant 2|A| - 1 + 2\left(b - \frac{1}{2}(u_1 - c) - 1 - \frac{1}{2}(u_0 - l_0)\right) + u_0 - u_1 \\ &= 2|A| - 1 + 2b - u_1 + c - 2 - u_0 + l_0 + u_0 - u_1 \\ &= 2|A| - 1 + 2b - 2u_1 + c - 2 + l_0. \end{aligned}$$

If $2u_1 \geqslant u_0$, then $c = 2u_1 - u_0 + 2$. Hence $H + 1 \leqslant 2|A| - 1 + 2b - 2u_1 + 2u_1 - u_0 + l_0 \leqslant 2|A| - 1 + 2b$. Otherwise, we have $c = 0$. Hence $H + 1 \leqslant 2|A| - 1 + 2b - 2u_1 - 2 + l_0 < 2|A| - 1 + 2b$. But both contradicts the assumption that (7) is false.

□ (Claim 6.1.1)

We are now proving the theorem under Case 6.1.1. Notice that by Claim 6.1.1 we have that $\frac{1}{2}H + 1 = |A_J| + s_J \leqslant |A_J| + b_J$.

Suppose $|A_2| \leqslant s_J$. By Theorem A.4 we have

$$\begin{aligned}|2A| &\geqslant |2A_J| + |A_J + A_2| \\ &\geqslant \frac{1}{2}H + |A_J| + |A_J| + 2|A_2| - 2 = 3|A| - 3 - |A| + \frac{1}{2}H + 1.\end{aligned}$$

Hence $b \geqslant \frac{1}{2}H + 1 - |A|$, which implies $H + 1 \leqslant 2|A| - 1 + 2b$. We have a contradiction to our assumption that (7) is false.

We can now assume that $|A_2| > s_J$. However, $\frac{1}{2}H + 1 = |A_J| + s_J < |A_J| + |A_2| = |A|$ implies $H + 1 < 2|A| - 1$. Hence (7) is true because $b \geqslant 0$ by (6), which contradicts our assumption that (7) is false. This completes the proof of Case 6.1.1.

□ (Case 6.1.1)

Case 6.1.2 $u_2 = H$.

Let J be a tight *a.p.* of difference 2 containing $I_0 \cup I_1$. Let $A_J = A_1 \cup A_0$. Notice that $0 = \min J$ and $u_0 = \max J$. Let $g_i = \gcd(A_i - l_i))$ for $i = 1, 2$ and $g_J = \gcd((A_1 - l_1) \cup (A_0 - l_0))$.

Claim 6.1.2 Either $g_1 = 2$ or $g_2 = 2$. If $g_i > 2$ for $i = 1$ or 2, then $|A_i| \preccurlyeq \frac{1}{3}|A|$.

Proof of Claim 6.1.2 Notice that $\gcd(g_1, g_2) = 2$ by the definition of almost subset of a *b.p.* of difference 2. If $g_1 > 2$ and $g_2 > 2$, then

$$\begin{aligned}|2A| &\succcurlyeq |2A_1| + |A_1 + A_2| + |2A_2| \\ &\succcurlyeq 2|A_1| + |A_1| + |A_2| + \max\{|A_1|, |A_2|\} + 2|A_2| \\ &\succcurlyeq 3|A| + \frac{1}{2}|A| \succ \frac{10}{3}|A|,\end{aligned}$$

which contradicts (6). Hence either $g_1 = 2$ or $g_2 = 2$.

If $g_1 > 2$, then $g_2 = 2$. Hence

$$|2A| \succcurlyeq |2A_1| + |A_1 + A_2| + |2A_2| \succcurlyeq 3|A| + |A_1|,$$

which implies $|A_1| \preccurlyeq \frac{1}{3}|A|$ by (6). By the same argument we can show that if $g_2 > 2$, then $|A_2| \preccurlyeq \frac{1}{3}|A|$.

□ (Claim 6.1.2)

Claim 6.1.3 If $g_2 = 2$, then $g_1 = 2$.

Proof of Claim 6.1.3 Assume the contrary that $g_1 > 2$ and $g_2 = 2$.

If $|2A_2| \succcurlyeq 3|A_2|$, then $|2A| \succcurlyeq 2|A_1| + 2|A_1| + |A_2| + 3|A_2| \sim 4|A| \succ \frac{10}{3}|A|$, a contradiction to (6). Hence we can assume $|2A_2| \sim 2|A_2| + b_2 \prec 3|A_2|$ with $s_2 \leqslant b_2 \prec |A_2|$ by Theorem A.1. This implies $|A_2| \succ \frac{1}{2}|I_2|$. In fact, since

$$|2A| \succcurlyeq 2|A_1| + 2|A_2| + b_2 + 2|A_1| + |A_2| \sim 3|A| + b_2 + |A_1|,$$

then

$$s_2 \leqslant b_2 + |A_1| - |A_1| \preccurlyeq b - |A_1| \preccurlyeq \frac{1}{3}|A| - \frac{1}{3}|A_1| \preccurlyeq \frac{1}{3}|I_2| - \frac{1}{3}s_2.$$

Hence $s_2 \preccurlyeq \frac{1}{4}|I_2|$, which implies $|A_2| \succcurlyeq \frac{3}{4}|I_2|$.

Subclaim 6.1.3.1 $|J| \prec |I_2|$.

Proof of Subclaim 6.1.3.1 Suppose $|J| \succcurlyeq |I_2|$. Then $|A_J + A_2| \succcurlyeq 2|A_2|$. Hence

$$\begin{aligned}|2A| &\succcurlyeq 2|A_1| + b_1 + 2|A_2| + b_2 + |u_0 + A_1[2u_1 - u_0, 2l_2 - u_0]| + 2|A_2| \\ &\succcurlyeq 3|A| + |A_2| - |A_1| + b_1 + b_2 + A_1(2u_1 - u_0, 2l_2 - u_0) \\ &\succcurlyeq \frac{10}{3}|A| + b_1 + b_2 + A_1(2u_1 - u_0, 2l_2 - u_0).\end{aligned}$$

By (6) we have $b_1 \sim 0$, $b_2 \sim 0$, $|A_1| \sim \frac{1}{3}|A|$, $|A_2| \sim \frac{2}{3}|A|$, $|A_J + A_2| \sim 2|A_2|$, and $A_1(2u_1 - u_0, 2l_2 - u_0) \sim 0$. Thus A_1 is full in an *a.p.* of difference g_1, A_2 is full in I_2, $l_2 \sim u_1$, and $|J| \sim |I_2|$. Clearly, $g_1 = 4$ because $g_1 > 4$ implies $|A_1 + A_2| \succcurlyeq 3|A_1| + |A_2|$. It is easy to see that $u_0 \sim u_1$ by the fact that $|J| \sim |I_2| \sim |A_2| \sim 2|A_1| \sim |I_1|$. It is also easy to see that $g_J = 4$ because otherwise we have $|A_J + A_2| \succcurlyeq \min\{|J| + |A_2|, |A_1| + 2|A_2|\} \succ 2|A_2|$ by Theorem A.4. Let J' be the *a.p.* of difference 4 tightly containing A_J.

We can assume that $|A|$ is maximal for $A \subseteq I_2 \cup J'$ satisfying (4), (5), and (6) but failing (7). Notice that the value of b will not increase when we add elements by Lemma 3.1. Hence the condition $b \preccurlyeq \frac{1}{3}|A|$ in (6) will not be violated when apply Lemma 3.1. It is now easy to see that if $x \in I_2$ and $l_2 \prec x \prec u_2$, then $x \in A$, and if $x \in J'$ and $0 \prec x \prec u_0$, then $x \in A$ by Lemma 3.1. Let $x_{2,L}$ and $x_{2,R}$ be defined in (29) and (30). If $x_{2,L} > l_2$, let $A' = A \cup \{x_{2,L}\}$. Then $(2A') \setminus (2A) \subseteq \{x_{2,L} + l_2, x_{2,L} + l_1\}$, which contradicts the maximality of $|A|$ by lemma 3.1. Similarly, we have that $x_{2,R} = u_2$. By the same argument we can show that $A_J = J'$. Now we have

$$\begin{aligned} |2A| &\geqslant 2|A_J| - 1 + 2|A_2| - 1 - \frac{1}{4}(2u_0 - 2l_2) - 1 \\ &\quad + |A_2| + 2|A_J| - 2 \\ &= 3|A| - 3 + |A_J| - 2 - \frac{1}{2}u_0 + \frac{1}{2}l_2. \end{aligned}$$

This implies $b \geqslant |A_J| - 2 - \frac{1}{2}u_0 + \frac{1}{2}l_2$. Therefore,

$$\begin{aligned} H + 1 &= u_2 - l_2 + 2 - 1 + l_2 \\ &\leqslant 2|A| - 2|A_J| - 1 + 2\left(b - |A_J| + 2 + \frac{1}{2}u_0\right) \\ &= 2|A| - 1 + 2b - 4|A_J| + 4 + u_0 = 2|A| - 1 + 2b \end{aligned}$$

because $|A_J| = |J'| = \frac{1}{4}u_0 + 1$. Hence we have a contradiction to the failure of (7).

□ (Subclaim 6.1.3.1)

Subclaim 6.1.3.2 $2|J| \succcurlyeq |I_2|$.

Proof of Subclaim 6.1.3.2 Suppose $2|J| \prec |I_2|$. Since $s_2 \preccurlyeq \frac{1}{4}|I_2|$, we have $L_2 - l_2 \preccurlyeq 4s_2 \preccurlyeq |I_2| \sim \frac{1}{2}(u_2 - l_2)$ and by the same reason we have $u_2 - R_2 \preccurlyeq \frac{1}{2}(u_2 - l_2)$ where L_2 and R_2 are defined by (13) and (14). Hence $L_2 - l_2 + u_2 - R_2 \preccurlyeq 4s_2 \preccurlyeq \frac{1}{2}(u_2 - l_2)$, which implies $R_2 - L_2 \succcurlyeq \frac{1}{2}(u_2 - l_2)$. By

Lemma 3.7 we have $|I_2| \leqslant |A_2| + b_2 - p_2$ where p_2 is defined in (17). Since

$$\begin{aligned}
|2A| &\geqslant |2A_J| + |2A_2| - \frac{1}{g_1}(2u_0 - 2l_2) - 1 + |A_2 + A_J| \\
&\geqslant 2|A_J| - 1 + 2|A_2| - 1 + b_2 - \frac{1}{g_1}(2u_0 - 2l_2) - 1 \\
&\quad + |A_2| + A_2(L_2 + 2, L_2 + u_0) \\
&= 3|A| - 3 + b_2 - |A_J| - \frac{1}{g_1}(2u_0 - 2l_2) + A_2(L_2 + 2, L_2 + u_0),
\end{aligned}$$

we have $b \geqslant b_2 - |A_J| - \frac{1}{g_1}(2u_0 - 2l_2) + A_2(L_2 + 2, L_2 + u_0)$. Since $u_0 \sim 2|J| \prec |I_2| \sim \frac{1}{2}(u_2 - l_2) \preccurlyeq R_2 - L_2$, we have

$$\begin{aligned}
\frac{1}{2}(H - l_2) + 1 = |I_2| &\leqslant |A_2| + b_2 - p_2 = |A| - p_2 + b_2 - |A_J| \\
&\leqslant |A| - p_2 + b + \frac{1}{g_1}(2u_0 - 2l_2) - A_2(L_2 + 2, L_2 + u_0) \\
&\leqslant |A| + b + \frac{2}{g_1}(u_0 - l_2) - \frac{1}{2}u_0.
\end{aligned}$$

Hence $H - l_2 + 2 \leqslant 2|A| + 2b - u_0 + \frac{4}{g_1}(u_0 - l_2)$, which implies

$$H + 1 \leqslant 2|A| - 1 + 2b + l_2 - u_0 + u_0 - l_2 = 2|A| - 1 + 2b$$

because $g_1 \geqslant 4$. This contradicts our assumption that A fails (7).

$\square$ (Subclaim 6.1.3.2)

By Subclaim 6.1.3.1 and Subclaim 6.1.3.2 we can assume that $\frac{1}{2}|I_2| \preccurlyeq |J| \prec |I_2|$. We now want to derive a contradiction under this assumption.

Suppose $|2A_1| \succcurlyeq 3|A_1|$. Then

$$|2A| \succcurlyeq 2|A_2| + s_2 + 3|A_1| + |A_2| + A_2(l_2, l_2 + u_0) \succcurlyeq 3|A| + \frac{1}{2}u_0.$$

This implies that

$$|J| \sim \frac{1}{2}u_0 \preccurlyeq b \preccurlyeq \frac{1}{3}|A| = \frac{1}{3}|A_J| + \frac{1}{3}|A_2| \preccurlyeq \frac{1}{6}|J| + \frac{1}{3}|I_2|.$$

Hence $5|J| \preccurlyeq 2|I_2|$ or $|J| \preccurlyeq \frac{2}{5}|I_2| \preccurlyeq \frac{4}{5}|J|$, which is absurd. Thus we can now assume that $|2A_1| \prec 3|A_1|$, which implies that $\frac{1}{g_1}u_1 + 1 \leqslant |A_1| + b_1$ by Theorem

A.1. Suppose $|A_1| \sim 0$. Then $|J| \sim \frac{1}{2}u_0 \preccurlyeq b \preccurlyeq \frac{1}{3}|A| \sim \frac{1}{3}|A_2| \preccurlyeq \frac{1}{3}|I_2|$, which contradicts the assumption that $|J| \succcurlyeq \frac{1}{2}|I_2|$. Thus we can now assume that $|A_1| \succ 0$.

Clearly, g_J is an even number and $g_J|g_1$. For $i \in [0, \frac{1}{2}g_J - 1]$ and $x \in I_2$ with $x + u_0 - 2 \leqslant u_2$ let

$$A_x^i = A_2 \cap (l_2 + 2i + g_J\mathbb{N}) \cap [x, x + u_0 - 2].$$

Let $k_x = |\{i \in [0, \frac{1}{2}g_J - 1] : A_x^i \neq \emptyset\}|$. Now we fix an $x \in I_2[l_2, u_2 - u_0 + 2]$ such that $k = k_x$ is maximal. Notice that $\frac{1}{2}g_J \geqslant k > \frac{1}{4}g_J$ because $|A_2| \succcurlyeq \frac{3}{4}|I_2|$.

Suppose there is $i_0 \in [0, \frac{1}{2}g_J - 1]$ such that $|A_J + A_x^{i_0}| \geqslant |A_J| + 2|A_x^{i_0}| - 2$. Since

$$\begin{aligned} |A_J + A_2[x, x + u_0 - 2]| &= \sum_{i \in \left[0, \frac{1}{2}g_J - 1\right]} |A_J + A_x^i| \\ &\geqslant |A_J| - 2 + 2 \sum_{i \in \left[0, \frac{1}{2}g_J - 1\right]} |A_x^i| \geqslant |A_J| - 2 + 2A_2(x, x + u_0 - 2), \end{aligned}$$

then $|A_J + A_2| \succcurlyeq |A_2| + |A_J| + A_2(x, x + u_0 - 2)$ and

$$\begin{aligned} |2A| &\succcurlyeq 2|A_2| + b_2 + 2|A_1| + |A_J| + |A_2| + A_2(x, x + u_0 - 2) \\ &\succcurlyeq 3|A| + b_2 + A_2(x, x + u_0 - 2) \succcurlyeq 3|A| + \frac{1}{2}u_0. \end{aligned}$$

Hence $\frac{1}{2}u_0 \preccurlyeq b \preccurlyeq \frac{1}{3}|A|$, which implies that $|J| \sim \frac{1}{2}u_0 \preccurlyeq \frac{1}{3}|A| \preccurlyeq \frac{1}{6}|J| + \frac{1}{3}|I_2|$. This implies $\frac{5}{6}|J| \preccurlyeq \frac{1}{3}|I_2|$ or $|J| \preccurlyeq \frac{2}{5}|I_2|$, which contradicts the assumption $|J| \succcurlyeq \frac{1}{2}|I_2|$.

Now we can assume that $|A_J + A_x^i| \geqslant \frac{1}{g_J}u_0 + |A_x^i|$ for every $i \in [0, \frac{1}{2}g_J - 1]$ with $A_x^i \neq \emptyset$ by Theorem A.4. Then we have

$$|2A| \succcurlyeq 2|A_2| + b_2 + 2|A_1| + b_1 + |A_2| + \frac{k}{g_J}u_0 \tag{61}$$

$$\succcurlyeq 3|A| + b_2 + b_1 - |A_1| + \frac{k}{g_J}u_0, \tag{62}$$

which implies $b \succcurlyeq b_2 + b_1 - |A_1| + \frac{k}{g_J}u_0$. If $\frac{k}{g_J} = \frac{1}{2}$, then

$$\begin{aligned} \frac{1}{2}(H - l_2) + 1 &\preccurlyeq |A_2| + b_2 - p_2 = |A| + b_2 - p_2 - |A_1| \\ &\preccurlyeq |A| + b - \frac{k}{g_J}u_0 - b_1 - p_2 \preccurlyeq |A| + b - b_1 - p_2 - \frac{1}{2}u_0. \end{aligned}$$

Hence $H+1 \preccurlyeq 2|A|-1+2b-2b_1-2p_2-u_0+l_2$. If $u_0 \succ l_2$, $b_1 \succ 0$, or $p_2 \succ 0$, then $H+1 \prec 2|A|-1+2b$, which contradicts the failure of (7). Hence we can assume that $u_0 \sim l_2$, $p_2 \sim 0$, and $b_1 \sim 0$. Furthermore, when $u_0 \sim l_2$, we can replace $2|A_1|+b_1$ by $2|A_J|+b_J$ in (61) and conclude that $b_J \sim 0$, which implies that A_J is full in $J \cap (\{g_J k : k \in {}^*\mathbb{N}\})$ and $g_J = g_1 \geqslant 4$. By the fact that $|J| \prec |I_2|$, $|A_2| \succcurlyeq \frac{3}{4}|I_2|$, and $A_2[L_2, R_2]$ is full in $I_2[L_2, R_2]$ because $p_2 \sim 0$, we can assume that $A_2 = I_2$ and $A_J = (J \cap \{g_J k : k \in {}^*\mathbb{N}\})$ by the following argument: Suppose that $|A|$ is maximal so that $A \subseteq (J \cap \{g_J k : k \in {}^*\mathbb{N}\}) \cup I_2$ and A is a counterexample of Theorem 6.1. Then we can show that $x \in A_2$ for every $x \in I_2$ with $L_2 \prec x \prec R_2$ by Lemma 3.1. Then we can show that $x_{2,L}, x_{2,R} \in A_2$ where $x_{2,L}$ and $x_{2,R}$ are defined in (29) and (30), and then we can show that $A_J = (J \cap \{g_J k : k \in {}^*\mathbb{N}\})$ again by Lemma 3.1. Hence

$$\begin{aligned}|2A| = {} & 2|A_J| - 1 + 2|A_2| - 1 - \frac{1}{g_J}(2u_0 - 2l_2) - 1 \\ & + |A_2| + \frac{1}{2}u_0 \\ = {} & 3|A| - 3 - |A_J| + \frac{1}{2}u_0 - \frac{2}{g_J}(u_0 - l_2).\end{aligned}$$

This implies that $b \geqslant -|A_J| + \frac{1}{2}u_0 - \frac{2}{g_J}(u_0 - l_2)$. Hence

$$\begin{aligned}H+1 &= H - l_2 + 2 - 1 + l_2 \\ &= 2|A_2| - 1 + l_2 = 2|A| - 1 + l_2 - 2|A_J| \\ &\leqslant 2|A| - 1 + l_2 + 2\left(b - \frac{1}{2}u_0 + \frac{2}{g_J}(u_0 - l_2)\right) \\ &= 2|A| - 1 + 2b + l_2 - u_0 + \frac{4}{g_J}(u_0 - l_2) \\ &\leqslant 2|A| - 1 + 2b\end{aligned}$$

because $4 \leqslant g_J$ and $u_0 > l_2$. This again contradicts our assumption that A fails (7). Hence we can now assume $\frac{k}{g_J} < \frac{1}{2}$. If $\frac{k}{g_J} < \frac{1}{2}$, then $g_J > 4$. Since $|A_2| \succcurlyeq \frac{3}{4}|I_2|$, we have $g_J \geqslant 8$. This is true because if $g_J = 6$, then $|A_2| \preccurlyeq \frac{2}{3}|I_2|$. Hence we have that $|A_J| \preccurlyeq \frac{2}{g_J}|J| \leqslant \frac{1}{4}|J|$. Let $w = |(I_2 \setminus A_2) \cap [l_2 + u_0 + 2, u_2]|$. Since $|J| \prec |I_2|$ and $k < \frac{g_J}{2}$, we have $w \succ 0$. We now have

$$|2A| \succcurlyeq 2|A_2| + b_2 + 2|A_1| + |A_2| + A_2(l_2, l_2 + u_0) \succcurlyeq 3|A| - |A_J| + |J| + w,$$

which implies that $-|A_J| + |J| + w \preccurlyeq b \preccurlyeq \frac{1}{3}|A_2| + \frac{1}{3}|A_J|$. Hence

$$|J| \prec |J| + w \preccurlyeq \frac{1}{3}|I_2| + \frac{4}{3}|A_J| \preccurlyeq \frac{2}{3}|J| + \frac{1}{3}|J| = |J|,$$

which is absurd. This completes the proof of Claim 6.1.3.

□ (Claim 6.1.3)

Claim 6.1.4 If $g_1 = 2$, then $g_2 = 2$.

Proof of Claim 6.1.4 Notice that Claim 6.1.4 is not symmetric to Claim 6.1.3 due to the relative position of A_0. Suppose $g_1 = 2$ and $g_2 > 2$.

We can assume that $u_0 \succ l_2$ because of the following: Suppose $l_2 \sim u_0$. Let $B_1 = A_2[u_0 + 2, u_2]$, $B_0 = A_2[l_2, u_2]$, and $B_2 = A_1 \cup A_0$. Let $A_i' = H - B_i$ for $i = 0, 1, 2$. Then $A' = H - A$ would be a counterexample of Claim 6.1.3.

By a similar argument as in the beginning of the proof of Claim 6.1.3 we can assume that $A_1 \succcurlyeq \frac{3}{4}|I_1|$, which implies that $s_1 \preccurlyeq \frac{1}{4}|I_1|$ and $R_1 - L_1 \succcurlyeq \frac{1}{2}u_1$. Notice that we have $|A_2| \preccurlyeq \frac{1}{3}|A|$ and $|A_1| \succcurlyeq \frac{2}{3}|A|$.

Again let J be an *a.p.* of difference 2 tightly containing $I_1 \cup I_0$ and let $A_J = A_1 \cup A_0$. If $|2A_J| \succ \frac{5}{2}|A_J|$, then

$$|2A| \succcurlyeq |2A_J| + |u_2 + A_2| + |A_J| + 2|A_2| \succ 3|A| + \frac{1}{2}|A_J| \succcurlyeq \frac{10}{3}|A|,$$

a contradiction to (6). Hence we can assume $|2A_J| = 2|A_J| - 1 + b_J$ with $b_J \preccurlyeq \frac{1}{2}|A_J|$ and $|J| \leqslant |A_J| + b_J$ by Theorem A.4. This implies $|A_J| \succcurlyeq \frac{2}{3}|J|$. Let $s_J = |J| - |A_J|$. Then $s_J \leqslant b_J$ and $s_J \preccurlyeq \frac{1}{3}|J|$.

Subclaim 6.1.4.1 $|I_2| \prec |I_1|$.

Proof of Subclaim 6.1.4.1 Suppose $|I_2| \succcurlyeq |I_1|$. Then $|A_1 + A_2| \succcurlyeq 2|A_1|$. Hence

$$\begin{aligned}
|2A| &\succcurlyeq |2A_1| + |2A_2| + |A_1 + A_2| + |u_0 + A_2[u_2 - (u_0 - u_1), u_2]| \\
&\succcurlyeq 2|A_1| + b_1 + 2|A_2| + b_2 + 2|A_1| + A_2(u_2 - (u_0 - u_1), u_2) \\
&\sim 3|A| + |A_1| - |A_2| + b_1 + b_2 + A_2(u_2 - (u_0 - u_1), u_2) \\
&\succcurlyeq \frac{10}{3}|A| + b_1 + b_2 + A_2(u_2 - (u_0 - u_1), u_2).
\end{aligned}$$

By (6) we have $b_1 \sim 0$, $b_2 \sim 0$, $A_2(u_2-(u_0-u_1), u_2) \sim 0$, $|A_1| \sim \frac{2}{3}|A| \sim 2|A_2|$. By Lemma A.1 we know that A_2 is full in $I_2 \cap (l_2 + \{g_2 k : k \in {}^*\mathbb{N}\})$. Since $u_0 - u_1 > u_0 - l_2 \succ 0$, then $A_2(u_2 - (u_0 - u_1), u_2) \succ 0$, which contradicts $A_2(u_2 - (u_0 - u_1), u_2) \sim 0$.

□ (Subclaim 6.1.4.1)

Subclaim 6.1.4.2 $|I_2| \prec \frac{1}{2}|I_1|$.

Proof of Subclaim 6.1.4.2 Assume the contrary. By Subclaim 6.1.4.1 we can assume that $\frac{1}{2}|I_1| \preccurlyeq |I_2| \prec |I_1|$. If $|A_2| \sim 0$, then

$$|2A| \succapprox 2|A_1| + b_1 + |A_1| + A_1(0, u_2 - l_2) \succapprox 3|A| + |I_2| \succapprox 3|A| + \frac{1}{2}|A|,$$

which contradicts (6). Thus we can assume that $|A_2| \succ 0$.

For $i \in \left[0, \frac{1}{2}g_2 - 1\right]$ let $A_x^i = A_1 \cap (2i + g_2\mathbb{N}) \cap [x, x + (u_2 - l_2) - 2]$ for some $x \in I_1$ such that $x + u_2 - l_2 - 2 \leqslant u_1$ and let $k_x = |\{i \in \left[0, \frac{1}{2}g_2 - 1\right] : A_x^i \neq \emptyset\}|$. Fix an $x \in I_1\,[0, u_1 - (u_2 - l_2) - 2]$ such that $k = k_x > \frac{1}{4}g_2$ is maximal. Notice that $b_1 \geqslant s_1 \geqslant \left(\left(\frac{g_2}{2} - k\right)/g_2\right) u_1$.

Suppose there is $i_0 \in \left[0, \frac{1}{2}g_2 - 1\right]$ such that $|A_2 + A_x^{i_0}| \geqslant |A_2| + 2|A_x^{i_0}| - 2$. Since

$$\begin{aligned} |A_2 + A_1[x, x + u_2 - l_2 - 2]| &= \sum_{i \in \left[0, \frac{1}{2}g_2 - 1\right]} |A_2 + A_x^i| \\ &\geqslant |A_2| - 2 + 2 \sum_{i \in \left[0, \frac{1}{2}g_2 - 1\right]} |A_x^i| \geqslant |A_2| - 2 + 2A_1(x, x + u_2 - l_2 - 2), \end{aligned}$$

then

$$\begin{aligned} |2A| &\succapprox 2|A_1| + b_1 + 2|A_2| + |A_1| + |A_2| + A_1(x, x + u_2 - l_2 - 2) \\ &\succapprox 3|A| + \frac{1}{2}(u_2 - l_2) \succapprox 3|A| + \frac{1}{3}(u_2 - l_2) + \frac{1}{6}(u_2 - l_2) \\ &\succapprox 3|A| + \frac{1}{3}|I_1| + \frac{1}{3}|I_2| \succ 3|A| + \frac{1}{3}|A_1| + \frac{1}{3}|A_2| \sim \frac{10}{3}|A|, \end{aligned}$$

which contradicts (6). By Theorem A.4 we can now assume that

$$|A_2 + A_x^i| \geqslant \frac{1}{g_2}(u_2 - l_2) + |A_x^i|$$

for every $i \in \left[0, \frac{1}{2}g_2 - 1\right]$ with $A_x^i \neq \emptyset$. Let $w = \max\{u_0 + u_1 - 2l_2, 0\}$. Suppose $\frac{k}{g_2} = \frac{1}{2}$. Then

$$\begin{aligned}|2A| &\succcurlyeq 2|A_J| + b_J + 2|A_2| - \frac{1}{g_2}w + |A_J| + \frac{k}{g_2}(u_2 - l_2)\\ &\sim 3|A| - |A_2| + b_J - \frac{1}{g_2}w + \frac{1}{2}(u_2 - l_2).\end{aligned}$$

Hence $b \succcurlyeq b_J - |A_2| - \frac{1}{g_2}w + \frac{1}{2}(u_2 - l_2)$. Thus

$$\begin{aligned}H + 1 &\preccurlyeq u_2 - u_0 + 2(|A_J| + b_J)\\ &\preccurlyeq u_2 - u_0 + 2|A| + 2\left(b + \frac{1}{g_2}w - \frac{1}{2}(u_2 - l_2)\right)\\ &= 2|A| + 2b - u_0 + u_2 + \frac{2}{g_2}w - u_2 + l_2 \prec 2|A| + 2b\end{aligned}$$

because $g_2 \geqslant 4$ and if $w = 0$, then $-u_0 + l_2 \prec 0$ and if $w > 0$, then

$$-u_0 + \frac{2}{g_2}w + l_2 \leqslant -u_0 + \frac{1}{2}(u_0 + u_1 - 2l_2) + l_2 = -\frac{1}{2}(u_0 - u_1) \prec 0.$$

This contradicts our assumption that A fails (7). Therefore, we can now assume that $\frac{k}{g_2} < \frac{1}{2}$. This implies that g_2 is at least 8 because $|A_1| \succcurlyeq \frac{3}{4}|I_1|$. Hence $|A_2| \preccurlyeq \frac{2}{g_2}|I_2| \leqslant \frac{1}{4}|I_2|$. Let $w = |(I_1 \setminus A_1) \cap [u_2 - l_2 + 2, u_1]|$. Since $|I_2| \prec |I_1|$ and $k < \frac{g_2}{2}$, we have $w \succ 0$. Therefore,

$$|2A| \succcurlyeq 2|A_1| + b_1 + 2|A_2| + |A_1| + A_1(0, u_2 - l_2) \succcurlyeq 3|A| - |A_2| + |I_2| + w,$$

which implies that $-|A_2| + |I_2| + w \preccurlyeq b \preccurlyeq \frac{1}{3}|A_1| + \frac{1}{3}|A_2|$. Hence

$$|I_2| \prec |I_2| + w \preccurlyeq \frac{1}{3}|I_1| + \frac{4}{3}|A_2| \preccurlyeq \frac{2}{3}|I_2| + \frac{1}{3}|I_2| = |I_2|,$$

which is absurd. This completes the proof of Subclaim 6.1.4.2.

$\square$ (Subclaim 6.1.4.2)

It is now sufficient to assume that $|I_2| \prec \frac{1}{2}|I_1|$ by Subclaim 6.1.4.2 and derive a contradiction under this assumption.

We first assume that $g_2 \geqslant 6$. We want to show that $R_J - L_R \succcurlyeq \frac{1}{2}u_0$ where L_J and R_J are defined as in (13) and (14) with I_i being replaced by J and $d = 2$. Since $|A_1| \succcurlyeq \frac{3}{4}|I_1|$, then $|A_1 + A_2| \succcurlyeq |A_1| + 3|A_2|$. Hence

$$|2A| \succcurlyeq 2|A_J| + b_J + |A_2| + |A_J| + 3|A_2| = 3|A| + b_J + |A_2|.$$

So $s_J + |A_2| \preccurlyeq b_J + |A_2| \preccurlyeq b \preccurlyeq \frac{1}{3}|A|$, which implies $s_J \preccurlyeq \frac{1}{3}|A_J| = \frac{1}{3}|J| - \frac{1}{3}s_J$. This shows $s_J \preccurlyeq \frac{1}{4}|J|$. Hence $L_J + u_0 - R_J \preccurlyeq 4s_J \preccurlyeq \frac{1}{2}u_0$. This implies $R_J - L_J \succcurlyeq \frac{1}{2}u_0$.

Notice that $R_J - L_J \succcurlyeq \frac{1}{2}u_0 \sim |J| \succ |I_1| \succ 2|I_2| \sim u_2 - l_2$. Let $p_J = \frac{1}{2}(R_J - L_J - 2) - A_1(L_J + 2, R_J - 2)$ and $w = \max\{u_0 + u_1 - 2l_2, 0\}$. Then

$$\begin{aligned}|2A| &\succcurlyeq 2|A_J| + b_J + 2|A_2| - \frac{1}{g_2}w + |A_J| + A_J(L_J, L_J + (u_2 - l_2)) \\ &\succcurlyeq 3|A| - |A_2| + b_J - \frac{1}{g_2}w + A_J(L_J, L_J + (u_2 - l_2)).\end{aligned}$$

Hence

$$\begin{aligned}H + 1 &\preccurlyeq u_2 - u_0 + 2(|A_J| + b_J - p_J) \\ &\preccurlyeq u_2 - u_0 + 2|A| + 2\left(b + \frac{1}{g_2}w - A_J(L_J, L_J + (u_2 - l_2))\right) - 2p_J \\ &\preccurlyeq 2|A| + 2b + u_2 - u_0 + \frac{2}{g_2}w - u_2 + l_2 = 2|A| + 2b - u_0 + \frac{2}{g_2}w + l_2.\end{aligned}$$

If $w = 0$, then $H + 1 \prec 2|A| + 2b$ because $u_0 \succ l_2$ assumed in the beginning of the proof of Claim 6.1.4. If $w > 0$, then

$$\begin{aligned}H + 1 &\preccurlyeq 2|A| + 2b - u_0 + \frac{1}{2}(u_0 + u_1 - 2l_2) + l_2 \\ &= 2|A| + 2b - \frac{1}{2}(u_0 - u_1) \prec 2|A| + 2b\end{aligned}$$

because $u_0 \succ l_2 \succcurlyeq u_1$ and $g_2 \geqslant 4$. Therefore, we have a contradiction to the assumption that A fails (7). Notice that the proof in this paragraph does not use the assumption that $g_2 \geqslant 6$.

We can now assume $g_2 = 4$. By the definition of L_1 we have

$$A^0_{L_1} = A_1[L_1, L_1 + (u_2 - l_2) - 2] \cap (L_1 + \{4k : k \in {}^*\mathbb{N}\}) \neq \emptyset \text{ and}$$

$$A^1_{L_1} = A_1[L_1, L_1 + (u_2 - l_2) - 2] \cap (L_1 + 2 + \{4k : k \in {}^*\mathbb{N}\}) \neq \emptyset.$$

Suppose there is $i = 0$ or 1 such that $|A_2 + A^i_{L_1}| \succcurlyeq |A_2| + 2|A^i_{L_1}|$, which implies that $|A_J + A_2| \succcurlyeq |A_J| + |A_2| + A_1(L_1, L_1 + (u_2 - l_2))$. Without loss of generality let $i = 0$.

If $|2A_2| \succcurlyeq 2|A_2| + \frac{1}{3}|A_2|$, then

$$|2A| \succcurlyeq 2|A_J| + b_J + 2|A_2| + \frac{1}{3}|A_2| - \frac{1}{4}w \tag{63}$$
$$+|A_J| + |A_2| + A_1(L_1, L_1 + (u_2 - l_2)) \tag{64}$$
$$\sim 3|A| + b_J + \frac{1}{3}|A_2| - \frac{1}{4}w + \frac{1}{4}(u_2 - l_2) \tag{65}$$
$$\succcurlyeq 3|A| + b_J + \frac{1}{3}|A_2| \tag{66}$$

where $w = \max\{u_0 + u_1 - 2l_2, 0\}$. Hence $s_J + \frac{1}{3}|A_2| \preccurlyeq b \preccurlyeq \frac{1}{3}|A|$, which implies that $s_J \preccurlyeq \frac{1}{4}|J|$. Therefore, we have $|A_J| \succcurlyeq \frac{3}{4}|J|$, which implies $R_J - L_J \succcurlyeq \frac{1}{2}u_0$. Now we can prove that $H + 1 \prec 2|A| + 2b$ by the same steps as in the proof of the case when we assume that $g_2 \geqslant 6$.

If $|2A_2| \prec 2|A_2| + \frac{1}{3}|A_2|$, then $\frac{1}{4}(u_2 - l_2) \prec \frac{4}{3}|A_2|$ by Theorem A.1. Notice that if $|A_2 + A^1_{L_1}| \succcurlyeq |A_2| + 2|A^1_{L_1}|$, then

$$\begin{aligned} |A_J + A_2| &\succcurlyeq |A_J| + 2|A_2| + A_J(L_1, L_1 + (u_2 - l_2)) \\ &\succ |A_J| + \frac{3}{4}|I_2| + \frac{1}{2}|I_2| \succ |A_J| + |I_2| \end{aligned}$$

and if $|A_2 + A^1_{L_1}| \succcurlyeq \frac{1}{4}(u_2 - l_2) + |A^1_{L_1}|$, then

$$b_2 + |A_J + A_2| \succcurlyeq b_2 + |A_J| + |A_2| + \frac{1}{4}(u_2 - l_2) \succcurlyeq |A_J| + |I_2|.$$

Hence substituting $\frac{1}{3}|A_2|$ by b_2 and substituting $|A_J| + |A_2| + A_1(L_1, L_1 + (u_2 - l_2))$ by $|A_J + A_2|$ in (63) and (64) we have $|2A| \succcurlyeq 3|A| - |A_2| + b_J + |I_2| - \frac{1}{4}w$. Hence $b \succcurlyeq -|A_2| + b_J + |I_2| - \frac{1}{4}w$ and

$$\begin{aligned} H + 1 &\preccurlyeq u_2 - u_0 + 2(|A_J| + b_J) \\ &\preccurlyeq 2|A| + 2\left(b - |I_2| + \frac{1}{4}w\right) + u_2 - u_0 \\ &\preccurlyeq 2|A| + 2b + l_2 - u_0 + \frac{1}{2}w \prec 2|A| + 2b \end{aligned}$$

no matter $w = 0$ or $w > 0$.

Now we can assume that $|A_2 + A^i_{L_1}| \succcurlyeq \frac{1}{4}(u_2 - l_2) + |A^i_{L_1}|$ for $i = 0, 1$, which implies that $|A_J + A_2| \succcurlyeq |A_J| + |I_2|$. Hence by the same argument as above we have $|2A| \succcurlyeq 3|A| - |A_2| + b_J + |I_2| - \frac{1}{4}w$ and

$$H + 1 \preccurlyeq 2|A| + 2b + l_2 - u_0 + \frac{1}{2}w \prec 2|A| + 2b,$$

which again contradicts the assumption that A fails (7). This completes the proof of Claim 6.1.4.

$\square$ (Claim 6.1.4)

We continue to prove Theorem 6.1 under Case 6.1.2 that $u_2 = H$. By Claim 6.1.2, Claim 6.1.3, and Claim 6.1.4 we can now assume that $g_1 = 2$ and $g_2 = 2$.

Claim 6.1.5 $|J| \succ |I_2|$.

Proof of Claim 6.1.5 Assume the contrary that $|J| \preccurlyeq |I_2|$.

First we prove that $|2A_2| \prec 3|A_2|$, which implies that $|I_2| \preccurlyeq |A_2| + b_2$. Suppose $|2A_2| \succcurlyeq 3|A_2|$. If $|2A_1| \succcurlyeq 3|A_1|$, then $|2A| \succcurlyeq 4|A|$, which contradicts (6). Hence we can assume that $|2A_1| = 2|A_1| - 1 + b_1$ with $b_1 \prec |A_1|$. If $|A_1 + A_2| \succcurlyeq 2|A_1| + |A_2|$, then $|2A| \succcurlyeq 2|A_1| + 3|A_2| + 2|A_1| + |A_2| \sim 4|A|$. Hence by Theorem A.4 we have that $|A_1 + A_2| \succcurlyeq |A_1| + |I_2|$, which implies that $|2A| \succcurlyeq 3|A| + |I_2|$. However, $|I_2| \succcurlyeq |J|$ implies that $|I_2| \succcurlyeq \frac{1}{2}(|J| + |I_2|) \succcurlyeq \frac{1}{2}|A|$, which again contradicts (6). Therefore, we have that $|2A_2| \prec 3|A_2|$.

Second we prove that $|2A_J| \succcurlyeq 3|A_J|$. Suppose $|2A_J| \prec 3|A_J|$. Then $|A_J| \succ 0$. Let $|2A_J| = 2|A_J| - 1 + b_J$ with $b_J \prec |A_J|$ and $|A_J + A_2| = |A_1| + |A_2| - 1 + b_{J,2}$. Let $w = \max\{u_0 + u_1 - 2l_2, 0\}$. Since

$$\begin{aligned}|2A| &\succcurlyeq 2|A_J| + b_J + 2|A_2| + b_2 - \frac{1}{2}w \\ &\quad + |A_J| + |A_2| + b_{J,2} \\ &\sim 3|A| + b_J + b_2 + b_{J,2} - \frac{1}{2}w,\end{aligned}$$

then

$$H + 1 \sim u_2 - l_2 + u_0 - (u_0 - l_2)$$

$$\begin{aligned}&\preccurlyeq 2|A_2| + 2b_2 + 2|A_J| + 2b_J - (u_0 - l_2)\\&\preccurlyeq 2|A| + 2(b - b_{J,2} + \frac{1}{2}w) - (u_0 - l_2)\\&\sim 2|A| + 2b - 2b_{J,2} + w - u_0 + l_2.\end{aligned}$$

If $w = 0$, then $H + 1 \prec 2|A| + 2b$ because $u_0 \succ l_2$ is assumed. If $w > 0$, then $w - u_0 + l_2 = u_0 + u_1 - 2l_2 - u_0 + l_2 = u_1 - l_2 < 0$, which implies that $b_{J,2} \sim 0$. However, by Theorem A.4, $b_{J,2} \sim 0$ implies that A_J is full in J, which contradicts the assumption that $l_0 \succ l_2 \succcurlyeq u_1$.

Now we can assume $|2A_2| \prec 3|A_2|$ and $|2A_J| \succcurlyeq 3|A_J|$. By Theorem A.1 we have $|A_2| \succ \frac{1}{2}|I_2|$. Notice that $|A_1| \sim |A_J| \sim 0$ is possible.

Subclaim 6.1.5.1 $|A_1| \sim 0$, $u_0 - l_2 \sim 0$, and $R_2 - L_2 \succ u_0$.

Proof of Subclaim 6.1.5.1 Assume first that $|A_J + A_2| \succcurlyeq |J| + |A_2|$. Since

$$|2A| \succcurlyeq 2|A_1| + 2|A_2| + b_2 + |J| + |A_2| \sim 3|A| + b_2 - |A_1| + |J|,$$

then

$$\begin{aligned}H + 1 &\preccurlyeq u_2 - l_2 + l_2 \preccurlyeq 2|A| - 2|A_1| + 2b_2 + l_2\\&\preccurlyeq 2|A| + 2b - 2|J| + l_2 \sim 2|A| + 2b - (u_0 - l_2).\end{aligned}$$

By the failure of (7) we have $u_0 \sim l_2$. Hence

$$|2A| \succcurlyeq 3|A_J| + 2|A_2| + b_2 + |A_2| + |J| \sim 3|A| + b_2 + |J|,$$

which implies

$$\begin{aligned}H + 1 &\preccurlyeq 2|A| - 2|A_1| + 2b_2 + l_2\\&\preccurlyeq 2|A| + 2b - 2|A_1| - 2|J| + l_2 \preccurlyeq 2|A| + 2b - 2|A_1|.\end{aligned}$$

Hence $|A_1| \sim 0$ by the failure of (7). However, this implies that $|A_2| \sim |A|$ and $b_2 + |J| \preccurlyeq \frac{1}{3}|A| \sim \frac{1}{3}|I_2| - \frac{1}{3}s_2$. Therefore, we have $\frac{4}{3}s_2 + |J| \preccurlyeq \frac{1}{3}|I_2|$, which shows $|J| \preccurlyeq \frac{1}{3}|I_2|$ and $s_2 \preccurlyeq \frac{1}{4}|I_2| - \frac{3}{4}|J|$. Hence $u_2 - R_2 + L_2 - l_2 \preccurlyeq 4s_2 \preccurlyeq |I_2| - 3|J|$, which implies $R_2 - L_2 \succcurlyeq |I_2| + 3|J| \succcurlyeq 6|J| \succcurlyeq 3u_0 \succ u_0$.

Second we assume that $|A_J + A_2| \prec |J| + |A_2|$. Let $x \in I_2[l_2, l_2 + u_0 - 2]$. If $|A_J + A_2[x, x + u_0 - 2]| \succcurlyeq |J| + A_2(x, x + u_0 - 2)$, then

$$\begin{aligned}
&|A_J + A_2| \\
&\quad \succcurlyeq A_2(l_2, x - 2) + |A_J + A_2[x, x + u_0 - 2]| + A_2(x + u_0, u_2) \\
&\quad \succcurlyeq |J| + |A_2|,
\end{aligned}$$

which is absurd. Hence we can assume that

$$|A_J + A_2[x, x + u_0 - 2]| \succcurlyeq |A_J| + 2A_2(x, x + u_0 - 2)$$

for any $x \in [l_2, u_2 - u_0 + 2]$ by Theorem A.4, which implies that

$$\begin{aligned}
&|A_J + A_2| \\
&\quad \succcurlyeq A_2(l_2, x - 2) + |A_J + A_2[x, x + u_0 - 2]| + A_2(x + u_0, u_2) \\
&\quad \succcurlyeq |A_J| + |A_2| + A_2(x, x + u_0 - 2),
\end{aligned}$$

which implies that

$$|A_J| + A_2(x, x + u_0 - 2) \prec |J| \tag{67}$$

for any $x \in [l_2, u_2 - u_0 + 2]$. Hence

$$\begin{aligned}
&|2A| \\
&\quad \succcurlyeq 2|A_1| + 2|A_2| + b_2 + |A_J| + |A_2| + A_2(x, x + u_0 - 2) \\
&\quad \sim 3|A| + b_2 + A_2(x, x + u_0 - 2).
\end{aligned}$$

If $|A_J| \succcurlyeq \frac{1}{2}|J|$, then $A_2(x, x + u_0 - 2) \prec \frac{1}{2}|J|$ by (67), which implies that $s_2 \succ \frac{1}{2}|J|$. Hence we have

$$\begin{aligned}
|A_2 + A_J| &\succcurlyeq \min\{|I_2| + |A_J|, |A_2| + 2|A_J|\} \\
&\sim |A_2| + |A_J| + \min\{s_2, |A_J|\} \succcurlyeq |A_2| + |J|
\end{aligned}$$

by Theorem A.4, which contradicts that $|A_2 + A_J| \prec |A_2| + |J|$. Therefore, we can assume that $|A_J| \prec \frac{1}{2}|J|$. Now we have that

$$\frac{1}{2}u_0 \preccurlyeq b_2 + A_2(x, x + u_0 - 2) \preccurlyeq \frac{1}{3}|A| \prec \frac{1}{3}|I_2| + \frac{1}{6}|J| \preccurlyeq \frac{1}{2}|I_2|.$$

Thus $u_0 \prec \frac{1}{2}(u_2 - l_2)$. Clearly, for every $x \in I_2[l_2, l_2 + u_0 - 2]$ we have $A_2(x, x + u_0 - 2) \succcurlyeq \frac{1}{3}|A_J|$ because otherwise we have $|A_J| \succ \frac{2}{3}|J|$ by (67), which implies that $|2A_J| \preccurlyeq 2|J| \prec 3|A_J|$, a contradiction to our assumption that $|2A_J| \succcurlyeq 3|A_J|$. Hence we have $b_2 + A_2(x, x+u_0-2) \preccurlyeq \frac{1}{3}|A|$, which implies that $b_2 \preccurlyeq \frac{1}{3}|A_2|$ and $s_2 \preccurlyeq \frac{1}{3}|I_2| - \frac{1}{3}s_2$, which in turn imply that $s_2 \preccurlyeq \frac{1}{4}|I_2|$. Hence $R_2 - L_2 \succcurlyeq \frac{1}{2}(u_2 - l_2) \succ u_0$. Now let $x = L_2$ and we have

$$\begin{aligned} H + 1 = H - l_2 + l_2 + 1 &\preccurlyeq 2|A_2| + 2b_2 - 2p_2 + l_2 \\ &\preccurlyeq 2|A| - 2|A_1| + 2(b - A_2(L_2, L_2 + u_0 - 2)) - 2p_2 + l_2 \\ &\preccurlyeq 2|A| + 2b - 2|A_1| - u_0 + l_2 \end{aligned}$$

where p_2 is defined in (17). This implies $|A_1| \sim 0$ and $u_0 \sim l_2$ by the failure of (7).

$\square$ (Subclaim 6.1.5.1)

We continue to prove Claim 6.1.5. Suppose

$$|A_J + A_2[L_2, L_2 + u_0 - 2]| \geqslant |A_J| + 2A_2(L_2, L_2 + u_0 - 2) - 2.$$

Then

$$\begin{aligned} |2A| \geqslant\ & A_J(0, 2l_2 - 2) + 2|A_2| - 1 + b_2 \\ & + |A_J| + |A_2| + A_2(L_2, L_2 + u_0 - 2) - 2 \\ =\ & 3|A| - 3 + b_2 - |A_J| - A_J(2l_2, u_0) + A_2(L_2, L_2 + u_0 - 2). \end{aligned}$$

Hence $b \geqslant b_2 - |A_J| - A_J(2l_2, u_0) + A_2(L_2, L_2 + u_0 - 2)$. Now we have

$$\begin{aligned} H + 1 \leqslant H - l_2 + 2 + l_2 - 1 &\leqslant 2|A_2| + 2b_2 - 2p_2 + l_2 - 1 \\ &\leqslant 2|A| - 1 + 2(b - A_2(L_2, L_2 + u_0 - 2) + A_J(2l_2, u_0)) - 2p_2 + l_2 \\ &\leqslant 2|A| - 1 + 2b - u_0 + l_2 + 2A_J(2l_2, u_0). \end{aligned}$$

If $2l_2 > u_0$, then $A_J(2l_2, u_0) = 0$. Hence

$$H + 1 \leqslant 2|A| - 1 + 2b - u_0 + l_2 < 2|A| - 1 + 2b.$$

If $2l_2 \leqslant u_0$, then

$$\begin{aligned} H + 1 &\leqslant 2|A| - 1 + 2b - u_0 + l_2 + u_0 - 2l_2 + 2 \\ &= 2|A| - 1 + 2b - l_2 + 2 < 2|A| - 1 + 2b \end{aligned}$$

because $l_2 > u_1 \geqslant 2$ which is implied by $g_1 = 2$. Hence we have a contradiction to the failure of (7). Thus we can now assume that

$$|A_J + A_2[L_2, L_2 + u_0 - 2]| \geqslant |J| + A_2(L_2, L_2 + u_0 - 2) - 1$$

by Theorem A.4, which implies that $|A_J + A_2| \geqslant |J| + |A_2| - 1$.

Suppose $p_2 \geqslant |A_J|$. Then we have

$$\begin{aligned}|2A| &\geqslant A_J(0, 2l_2 - 2) + 2|A_2| - 1 + b_2 + |J| + |A_2| - 1 \\ &= 3|A| - 3 + 2 - A_J(2l_2, u_0) - 2|A_J| + b_2 + \frac{1}{2}u_0\end{aligned}$$

which implies that

$$\begin{aligned}H + 1 &= u_2 - l_2 + 2 + l_2 - 1 \\ &\leqslant 2|A_2| + 2b_2 - 2p_2 + l_2 - 1 \leqslant 2|A| - 2|A_J| + 2b_2 - 2p_2 + l_2 - 1 \\ &\leqslant 2|A| - 1 + 2\left(b - 2 + A_J(2l_2, u_0) + |A_J| - \frac{1}{2}u_0\right) - 2p_2 + l_2 \\ &\leqslant 2|A| - 1 + 2b - 4 + 2A_J(2l_2, u_0) - u_0 + l_2.\end{aligned}$$

If $2l_2 > u_0$, then $H + 1 \leqslant 2|A| - 1 + 2b - 4 - u_0 + l_2 < 2|A| - 1 + 2b$ and if $2l_2 \leqslant u_0$, then

$$\begin{aligned}H + 1 &\leqslant 2|A| - 1 + 2b - 4 + u_0 - 2l_2 + 2 - u_0 + l_2 \\ &= 2|A| - 1 + 2b - 2 - l_2 < 2|A| - 1 + 2b.\end{aligned}$$

Hence we have a contradiction to that A fails (7).

Suppose $p_2 < |A_J|$. Notice that in the proof of Subclaim 6.1.5.1 under the assumption that $|A_J + A_2| \not\geqslant |J| + |A_2|$, which is implies by $|A_J + A_2| \geqslant |J| + |A_2| - 1$, we have $R_2 - L_2 \not\geqslant 3u_0$. Now we enlarge A_2 in I_2 by applying Lemma 3.1. Suppose $|A_2|$ is maximal so that $A \subseteq J \cup I_2$, $g_1 = g_2 = 2$, and A is a counterexample of Theorem 6.1. We want to show that $A_2 = I_2$. First if $x \in I_2$, $x \sim \frac{1}{2}(L_2 + R_2)$, then $x \in A_2$ because otherwise let $A' = A \cup \{x\}$ and we can show that $2A' = 2A$ by the fact that $R_2 - L_2 \not\geqslant 3u_0$ and $p_2 < |A_J|$, which contradicts the maximality of $|A_2|$ by Lemma 3.1. Let $x_{2,L}$ and $x_{2,R}$ be defined by (29) and (30). If $x_{2,L} > l_2$, let $A' = A \cup \{x_{2,L}\}$. Then $(2A') \setminus (2A) \subseteq \{x_{2,L}, x_{2,L} + l_2\}$, which again contradicts the maximality of

$|A_2|$ by Lemma 3.1. By the same reason we have $x_{2,R} = u_2$. Thus $A_2 = I_2$. If $|J| \sim 0$, then $|2A| \sim 3|A|$, which, combined with Theorem 1.3, contradicts the assumption that A fails (7). Hence we can assume that $|J| \succ 0$. Recall that $|A_J| \sim 0$ and $l_2 \sim u_0$ by Subclaim 6.1.5.1. Notice that $g_1 = 2$ implies that $|A_J| > 2$. By Theorem A.1 and the facts that $|J| \succ 0$ and $|A_J| \sim 0$ we have $|2A_J| \geqslant 3|A_J| - 3$. Suppose $u_0 - l_2 \leqslant 2|A_J| - 4$. Since

$$\begin{aligned} |2A| &\geqslant 3|A_J| - 3 + 2|A_2| - 1 - \frac{1}{2}(2u_0 - 2l_2) - 1 + \frac{1}{2}u_0 + |A_2| \\ &\geqslant 3|A| - 5 - (u_0 - l_2) + \frac{1}{2}u_0 \geqslant 3|A| - 3 - |A_J| - \frac{1}{2}(u_0 - l_2) + \frac{1}{2}u_0, \end{aligned}$$

we have that

$$\begin{aligned} H + 1 = H - l_2 + 2 + l_2 - 1 &\leqslant 2|A| - 1 - 2|A_J| + l_2 \\ &\leqslant 2|A| - 1 + 2\left(b + \frac{1}{2}(u_0 - l_2) - \frac{1}{2}u_0\right) + l_2 \\ &\leqslant 2|A| - 1 + 2b - l_2 + l_2 = 2|A| - 1 + 2b, \end{aligned}$$

which contradicts that A fails (7). Hence we can assume that $u_0 - l_2 > 2|A_J| - 4$. Since

$$|2A| \geqslant 2|A_2| - 1 + |A_J| + |A_2| + \frac{1}{2}u_0 = 3|A| - 3 + 2 - 2|A_J| + \frac{1}{2}u_0,$$

we have that

$$\begin{aligned} H + 1 &= H - l_2 + 2 - 1 + l_2 = 2|A_2| - 1 + l_2 \\ &= 2|A| - 1 - 2|A_J| + l_2 \leqslant 2|A| - 1 + 2\left(b - 2 + |A_J| - \frac{1}{2}u_0\right) + l_2 \\ &= 2|A| - 1 + 2b - 4 + 2|A_J| - (u_0 - l_2) < 2|A| - 1 + 2b, \end{aligned}$$

which again contradicts that A fails (7). This completes the proof of Claim 6.1.5.

□ (Claim 6.1.5)

We continue to prove Theorem 6.1 under Case 6.1.2 that $u_2 = H$. Because of Claim 6.1.5 we can now assume that $|J| \succ |I_2|$. By symmetry we can assume that $u_0 \succ l_2$. The condition $|J| \succ |I_2|$ here is not symmetric to the condition $|J| \prec |I_2|$ in Claim 6.1.5 due to the relative position of A_0 although the proof here is similar to the proof of Claim 6.1.5.

We first prove that $|2A_1| \prec 3|A_1|$. Suppose $|2A_1| \succcurlyeq 3|A_1|$. If $|A_J + A_2| \succcurlyeq |A_J| + 2|A_2|$, then

$$|2A| \succcurlyeq 3|A_1| + 2|A_2| + |A_J| + 2|A_2| \succcurlyeq 4|A| \succ \frac{10}{3}|A|.$$

If $|A_J + A_2| \succcurlyeq |J| + |A_2|$, then

$$|2A| \succcurlyeq 3|A_1| + 2|A_2| + |J| + |A_2| \succcurlyeq 3|A| + |J| \succ \frac{10}{3}|A|$$

by the fact that $|J| \succ \frac{1}{2}(|J| + |I_2|) \succcurlyeq \frac{1}{2}|A|$.

Second we prove that $|2A_J| \prec 3|A_J|$. Suppose $|2A_J| \succcurlyeq 3|A_J|$. If $|A_J + A_2| \succcurlyeq |A_J| + 2|A_2|$, then

$$|2A| \succcurlyeq 3|A_J| + |A_2| + |A_J| + 2|A_2| \sim 3|A| + |A_J|$$

implies that $|A_J| \preccurlyeq \frac{1}{3}|A|$ and

$$|2A| \succcurlyeq 2|A_1| + 2|A_2| + |A_J| + 2|A_2| \sim 3|A| + |A_2|$$

implies that $|A_2| \preccurlyeq \frac{1}{3}|A|$. Hence we have $|A| = |A_J| + |A_2| \preccurlyeq \frac{2}{3}|A|$, which is absurd. Suppose that $|A_J + A_2| \succcurlyeq |J| + |A_2|$. If $|2A_2| \succcurlyeq 3|A_2|$, then

$$|2A| \succcurlyeq 3|A_J| + 3|A_2| - \frac{1}{2}w + |J| + |A_2| \sim 3|A| + |J| - \frac{1}{2}w + |A_2| \succcurlyeq 4|A|$$

because $w = \max\{u_0 + u_1 - 2l_2, 0\} = 0$ implies that $|J| + |A_2| \succ |A|$ and $w > 0$ implies that

$$|J| - \frac{1}{2}w + |A_2| \succcurlyeq \frac{1}{2}(u_0 - u_0 - u_1 + 2l_2) + |A_2| \succcurlyeq \frac{1}{2}l_2 + |A_2| \succcurlyeq |A|.$$

If $|2A_2| \prec 3|A_2|$, then

$$|2A| \succcurlyeq 2|A_1| + b_1 + 2|A_2| + b_2 + |J| + |A_2| \sim 3|A| + b_1 + b_2 + |J| - |A_1|.$$

Hence

$$\begin{aligned} H+1 &\sim u_2 - l_2 + u_1 + (l_2 - u_1) \\ &\preccurlyeq 2|A_2| + 2b_2 + 2|A_1| + 2b_1 + (l_2 - u_1) \\ &\preccurlyeq 2|A| + 2b + 2(|A_1| - |J|) + (l_2 - u_1) \\ &\preccurlyeq 2|A| + 2b + u_1 - u_0 + l_2 - u_1 \prec 2|A| + 2b \end{aligned}$$

by the assumption that $l_2 \prec u_0$.

Third we prove that $|2A_2| \prec 3|A_2|$. Suppose $|2A_2| \succcurlyeq 3|A_2|$.

Assume that $|A_2| \succcurlyeq \frac{1}{2}|I_2|$. If $|A_J + A_2| \succcurlyeq |A_J| + 2|A_2|$, then

$$|2A| \succcurlyeq 2|A_J| + b_J + 3|A_2| - \frac{1}{2}w + |A_J| + 2|A_2| \succcurlyeq 3|A| + b_J + 2|A_2| - \frac{1}{2}w.$$

Hence

$$\begin{aligned} H+1 &\sim u_2 - l_2 + u_0 - (u_0 - l_2) \\ &\preccurlyeq 4|A_2| + 2|A_J| + 2b_J - u_0 + l_2 \preccurlyeq 2|A| + 2b - 2|A_2| + w - u_0 + l_2. \end{aligned}$$

If $w = \max\{u_0 + u_1 - 2l_2, 0\} = 0$, then $H + 1 \prec 2|A| + 2b$. If $w > 0$, then

$$\begin{aligned} H+1 &\preccurlyeq 2|A| + 2b - |I_2| + u_0 + u_1 - 2l_2 - u_0 + l_2 \\ &\preccurlyeq 2|A| + 2b - |I_2| + u_1 - l_2 \prec 2|A| + 2b. \end{aligned}$$

If $|A_J + A_2| \succcurlyeq |J| + |A_2|$, then

$$\begin{aligned} |2A| &\succcurlyeq 2|A_J| + b_J + 3|A_2| - \frac{1}{2}w + |J| + |A_2| \\ &\succcurlyeq 3|A| + b_J + |A_2| + |J| - |A_J| - \frac{1}{2}w. \end{aligned}$$

Hence

$$\begin{aligned} H+1 &\sim u_2 - l_2 + u_0 - u_0 + l_2 \\ &\preccurlyeq 4|A_2| + 2|A_J| + 2b_J - u_0 + l_2 \\ &\preccurlyeq 2|A| + 2\left(b - |J| + |A_J| + \frac{1}{2}w\right) - u_0 + l_2 \\ &\preccurlyeq 2|A| + 2b - u_0 + u_1 + w - u_0 + l_2. \end{aligned}$$

If $w = \max\{u_0 + u_1 - 2l_2, 0\} = 0$, then $H + 1 \prec 2|A| + 2b - u_0 + l_2 \prec 2|A| + 2b$. If $w > 0$, then

$$\begin{aligned} H + 1 &\preccurlyeq 2|A| + 2b - 2u_0 + u_1 + l_2 + (u_0 + u_1 - 2l_2) \\ &= 2|A| + 2b - u_0 - l_2 + 2u_1 \prec 2|A| + 2b. \end{aligned}$$

Both contradicts the failure of (7).

Thus we can assume that $|A_2| \prec \frac{1}{2}|I_2|$.

Suppose $|A_J + A_2| \succcurlyeq |A_J| + |I_2|$. Then

$$|2A| \succcurlyeq 2|A_J| + b_J + 3|A_2| - \frac{1}{2}w + |A_J| + |I_2| \sim 3|A| + b_J - \frac{1}{2}w + |I_2|,$$

which implies that

$$\begin{aligned} H + 1 &\preccurlyeq u_2 - u_0 + u_0 \preccurlyeq 2|A_J| + 2b_J + u_2 - u_0 \\ &\preccurlyeq 2|A| + 2b - 2|A_2| + w - 2|I_2| + u_2 - u_0. \end{aligned}$$

If $w = \max\{u_0 + u_1 - 2l_2, 0\} = 0$, then

$$H + 1 \preccurlyeq 2|A| + 2b - u_2 + l_2 + u_2 - u_0 = 2|A| + 2b + l_2 - u_0 \prec 2|A| + 2b$$

because of the assumption that $u_0 \succ l_2$. If $w > 0$, then

$$\begin{aligned} &H + 1 \\ &\quad \preccurlyeq 2|A| + 2b - 2|A_2| + u_0 + u_1 - 2l_2 - u_2 + l_2 + u_2 - u_0 \\ &\quad \preccurlyeq 2|A| + 2b - 2|A_2| + u_1 - l_2. \end{aligned}$$

Since the failure of (7), we have that $|A_2| \sim 0$ and $l_2 \sim u_1$. Therefore, $|2A| \succcurlyeq 2|A_J| + b_J + |A_J| + |I_2| \sim 3|A| + b_J + |I_2|$, which implies that

$$\begin{aligned} H + 1 &\preccurlyeq u_2 - u_0 + 2|A_J| + 2b_J \\ &\preccurlyeq 2|A| + 2b - 2|I_2| + u_2 - u_0 \sim 2|A| + 2b - u_2 + l_2 + u_2 - u_0 \\ &= 2|A| + 2b + l_2 - u_0 \prec 2|A| + 2b. \end{aligned}$$

Now we have a contradiction to the failure of (7).

Suppose $|A_J + A_2| \prec |A_J| + |I_2|$. Then

$$\begin{aligned}|A_J + A_2| &\succcurlyeq |A_J[0,x] + l_2| + |A_J[x+2, x+(u_2-l_2)] + A_2| \\ &\quad + |A_J[x+(u_2-l_2)+2, u_0] + u_2| \\ &\succcurlyeq A_J(0,x) + 2A_J(x+2, x+(u_2-l_2)) + |A_2| \\ &\quad + A_J(x+(u_2-l_2)+2, u_0) \\ &\succcurlyeq |A_J| + |A_2| + A_J(x+2, x+(u_2-l_2))\end{aligned}$$

for every $x \in [0, u_0 - (u_2 - l_2)]$. Since

$$\begin{aligned}|2A| &\succcurlyeq 2|A_1| + b_1 + 3|A_2| \\ &\quad + |A_J| + |A_2| + A_J(x, x+(u_2-l_2)) \\ &\sim 3|A| + b_1 + |A_2| + A_J(x, x+(u_2-l_2)),\end{aligned}$$

then $b_1 + |A_2| + A_J(x, x+(u_2-l_2)) \preccurlyeq \frac{1}{3}|A|$, which implies the following: (1) $b_1 \prec \frac{1}{3}|A_1| \sim \frac{1}{3}|I_1| - \frac{1}{3}s_1$, which implies $s_1 \prec \frac{1}{4}|I_1|$. Hence $R_1 - L_1 \succ \frac{1}{2}u_1$. (2) $|I_2| + |A_2| \preccurlyeq b_1 + |A_2| + A_J(L_1, L_1 + (u_2 - l_2)) \preccurlyeq \frac{1}{3}|A|$, which implies $|I_2| \preccurlyeq \frac{1}{3}|I_1|$. Hence $L_1 + (u_2 - l_2) \prec R_1$. (3) $3|A_2| \preccurlyeq |I_2| + |A_2| \preccurlyeq \frac{1}{3}|A|$, which implies $|A_2| \preccurlyeq \frac{1}{9}|A|$ and $|A_1| \succcurlyeq \frac{8}{9}|A|$.

If $A_1(L_1, L_1 + (u_2 - l_2)) \succ \frac{4}{3}|A_2|$, then

$$\begin{aligned}|2A| &\succcurlyeq 2|A_J| + b_J + |A_2| \\ &\quad + |A_J| + |A_2| + A_1(L_1, L_1 + (u_2-l_2)) \\ &\sim 3|A| + b_J - |A_2| + A_1(L_1, L_1 + (u_2-l_2)).\end{aligned}$$

Hence $b_J + \frac{1}{3}|A_2| \prec b_J - |A_2| + A_1(L_1, L_1 + (u_2 - l_2)) \preccurlyeq \frac{1}{3}|A|$, which implies $s_J \preccurlyeq b_J \prec \frac{1}{3}|J| - \frac{1}{3}s_J$. This shows that $s_J \prec \frac{1}{4}|J|$, which implies $R_J - L_J \succ \frac{1}{2}u_0$. Hence $|2A| \succcurlyeq 3|A| + b_J - |A_2| + A_J(L_J, L_J + (u_2 - l_2))$. Now we have that

$$\begin{aligned}H + 1 &\preccurlyeq u_2 - u_0 + 2|A| - 2|A_2| + 2b_J - 2p_J \\ &\preccurlyeq 2|A| + 2b - 2A_J(L_J, L_J + (u_2 - l_2)) - 2p_J + u_2 - u_0 \\ &\preccurlyeq 2|A| + 2b - 2|I_2| + u_2 - u_0 \preccurlyeq 2|A| + 2b - (u_0 - l_2) \prec 2|A| + 2b,\end{aligned}$$

which is a contradiction to the assumption that (7) is false.

If $A_1(L_1, L_1 + (u_2 - l_2)) \preccurlyeq \frac{4}{3}|A_2|$, then

$$\begin{aligned}
&|2A| \\
&\quad \succcurlyeq 2|A_J| + b_J + 3|A_2| - \frac{1}{2}w + |A_J| + |A_2| + A_1(L_1, L_1 + (u_2 - l_2)) \\
&\quad \sim 3|A| + b_J - \frac{1}{2}w + |A_2| + A_1(L_1, L_1 + (u_2 - l_2)).
\end{aligned}$$

Hence

$$\begin{aligned}
H + 1 &\preccurlyeq u_2 - u_0 + 2|A| - 2|A_2| + 2b_J \\
&\quad \preccurlyeq 2|A| + 2b - 4|A_2| - 2A_1(L_1, L_1 + (u_2 - l_2)) + w + u_2 - u_0.
\end{aligned}$$

If $w = \max\{u_0 + u_1 - 2l_2, 0\} = 0$, then

$$\begin{aligned}
H + 1 &\preccurlyeq 2|A| + 2b - 5A_1(L_1, L_1 + (u_2 - l_2)) + (u_2 - l_0) \\
&\preccurlyeq 2|A| + 2b - \frac{5}{4}(u_2 - l_2) + (u_2 - u_0) \prec 2|A| + 2b + l_2 - u_0 \prec 2|A| + 2b.
\end{aligned}$$

If $w > 0$, then

$$\begin{aligned}
H + 1 &\preccurlyeq 2|A| + 2b - 5A_1(L_1, L_1 + (u_2 - l_2)) + u_0 + u_1 - 2l_2 + u_2 - u_0 \\
&\preccurlyeq 2|A| + 2b - \frac{5}{4}(u_2 - l_2) + u_1 + u_2 - 2l_2 \\
&\prec 2|A| + 2b - u_2 + l_2 + u_1 + u_2 - 2l_2 = 2|A| + 2b + u_1 - l_2 \preccurlyeq 2|A| + 2b.
\end{aligned}$$

Hence we have a contradiction to the failure of (7).

Now we can assume that $|2A_J| \prec 3|A_J|$ and $|2A_2| \prec 3|A_2|$. Clearly, we have that $|A_2| \succ 0$.

Notice that

$$|2A| \succcurlyeq 2|A_J| + b_J + 2|A_2| + b_1 - \frac{1}{2}w + |A_J + A_2| \tag{68}$$

$$\succcurlyeq 3|A| + b_J + b_2 - \frac{1}{2}w + |A_J + A_2| - |A|. \tag{69}$$

Suppose that $|A_J + A_2| \succcurlyeq |A_J| + 2|A_2|$. Then by (68) and (69) we have $|2A| \succcurlyeq 3|A| + b_J + b_2 - \frac{1}{2}w + |A_2|$. Hence

$$\begin{aligned}
H + 1 &\sim u_2 - l_2 + u_0 - u_0 + l_2 \\
&\preccurlyeq 2|A_2| + 2b_2 + 2|A_J| + 2b_J - u_0 + l_2 \\
&\preccurlyeq 2|A| + 2b + w - 2|A_2| - u_0 + l_2.
\end{aligned}$$

If $w = \max\{u_0 + u_1 - 2l_2, 0\} = 0$, then $H + 1 \prec 2|A| + 2b - 2|A_2| \prec 2|A| + 2b$. If $w > 0$, then

$$\begin{aligned} H + 1 &\preccurlyeq 2|A| + 2b + u_0 + u_1 - 2l_2 - 2|A_2| - u_0 + l_2 \\ &= 2|A| + 2b + u_1 - l_2 - 2|A_2| \prec 2|A| + 2b. \end{aligned}$$

Suppose that $|A_J + A_2| \succcurlyeq |J| + |A_2|$. Then by (68) and (69) we have $|2A| \succcurlyeq 3|A| + b_J + b_2 - \frac{1}{2}w + |J| - |A_J|$. Hence

$$\begin{aligned} H + 1 &\preccurlyeq 2|A_J| + 2b_J + 2|A_2| + 2b_2 - u_0 + l_2 \\ &\preccurlyeq 2|A| + 2b + w - 2|J| + 2|A_J| - u_0 + l_2 \\ &\preccurlyeq 2|A| + 2b + w - u_0 + u_1 - u_0 + l_2. \end{aligned}$$

If $w = \max\{u_0 + u_1 - 2l_2, 0\} = 0$, then $H + 1 \prec 2|A| + 2b$ because $u_0 \succ l_2$. If $w > 0$, then

$$\begin{aligned} H + 1 &\preccurlyeq 2|A| + 2b + u_0 + u_1 - 2l_2 - u_0 + u_1 - u_0 + l_2 \\ &= 2|A| + 2b + 2u_1 - u_0 - l_2 \prec 2|A| + 2b. \end{aligned}$$

This completes the proof of Case 6.1.2 as well as the proof of Theorem 6.1.

□

7 Proof of Theorem 1.7 when $d = 1$

In this section we prove Theorem 1.7 when $d = 1$.

Theorem 7.1 *Suppose $A \subseteq [0, H]$ is an almost tight subset of a* b.p. $I_1 \cup I_2$ *of difference $d = 1$ satisfying (4), (5), and (6). Then A satisfies (7).*

Proof Assume that Theorem 7.1 is not true and let $A \subseteq [0, H]$ be a counterexample of the theorem. Let $A_0 = A \setminus (I_1 \cup I_2)$ be tightly contained in an interval I_0. Recall that $I_1 \cup I_2$ is a *b.p.* Without loss of generality let $I_1 = [0, u_1]$, $I_2 = [l_2, u_2]$, and $I_0 = [l_0, u_0]$ where $u_1 < l_2$ and $|I_0|/|A| \approx 0$. Without loss of generality again we can assume that I_0 is either between

u_1 and l_2 or above u_2. Recall that $[a, b] \cup [c, d]$ with $b < c$ is a *b.p.* iff $c - b > \max\{b - a, d - c\}$. Let $g_1 = \gcd(A_1)$ and $g_2 = \gcd(A_2 - l_2)$.

Claim 7.1.1 $g_1 = 1$ or $g_2 = 1$. If $g_i > 1$, then $|A_i| \preccurlyeq \frac{1}{3}|A|$.

Proof of Claim 7.1.1 Similar to the proof of Claim 6.1.2.

$\square$ (Claim 7.1.1)

Case 7.1.1 $u_2 < l_0$.

We have $l_1 = 0$ and $u_0 = H$. Since $I_1 \cup I_2$ is a *b.p.* and $I_1 \cup I_2 \cup I_0$ is not a *b.p.*, we have $\max\{u_1, u_2 - l_2\} < l_2 - u_1 \leqslant u_0 - l_2$. Since A is not a subset of a *t.p.*, we have that $l_0 - u_2 \leqslant \max\{u_1, u_2 - l_2\}$. Let $J = [l_2, u_0]$, $A_J = A_2 \cup A_0$, and $g_J = \gcd(A_J - l_2)$. The reader should be aware that J has been used for combining I_1 and I_0 before. We use J to combine I_2 and I_0 only for this case.

Claim 7.1.2 $u_1 \succ 0$.

Proof of Claim 7.1.2 Suppose $u_1 \sim 0$. By Claim 7.1.1 we can assume $g_2 = 1$. Since $|2A| \succcurlyeq 3|A| + b_J$, then $b_J \preccurlyeq \frac{1}{3}|A| \sim \frac{1}{3}|A_2| \preccurlyeq \frac{1}{3}|J| - \frac{1}{3}s_J$, which implies $s_J \preccurlyeq \frac{1}{4}|J|$ and $R_J - L_J \succcurlyeq \frac{1}{2}|J|$ where b_J, s_J, L_J, and R_J are defined in (11), (10), (13), and (14), respectively, with I_i or i being replaced by J.

Suppose that $u_0 - l_2 \geqslant l_2$. If $u_1 > 0$, let $A' = A \setminus \{u_1\}$. We have $|2A| - |2A'| \geqslant 2$ because $2u_1, u_1 + u' \notin 2A'$ where u' is the second largest element in A_1. By Lemma 3.2 A' is still a counterexample of Theorem 7.1. By repeating this step, we can assume that $A_1 = \{0\}$. Notice that since $|A_1| \sim 0$, (6) will not be violated in this process. Suppose that $|A_J|$ is maximal such that $A_J \subseteq J$ and $A_1 \cup A_J$ is a counterexample of Theorem 7.1. Since $R_J - L_J \succcurlyeq \frac{1}{2}|J|$ we can show that $[L_J, R_J] \subseteq A_J$ by Lemma 3.1. Then we can show that $x_{J,L} = l_2$ and $x_{J,R} = H$ by Lemma 3.1 again where $x_{J,L}, x_{J,R}$ are defined in (29) and (30) with i and I_i being replaced by J. Now we have that $|2A| \sim 3|A|$, which contradicts, by Theorem 1.3, that A is a counterexample of Theorem 7.1.

Suppose $u_0 - l_2 < l_2$. Since

$$|2A| \geqslant 2|A_1| - 1 + |0 + A_J[l_2, L_J - 1]| + |A_1 + A_J[L_J, L_J + u_1 - 1]|$$

$$
\begin{aligned}
&+|u_1+A_J[L_J+u_1,u_0]|+2|A_J|-1+b_J-|u_1+A_J[2l_2-u_1,u_0]|\\
&\geqslant 2|A_1|+|A_J|+A_J(L_J,L_J+u_1-1)+2|A_J|+b_J-A_J(2l_2-u_1,u_0)-2\\
&=3|A|-3-|A_1|+1+A_J(L_J,L_J+u_1-1)+b_J-A_J(2l_2-u_1,u_0),
\end{aligned}
$$

we have that $b \geqslant -|A_1|+1+A_J(L_J,L_J+u_1-1)+b_J-A_J(2l_2-u_1,u_0)$. Hence by the assumption that $u_0-l_2<l_2$ we have that

$$
\begin{aligned}
H+1 &= 2(u_0-l_2+1)+2l_2-u_0-1\\
&\leqslant 2|A_J|-1+2b_J-2p_J+2l_2-u_0\\
&= 2|A|-1+2b-2-2A_J(L_J,L_J+u_1-1)\\
&\quad +2A_J(2l_2-u_1,u_0)-2p_J+2l_2-u_0\\
&\leqslant 2|A|-1+2b-2-2u_1+2(u_0+u_1-2l_2+1)+2l_2-u_0\\
&= 2|A|-1+2b+u_0-2l_2<2|A|-1+2b,
\end{aligned}
$$

which contradicts the failure of (7).

□ (Claim 7.1.2)

Claim 7.1.3 $g_1=1$ implies $g_2=1$.

Proof of Claim 7.1.3 Suppose $g_1=1$ and $g_2=d>1$. If $|2A_1|\succ\frac{7}{3}|A_1|$, then

$$
|2A|\succ\frac{7}{3}|A_1|+2|A_2|+|A_1|+2|A_2|\sim 3|A|+\frac{1}{3}|A_1|+|A_2|\succcurlyeq\frac{10}{3}|A|,
$$

which contradicts (6). If $|2A_1|\sim\frac{7}{3}|A_1|$, then $|2A|\succcurlyeq 3|A|+\frac{1}{3}|A_1|+|A_2|$, which implies that $|A_2|\sim 0$ by (6). Hence

$$
|2A|\succcurlyeq\frac{7}{3}|A|+|A_1+A_J|\succcurlyeq\frac{10}{3}|A|+A_1(L_1,L_1+(u_0-l_2))\succ\frac{10}{3}|A|
$$

because $L_1\prec u_1$ and $u_0-l_2\geqslant l_2-u_1\succ 0$. Hence we have a contradiction to (6).

We can now assume that $|2A_1|\prec\frac{7}{3}|A_1|$. This implies $b_1\prec\frac{1}{3}|A_1|=\frac{1}{3}|I_1|-\frac{1}{3}s_1$ and $s_1\prec\frac{1}{4}u_1$. Hence we have that $|A_1|\succ\frac{3}{4}|I_1|$ and $R_1-L_1\succ\frac{1}{2}u_1$.

Suppose $|I_2|\preccurlyeq\frac{1}{2}|I_1|$. Then $|A_1|\succ\frac{3}{4}|I_1|\succcurlyeq\frac{3}{2}|I_2|\succcurlyeq 3|A_2|$, which implies that $|A_1|\succcurlyeq\frac{3}{4}|A|$ and $|A_2|\preccurlyeq\frac{1}{4}|A|$. Hence

$$
|2A|\succcurlyeq 2|A_1|+|\{l_2,l_0\}+A_1|+|\{u_2,l_0\}+A_2|\succcurlyeq 3|A|+|A_1|-|A_2|\succ\frac{10}{3}|A|,
$$

which contradicts (6).

Suppose $|I_2| \succcurlyeq |I_1|$. Then

$$\begin{aligned}|2A| &\succcurlyeq 2|A_1| + b_1 + 2|A_2| + b_2 \\ &\quad + |A_1 + \{l_2, u_2\}| + |u_0 + A_2[2u_2 - u_0, u_2]| \\ &\succcurlyeq 3|A| + |A_1| - |A_2| + b_1 + b_2 + A_2(2u_2 - u_0, u_2) \\ &\succcurlyeq 3|A| + \frac{1}{3}|A| + b_1 + b_2 + A_2(2u_2 - u_0, u_2).\end{aligned}$$

By (6) we have $|A_1| \sim \frac{2}{3}|A|$, $|A_2| \sim \frac{1}{3}|A|$, $b_1 \sim 0$, and $b_2 \sim 0$, which imply that A_1 is full in I_1, A_2 is full in $I_2 \cap \{l_2 + 2k : k \in {}^*\mathbb{N}\}$, $u_0 \sim u_2$, and $|I_1| \sim |I_2| \sim l_2 - u_1$. Now we have

$$|2A| \succcurlyeq 4|A_1| + 2|A_2| \sim 3|A| + |A_1| - |A_2| \sim 3|A| + \frac{1}{3}|A|$$

and $b \succcurlyeq \frac{1}{3}|A|$. But

$$H + 1 \sim u_2 - l_2 + l_2 - u_1 + u_1 \sim 2|A_2| + |A_1| + |A_1| \sim 2|A| \prec 2|A| + 2b,$$

which contradicts the failure of (7).

So we can now assume that $|I_2| \prec |I_1| \prec 2|I_2|$. Suppose $g_2 > 2$. Then $|A_1 + A_2| \succcurlyeq |A_1| + 3|A_2|$. This implies that

$$|2A| \succcurlyeq 2|A_1| + b_1 + 2|A_2| + |A_1| + 3|A_2| \sim 3|A| + b_1 + 2|A_2|.$$

Hence $|A_2| \preccurlyeq \frac{1}{6}|A|$ and $|A_1| \succcurlyeq \frac{5}{6}|A|$, which imply that

$$\begin{aligned}|2A| &\succcurlyeq 2|A_1| + |A_1 + A_J| + |u_0 + A_2| \\ &\succcurlyeq 2|A_1| + 2|A_1| + |A_2| \succcurlyeq 3|A| + |A_1| - 2|A_2| \\ &\succcurlyeq 3|A| + \frac{5}{6}|A| - \frac{2}{6}|A| \succ \frac{10}{3}|A|.\end{aligned}$$

This contradicts (6). We can now assume $g_2 = 2$. First we show that $|A_J + A_1| \succcurlyeq \min\{|J| + |A_1|, |A_J| + 2|A_1|\}$. This is true by Theorem A.4 if $g_J = 1$. So we can assume that $g_J = g_2 = 2$ and let $A_1^i = A_1 \cap \{i + 2k : k \in {}^*\mathbb{N}\}$ for $i = 0, 1$. If $|A_J + A_1^0| \succcurlyeq \frac{1}{2}|J| + |A_1^0|$ and $|A_J + A_1^1| \succcurlyeq \frac{1}{2}|J| + |A_1^1|$, then $|A_J + A_1| \succcurlyeq |J| + |A_1|$. If $|A_J + A_1^0| \succcurlyeq |A_2| + 2|A_1^0|$, then

$$|A_J + A_1| \succcurlyeq |A_J + A_1^0| + |A_J + A_1^1| \succcurlyeq |A_2| + 2|A_1^0| + 2|A_1^1| = |A_2| + 2|A_1|.$$

Hence we always have that $|A_J+A_1| \succcurlyeq \min\{|J|+|A_1|, |A_2|+2|A_1|\}$. Suppose $|A_J+A_1| \succcurlyeq |J|+|A_1|$. Then

$$\begin{aligned}|2A| &\succcurlyeq 2|A_1| + b_1 + |J| + |A_1| + 2|A_2| - (2A_2)(2l_2, u_1+u_0)\\ &\sim 3|A| - |A_2| + b_1 + |J| - \frac{1}{2}(u_1+u_0-2l_2).\end{aligned}$$

Hence

$$\begin{aligned}H+1 &= u_0 - 2u_1 + 2u_1 \preccurlyeq u_0 - 2u_1 + 2|A_1| + 2b_1\\ &\preccurlyeq u_0 - 2u_1 + 2|A| + 2b - 2|J| + u_1 + u_0 - 2l_2\\ &\sim 2|A| + 2b + u_0 - 2u_1 - 2u_0 + 2l_2 + u_1 + u_0 - 2l_2\\ &= 2|A| + 2b - u_1 \prec 2|A| + 2b,\end{aligned}$$

which contradicts the failure of (7). Suppose $|A_J+A_1| \succcurlyeq |A_2|+2|A_1|$. Then

$$\begin{aligned}|2A| &\succcurlyeq 2|A_1| + b_1 + |A_2| + 2|A_1| + |u_0 + A_2|\\ &\succcurlyeq 3|A| + |A_1| - |A_2| + b_1 \succcurlyeq \frac{10}{3}|A| + b_1.\end{aligned}$$

Since $|A_1| - |A_2| \succcurlyeq \frac{1}{3}|A|$, by (6) we have $|A_1| \sim \frac{2}{3}|A| \sim 2|A_2|$ and $b_1 \sim 0$. This implies that A_1 is full in I_1. Hence $|I_1| \sim |A_1| \sim 2|A_2| \preccurlyeq |I_2|$, which contradicts the assumption that $|I_2| \prec |I_1|$. This completes the proof of Claim 7.1.3.

□ (Claim 7.1.3)

Claim 7.1.4 $g_2 = 1$ implies $g_1 = 1$.

Proof of Claim 7.1.4 Suppose $g_1 > 1$ and $g_2 = 1$. If $|I_1| \succcurlyeq |I_2|$, then $|A_1| \preccurlyeq \frac{1}{3}|A|$ by Claim 7.1.1 and

$$\begin{aligned}|2A| &\succcurlyeq 2|A_1| + b_1 + 2|A_2| + 2|A_J| + b_J\\ &\succcurlyeq 3|A| + |A_2| - |A_1| + b_1 + b_J \succcurlyeq \frac{10}{3}|A| + b_1 + b_J.\end{aligned}$$

By (6) we conclude that $b_1 \sim b_J \sim 0$, which implies that A_1 is full in $I_1 \cap \{g_1 k : k \in {}^*\mathbb{N}\}$ and A_J is full in J. We also have $|J| \sim |A_J| \sim |I_2| \sim \frac{2}{3}|A| \sim 2|A_1| \sim \frac{2}{g_1}|I_1|$ and $b \sim \frac{1}{3}|A|$. Since $|I_2| \preccurlyeq |I_1|$, then $g_1 = 2$. Hence

$H+1 \sim u_0 - l_2 + l_2 - u_1 + u_1 \sim 2|A_J| + 2|A_1| = 2|A| \prec 2|A| + 2b$, which contradicts the failure of (7).

We can now assume $|I_1| \prec |I_2|$. Since $|2A| \succcurlyeq 3|A| + |A_1| + b_J$, then $b_J \preccurlyeq \frac{1}{3}|A_2| = \frac{1}{3}|J| - \frac{1}{3}s_J$, which implies $s_J \preccurlyeq \frac{1}{4}|J|$. Hence $R_J - L_J \succcurlyeq \frac{1}{2}|J|$.

By Claim 7.1.2 we can assume that $u_1 \succ 0$. If $|A_1| \sim 0$, then

$$\begin{aligned} |2A| &\succcurlyeq |A_2 + A_1| + |2A_J| \\ &\succcurlyeq |A_2| + A_2(l_2, l_2 + u_1) + 2|A_J| + b_J \\ &\succcurlyeq 3|A| + A_2(l_2, l_2 + u_1) + b_J \succcurlyeq 3|A| + |I_1|. \end{aligned}$$

Hence $|I_1| \preccurlyeq \frac{1}{3}|A_J| \preccurlyeq \frac{1}{3}|J|$. One the other hand, we have $|2A| \succcurlyeq 3|A| + A_2(L_2, L_2 + u_1) + b_J$. Hence

$$\begin{aligned} H+1 &\preccurlyeq 2(u_0 - l_2) - u_0 + 2l_2 \\ &\preccurlyeq 2|A_J| + 2b_J - 2p_J - (u_0 - l_2) + l_2 \\ &\preccurlyeq 2|A| + 2b - 2A_2(L_2, L_2 + u_1) - 2p_J - (l_2 - u_1) + l_2 \\ &\preccurlyeq 2|A| + 2b - 2u_1 + u_1 \prec 2|A| + 2b, \end{aligned}$$

which contradicts the failure of (7). Thus we can now assume that $|A_1| \succ 0$.

If $|I_1| \preccurlyeq \frac{1}{2}|J|$, then

$$\begin{aligned} |2A| &\succcurlyeq 2|A_1| + b_1 + |A_2| + A_2(L_J, L_J + u_1) + 2|A_J| + b_J \\ &\succcurlyeq 3|A| - |A_1| + b_J + A_2(L_J, L_J + u_1). \end{aligned}$$

Hence

$$\begin{aligned} H+1 &\preccurlyeq 2(u_0 - l_2) - u_0 + 2l_2 \\ &\preccurlyeq 2|A_J| + 2b_J - 2p_J - (u_0 - l_2) + l_2 \\ &\preccurlyeq 2|A| - 2|A_1| + 2b_J - 2p_J - (l_2 - u_1) + l_2 \\ &\preccurlyeq 2|A| + 2b - 2A_2(L_J, L_J + u_1) - 2p_J + u_1 \\ &\preccurlyeq 2|A| + 2b - 2u_1 + u_1 \prec 2|A| + 2b. \end{aligned}$$

Suppose $|J| \prec 2|I_1| \prec 2|I_2|$. For each $x \in [l_2, u_0 - u_1]$ let

$$A_x^i = A_J[x, x + u_1] \cap (\{i + g_1 n : n \in {}^*\mathbb{N}\}$$

and $k_x = |X|$ where $X = \{i \in [0, g_1 - 1] : A_x^i \neq \emptyset\}|$. Fix $x \in [l_2, u_0 - u_1]$ such that $k = k_x$ is maximal. Clearly, $k > \frac{1}{2} g_1$ because otherwise $|A_J| \preccurlyeq \frac{1}{2}|J|$. If there is $i \in X$ such that $|A_1 + A_x^i| \succcurlyeq |A_1| + 2|A_x^i|$, then

$$\begin{aligned}|2A| &\succcurlyeq 2|A_1| + |A_1| + |A_2| + A_2(x, x+u_1) + 2|A_J| + b_J \\ &\sim 3|A| + A_2(x, x+u_1) + b_J \succcurlyeq 3|A| + u_1.\end{aligned}$$

Hence

$$|I_1| \sim u_1 \preccurlyeq \frac{1}{3}|A| \preccurlyeq \frac{1}{3}|A_2| + \frac{1}{3}|A_1| \preccurlyeq \frac{1}{3}|I_2| + \frac{1}{6}|I_1| \prec \frac{2}{3}|I_1| + \frac{1}{6}|I_1| = \frac{5}{6}|I_1|,$$

which is absurd. So we can assume that for every $i \in X$, $|A_1 + A_x^i| \succcurlyeq \frac{1}{g_1}|I_1| + |A_x^i|$. Hence

$$|2A| \succcurlyeq 2|A_1| + \frac{k}{g_1}|I_1| + |A_2| + 2|A_J| + b_J \sim 3|A| - |A_1| + b_J + \frac{k}{g_1}u_1.$$

Hence

$$\begin{aligned}H + 1 &\sim 2(u_0 - l_2) - u_0 + 2l_2 \\ &\preccurlyeq 2|A_J| + 2b_J + u_1 \preccurlyeq 2|A| + 2b - \frac{2k}{g_1}u_1 + u_1 \prec 2|A| + 2b\end{aligned}$$

because $\frac{k}{g_1} > \frac{1}{2}$, which contradicts the failure of (7). This completes the proof of the claim.

$\square$ (Claim 7.1.4)

We continue to prove Theorem 7.1 under Case 7.1.1. By Claim 7.1.1, Claim 7.1.3, and Claim 7.1.4 we can assume that $g_1 = 1$ and $g_2 = 1$. By Claim 7.1.2 we have $|I_1| \succ 0$.

Suppose $|I_1| \preccurlyeq |I_2|$.

If $|2A_1| \succcurlyeq 3|A_1|$, then $|2A| \succcurlyeq 3|A| + |A_1| + b_J$. Hence $|A_1| \preccurlyeq \frac{1}{3}|A|$, $|A_2| \succcurlyeq \frac{2}{3}|A|$, $b_J \preccurlyeq \frac{1}{3}|A_2| \sim \frac{1}{3}|J| - \frac{1}{3}s_J$, and $s_J \preccurlyeq \frac{1}{4}|J|$, which implies $R_J - L_J \succcurlyeq \frac{1}{2}|J|$. If $|A_1 + A_2| \succcurlyeq |A_1| + |A_2| + A_2(l_2, l_2 + u_1)$, then

$$|2A| \succcurlyeq 3|A| + |A_1| + A_2(l_2, l_2 + u_1) + b_J \succcurlyeq 3|A| + |A_1| + u_1.$$

Hence $u_1 \preccurlyeq \frac{1}{3}|A_2| \preccurlyeq \frac{1}{3}|I_2|$. We have now

$$|2A| \succcurlyeq 3|A| + |A_1| + A_2(L_J, L_J + u_1) + b_J,$$

which implies that

$$\begin{aligned} H + 1 \sim 2(u_0 - l_2) - u_0 + 2l_2 &\preccurlyeq 2|A_J| + 2b_J - 2p_J + u_1 \\ &\preccurlyeq 2|A| + 2b - 4|A_1| - 2A_2(L_J, L_J + u_1) - 2p_J + u_1 \\ &\preccurlyeq 2|A| + 2b - u_1 \prec 2|A| + 2b \end{aligned}$$

where p_J is defined in (17) with i and I_i being replace by J. If $|A_1 + A_2| \succcurlyeq |I_1| + |A_2|$, then

$$|2A| \succcurlyeq 3|A_1| + |I_1| + |A_2| + 2|A_J| + b_J = 3|A| + |I_1| + b_J.$$

Hence

$$\begin{aligned} H + 1 \preccurlyeq 2(u_0 - l_2) - u_0 + 2l_2 &\preccurlyeq 2|A_2| + 2b_J + u_1 \\ &\preccurlyeq 2|A| + 2b - 2|A_1| - 2u_1 + u_1 \prec 2|A| + 2b. \end{aligned}$$

We can assume now that $|2A_1| \sim 2|A_1| + b_1 \prec 3|A_1|$ and $u_1 \preccurlyeq |A_1| + b_1$. If $|2A_J| \succcurlyeq 3|A_J|$, then $|2A| \succcurlyeq 2|A_1| + |A_1| + |A_2| + 3|A_J| \sim 3|A| + |A_2|$, which implies $|A_2| \preccurlyeq \frac{1}{3}|A|$ and $|A_1| \succcurlyeq \frac{2}{3}|A|$. On the other hand, we have $|2A| \succcurlyeq 2|A_1| + 2|A_1| + 3|A_2| \sim 3|A| + |A_1| \succ \frac{10}{3}|A|$, which contradicts (6). Hence we can assume that $|2A_J| \sim 2|A_J| + b_J \prec 3|A_J|$, which implies $u_0 - l_2 \preccurlyeq |A_J| + b_J$. Now we have

$$|2A| \succcurlyeq 2|A_1| + b_1 + |A_1| + |A_2| + 2|A_J| + b_J \sim 3|A| + b_1 + b_J.$$

Hence

$$\begin{aligned} H + 1 &\sim u_0 - l_2 + l_2 - u_1 + 2u_1 - u_1 \\ &\preccurlyeq 2|A_J| + 2b_J + 2|A_1| + 2b_1 - u_1 \sim 2|A| + 2b - u_1 \prec 2|A| + 2b, \end{aligned}$$

which contradicts the failure of (7).

Suppose $|I_2| \prec |I_1|$.

Notice that we have $u_2 - l_2 \prec u_1 < l_2 - u_1$ and $u_0 - l_2 \geqslant l_2 - u_1$. If $|2A_1| \succcurlyeq 3|A_1|$. Then $|2A| \succcurlyeq 3|A_1| + |A_1| + |A_2| + 2|A_2| \sim 3|A| + |A_1|$. Hence $|A_1| \preccurlyeq \frac{1}{3}|A|$ and $|A_2| \succcurlyeq \frac{2}{3}|A|$. One the other hand, we have

$$|2A| \succcurlyeq 3|A_1| + |\{0, u_1\} + A_2| + 2|A_2| \sim 3|A| + |A_2| \succ \frac{10}{3}|A|,$$

which contradicts the failure of (7). Thus we can now assume that $|2A_1| \sim 2|A_1| + b_1 \prec 3|A_1|$ and $|I_1| \preccurlyeq |A_1| + b_1$. Suppose $|2A_J| \succcurlyeq 3|A_J|$. Then $|2A| \succcurlyeq 2|A_1| + |A_1| + |A_2| + 3|A_2| \sim 3|A| + |A_2|$, which implies $|A_2| \preccurlyeq \frac{1}{3}|A|$ and $|A_1| \succcurlyeq \frac{2}{3}|A|$. If $|A_1 + A_J| \succcurlyeq |A_1| + |J|$, then

$$\begin{aligned}|2A| &\succcurlyeq 2|A_1| + |A_1| + |J| + 3|A_2| - (u_0 + u_1 - 2l_2)\\ &\sim 3|A| + (u_0 - l_2) - (u_0 + u_1 - 2l_2) = 3|A| + l_2 - u_1\\ &\succcurlyeq 3|A| + u_1 \succcurlyeq 3|A| + |A_1| \succ \frac{10}{3}|A|,\end{aligned}$$

which contradicts the failure of (7). Thus we can assume that $|A_1 + A_J| \succcurlyeq 2|A_1| + |A_J|$. Then

$$|2A| \succcurlyeq 2|A_1| + b_1 + 2|A_1| + |A_J| + |A_J| \sim 3|A| + b_1 + |A_1| - |A_2|.$$

By (6) we have $|A_1| \sim \frac{2}{3}|A|$, $|A_2| \sim \frac{1}{3}|A|$, and $b_1 \sim 0$. This implies that A_1 is full in I_1. Since

$$|2A| \succcurlyeq 2|A_1| + |A_1 + A_2| + 3|A_J| \sim 3|A| + b_{1,2} + |A_2| \succcurlyeq \frac{10}{3}|A| + b_{1,2},$$

we have that $b_{1,2} \sim 0$, which implies that A_2 is full in I_2. Hence $|I_1| \sim 2|I_2|$. Suppose $u_0 - u_2 \preccurlyeq u_2 - l_2$. Notice that $u_0 - l_2 \succcurlyeq l_2 - u_1 \succcurlyeq u_1 \sim 2(u_2 - l_2)$. Then $u_0 - u_2 \sim u_2 - l_2$. Hence

$$|2A| \succcurlyeq 2|A_1| + 2|A_1| + 3|A_2| \sim 3|A| + |A_1| \succ \frac{10}{3}|A|,$$

which contradicts the failure of (7). Suppose $u_0 - u_2 \succ u_2 - l_2$. Notice that $u_0 - u_2 \preccurlyeq u_1$ because A is not a subset of a *t.p.* Then

$$\begin{aligned}|2A| &\succcurlyeq 2|A_1| + |A_1 + A_J| + |2A_J| - u_0 - u_1 + 2l_2\\ &\succcurlyeq 3|A| + u_0 - l_2 - u_0 - u_1 + 2l_2\\ &= 3|A| + l_2 - u_1 \succcurlyeq 3|A| + |A_1| \succ \frac{10}{3}|A|,\end{aligned}$$

which contradicts (6). Therefore, we can now assume that $|2A_J| \sim 2|A_J| + b_J \prec 3|A_J|$ and $|J| \preccurlyeq |A_J| + b_J$.

Since $|2A| \succcurlyeq 2|A_1| + b_1 + |A_1| + |A_2| + 2|A_J| + b_J \sim 3|A| + b_1 + b_J$, then

$$\begin{aligned} H + 1 &\sim u_0 - l_2 + l_2 - u_1 + 2u_1 - u_1 \\ &\preccurlyeq 2|A_2| + 2b_J + 2|A_1| + 2b_1 - u_1 \preccurlyeq 2|A| + 2b - u_1 \prec 2|A| + 2b, \end{aligned}$$

which contradicts the failure of (7). This completes the proof of Theorem 7.1 under Case 7.1.1.

$\square$ (Case 7.1.1)

Case 7.1.2 $u_1 < l_0 \leqslant u_0 < l_2$.

Without loss of generality we assume that $|I_1| \geqslant |I_2|$. By Case 7.1.1 we can assume that $l_0 \succ u_1$. This is because otherwise we can find $a \sim 0$ such that $I_1' \cup I_2'$ is a *b.p.* where $I_2' = H - [a+1, u_0]$ and $I_1' = H - [l_2, u_2]$. Let $I_0' = H - [0, a]$. Then we apply Case 7.1.1 to $H - A \subseteq I_0' \cup I_1' \cup I_2'$. By the same reason we can assume that $u_0 \prec l_2$. If $2u_0 < l_2$, then A is a subset of the *b.p.* $[0, u_0] \cup [l_2, u_2]$. Hence we can assume that $2u_0 \geqslant l_2$.

Claim 7.1.5 $|I_2| \succ 0$.

Proof of Claim 7.1.5 Assume the contrary. Clearly, $|A_1| \sim |A|$, which implies that $g_1 = 1$ by Claim 7.1.1.

If $|2A_1| \succ \frac{7}{3}|A_1|$, then $|2A| \succ \frac{7}{3}|A_1| + |A_1| \sim \frac{10}{3}|A|$, which contradicts (6). Hence we can assume $|2A_1| \preccurlyeq \frac{7}{3}|A_1| = 2|A_1| + \frac{1}{3}|A_1|$, which implies that $|I_1| \preccurlyeq |A_1| + \frac{1}{3}|A_1| = \frac{4}{3}|A_1|$. Therefore, $s_1 \preccurlyeq \frac{1}{4}|I_1|$, which implies $R_1 - L_1 \succcurlyeq \frac{1}{2}|I_1|$.

If $l_0 > 2u_1$, then A is a subset of a *b.p.* $[0, u_1] \cup [l_0, u_2]$ unless $u_2 - l_0 \geqslant l_0 - u_1$. But $u_2 - l_0 \geqslant l_0 - u_1 > u_1$ implies that $|2A| \succcurlyeq |2A_1| + |\{u_0, u_2\} + A_1| \succcurlyeq 4|A|$. Hence we can assume that $l_0 \leqslant 2u_1$.

Suppose $2u_0 \sim u_2$. Let $a = 2u_0 - l_2$, $I_0' = [0, a]$, $I_1' = [a+1, u_0]$, and $I_2' = [l_2, u_2]$. Then $A \subseteq I_0' \cup I_1' \cup I_2'$, $I_1' \cup I_2'$ is a *b.p.* of difference $d = 1$, and $|I_0'| \sim 0$. This is now a symmetric case of Case 7.1.1.

We can now assume $2u_0 \succ u_2$. If $l_2 < u_2$, let $A' = A \setminus \{l_2\}$. Then $2l_2, l_2 + l_2' \in (2A) \setminus (2A')$ where $l_2' = \min(A_2 \setminus \{l_2\})$. Since $2u_0 \succ u_2$, A' is not a subset of a *b.p.* By Lemma 3.2 we can replace A by A'. By Repeating this step we can assume that $A_2 = \{u_2\}$.

Let $I_J = [0, u_0]$ and $A_J = A_1 \cup A_0$.

Suppose that $|2A_J| \succcurlyeq 3|A_J|$. If $u_2 - u_0 \succcurlyeq \frac{1}{2}u_1$, then

$$|2A| \succcurlyeq 3|A_J| + |A_1[u_1 - (u_2 - u_0), u_1] + u_2| \sim 3|A| + A_1(u_1 - (u_2 - u_0), u_1),$$

which implies that $A_1(u_1-(u_2-u_0), u_1) \preccurlyeq \frac{1}{3}|A_1|$. Hence $A_1(0, u_1-(u_2-u_0)) \succcurlyeq \frac{2}{3}|A_1|$. Since $|A_1| \succcurlyeq \frac{3}{4}u_1$, we have that

$$A_1(0, u_1 - (u_2 - u_0)) \succcurlyeq \frac{2}{3} \cdot \frac{3}{4}u_1 = \frac{1}{2}u_1 \succcurlyeq u_1 - (u_2 - u_0) \succcurlyeq A_1(0, u_1 - (u_2 - u_0)).$$

The consequence of this is that we have

$$u_2 - u_0 \sim \frac{1}{2}u_1 \sim A_1\left(0, \frac{1}{2}u_1\right) \sim 2A_1\left(\frac{1}{2}u_1, u_1\right).$$

Notice that $u_0 - u_1 \succcurlyeq \frac{1}{2}u_1$ because $u_2 - u_1 \succcurlyeq u_1$. Hence

$$\begin{aligned} |2A| \succcurlyeq u_1 + \left|u_1 + A_1\left[0, \frac{1}{2}u_1\right]\right| \\ + \left|u_0 + A_1\left[0, \frac{1}{2}u_1\right]\right| + |u_2 + A_1| \succcurlyeq 2u_1 + |A_1| \sim \frac{11}{3}|A|, \end{aligned}$$

which contradicts (6). Therefore, we can assume that $u_2 - u_0 \prec \frac{1}{2}u_1$ and this also implies that $u_2 - u_0 \prec u_0 - u_1$. If $l_0 < u_0$, let $A' = A \setminus \{u_0\}$. Then $2u_0, u_0 + u_0' \in (2A) \setminus (2A')$. Notice that A' is not a subset of a *b.p.* By Lemma 3.2, we can replace A by A'. By repeating this step we can assume that $A_0 = \{l_0\}$. Now we assume that $|A|$ is maximal such that $A_0 = \{l_0\}$, $A_2 = \{l_2\}$, $A_1 \subseteq I_1$, and A is a counterexample of Theorem 7.1. Because $|A_0| = 1$ and $|A_2| = 1$, we can, by Lemma 3.1, show that for every x, $L_1 \prec x \prec R_1$ implies $x \in A_1$. Notice that $R_1 - L_1 \succcurlyeq \frac{1}{2}u_1 \succ u_2 - l_0$. Let $x_{1,L}$ and $x_{1,R}$ be defined by (29). Then $x_{1,L} \preccurlyeq L_1$ and $x_{1,R} \succcurlyeq R_1$. By Lemma 3.1 and the maximality of $|A|$ we can show that $x_{1,L}, x_{1,R} \in A_1$. Hence we have that $A_1 = [0, u_1]$ and

$$|2A| \geqslant u_2 + u_1 + 3 = 3(u_1 + 3) - 3 + u_2 - 2u_1 - 3 = 3|A| - 3 + u_2 - 2u_1 - 3.$$

So $b \geqslant u_2 - 2u_1 - 3$ and

$$H + 1 = u_2 - 2(u_1 + 3) + 2|A| + 1 = 2|A| - 1 + b - 1 < 2|A| - 1 + 2b,$$

which contradicts the failure of (7).

Suppose $|2A_J| \prec 3|A_J|$. Let $|2A_J| \sim 2|A_J| + b_J$. Then $b_J \prec |A_J|$, $s_1 + (u_0 - u_1 - 1) = s_J \preccurlyeq b_J$, and $u_0 \preccurlyeq |A_J| + b_J$. Hence

$$\begin{aligned}|2A| &\succcurlyeq 2|A_J| + b_J + A_1(u_1 - (u_2 - u_0), u_1)\\ &\succcurlyeq 3|A| + s_1 + (u_0 - u_1) - A_1(0, u_1 - (u_2 - u_0)).\end{aligned}$$

This implies that

$$\begin{aligned}H + 1 &\preccurlyeq u_2 - 2u_1 + 2|A| + 2(b - (u_0 - u_1) + A_1(0, u_1 - (u_2 - u_0)))\\ &= 2|A| + 2b + u_2 - 2u_1 - 2u_0 + 2u_1 + 2A_1(0, u_1 - (u_2 - u_0))\\ &\preccurlyeq 2|A| + 2b + u_2 - 2u_0 + 2(u_1 - u_2 + u_0) = 2|A| + 2b + 2u_1 - u_2.\end{aligned}$$

Since $I_1 \cup I_2$ is a *b.p.*, we have $2u_1 < u_2$. If $2u_1 \prec u_2$, then we have $H + 1 \prec 2|A| + 2b$. Hence we can assume that $2u_1 \sim u_2$. If $A_1(0, u_1 - (u_2 - u_0)) \prec u_1 - (u_2 - u_0)$ we also have $H + 1 \prec 2|A| + 2b$. Hence we can assume that $A_1(0, u_1 - (u_2 - u_0)) \sim u_1 - (u_2 - u_0)$.

Let p_1 be defined in (17) with $i = 1$. If $p_1 \succ 0$, then

$$|2A| \succcurlyeq 2|A_1| + b_1 + |A_1| \sim 3|A| + b_1$$

implies

$$\begin{aligned}H + 1 &\sim u_2 - 2u_1 + 2A_1 + 2b_1 - 2p_1\\ &\preccurlyeq 2|A| + 2b + u_2 - 2u_1 - 2p \prec 2|A| + 2b\end{aligned}$$

because $u_2 - 2u_1 \sim 0$. Hence we can assume $A_1(L_1, R_1) \sim R_1 - L_1$.

Notice that $L_1 = L_J$ because $|A_J| \succ \frac{1}{2}|J|$. If $u_2 - u_0 \prec \frac{3}{4}u_1$, then $u_1 - (u_2 - u_0) \succ \frac{1}{4}u_1$. Notice that $A_1(0, u_1 - (u_2 - u_0)) \sim u_1 - (u_2 - u_0) \succ \frac{1}{4}u_1$ and $A_1(L_1, R_1) \sim R_1 - L_1 \succcurlyeq \frac{1}{2}u_1$. Hence $L_J \sim L_1 \sim 0$, which implies that $R_J \succ L_J$. Assume that $|A|$ is maximal for $A \subseteq J \cup I_2$ being a counterexample of Theorem 7.1 after adding elements one by one according to Lemma 3.1. Then we can show first that x is in A_1 when $L_J \prec x \prec R_J$ and show second that for any $x \in J$, $x \in A_1$. Thus we can assume that $A_J = [0, u_0]$. However, this process should stop in the middle when the conditions $|2A| \sim 3|A|$ but $|2A| \geqslant 3|A| - 3$ are achieved. Then the contradiction follows from Theorem 1.3. Hence we can now assume that $u_2 - u_0 \succcurlyeq \frac{3}{4}u_1$. Then $u_0 - u_1 \preccurlyeq \frac{1}{4}u_1$. But this implies $R_J \succcurlyeq R_1 - \frac{1}{4}u_1 \succ L_1 \sim L_J$. By the same argument as above we can add elements to A so that at a certain point we will have $|2A| \sim 3|A|$, which contradicts the failure of (7) by Theorem 1.3. This completes the proof of the claim.

□ (Claim 7.1.5)

Claim 7.1.6 $g_1 = \gcd(A_1) = 1$.

Proof of Claim 7.1.6 Assume the contrary that $g_1 > 1$.

Let $g_2 = \gcd(A_2 - l_2)$. Since $A_1 \cup A_2$ is a tight subset of a *b.p.* of difference 1 we have that $\gcd(g_1, g_2) = g_{1,2} = 1$. If $g_2 > 1$, then

$$|2A| \succcurlyeq 2|A_1| + |A_1| + |A_2| + \max\{|A_1|, |A_2|\} + 2|A_2| \succcurlyeq 3|A| + \frac{1}{2}|A| \succ \frac{10}{3}|A|,$$

which contradicts (6). Hence we can now assume $g_2 = 1$. Notice that we have $|A_1| \preccurlyeq \frac{1}{3}|A|$ and $|A_2| \succcurlyeq \frac{2}{3}|A|$ by the same argument as in the second part of the proof of Claim 6.1.2. Since

$$\begin{aligned}|2A| &\succcurlyeq |2A_1| + |\{0, u_1\} + A_2| + |2A_2| \\ &\succcurlyeq 2|A_1| + b_1 + 2|A_2| + 2|A_2| + b_2 \succcurlyeq 3|A| + |A_2| - |A_1| \succcurlyeq 3|A| + \frac{1}{3}|A|,\end{aligned}$$

we have, by (6), that $|2A_1| \sim 2|A_1|$, $|2A_2| \sim 2|A_2|$, $|A_1| \sim \frac{1}{3}|A|$, and $|A_2| \sim \frac{2}{3}|A|$. Hence we can conclude that A_2 is full in I_2, $g_1 = 2$, A_1 is full in $I_1 \cap (2\,{}^*\mathbb{N})$, and $u_2 - l_2 \sim u_1$. If $l_2 \prec 3u_1$, then $|2A| \succcurlyeq 3|A| + |A_1| + b_1 + b_2$ implies

$$\begin{aligned}H + 1 &\sim 2(u_2 - l_2) + 2\left(\frac{u_1}{2}\right) + (l_2 - u_1) - (u_2 - l_2) \\ &\prec 2(|A_2| + b_2) + 2(|A_1| + b_1) + u_1 \preccurlyeq 2|A| + 2b - 2|A_1| + u_1 \sim 2|A| + 2b,\end{aligned}$$

which contradicts the failure of (7). Hence we can assume that $l_2 \succcurlyeq 3u_1$. If $l_0 \prec \frac{3}{2}u_1$, then A is a subset of the *b.p.* $[0, u_0] \cup [l_2, u_2]$. If $l_0 \succ \frac{1}{2}(u_1 + u_2)$, then A is a subset of the $b.p.$ $[0, u_1] \cup [l_0, u_2]$. Hence we can assume $\frac{3}{2}u_1 \preccurlyeq l_0 \preccurlyeq \frac{1}{2}(u_1 + u_2)$. If $l_0 \preccurlyeq \frac{1}{2}u_2$, then

$$\begin{aligned}|2A| &\succcurlyeq 3|A| + |A_1| + |l_0 + A_1[u_1 - (l_0 - u_1), u_1]|\\ &\succcurlyeq \frac{10}{3}|A| + A_1(u_1 - (l_0 - u_1), u_1) \succ \frac{10}{3}|A|,\end{aligned}$$

which contradicts (6). If $l_0 \succ \frac{1}{2}u_2$, then

$$|2A| \succcurlyeq 3|A| + |A_1| + A_2(l_2, 2l_2 - u_0) \succ \frac{10}{3}|A|,$$

which again contradicts (6). This completes the proof of the claim.

□ (Claim 7.1.6)

Claim 7.1.7 We can assume $2u_1 \geqslant l_0$.

Proof of Claim 7.1.7 If $2u_1 < l_0$, then A is a subset of a $b.p.$ $[0, u_1] \cup [l_0, u_2]$ unless $u_2 - l_0 \geqslant l_0 - u_1$. Assume $2u_1 < l_0$ and $u_2 - l_0 \geqslant l_0 - u_1$.

If $g_2 > 1$, then $|A_2| \preccurlyeq \frac{1}{3}|A|$ and $|A_1| \succcurlyeq \frac{2}{3}|A|$. Hence

$$\begin{aligned}|2A| &\succcurlyeq |2A_1| + |(A_0 \cup A_2) + A_1| + |2A_2|\\ &\succcurlyeq 2|A_1| + b_1 + 2|A_1| + 2|A_2| + b_2\\ &\sim 3|A| + |A_1| - |A_2| + b_1 + b_2 \succcurlyeq \frac{10}{3}|A| + b_1 + b_2.\end{aligned}$$

By (6) we have $b_1 \sim b_2 \sim 0$, which implies that A_1 is full in I_1, A_2 is full in $I_2 \cap (l_2 + g_2 {}^*\mathbb{N})$, $|I_1| \sim |A_1| \sim \frac{2}{3}|A| \sim 2|A_2| \sim \frac{2}{g_2}|I_2| \leqslant \frac{2}{g_2}|I_1|$. This implies that $g_2 = 2$ and $|I_1| \sim |I_2|$. Now

$$|2A| \succcurlyeq |2(A_1 \cup A_2)| + |l_0 + A_1[0, l_2 - l_0]| \succcurlyeq \frac{10}{3}|A| + A_1(0, l_2 - l_0) \succ \frac{10}{3}|A|.$$

Suppose $g_2 = 1$. If $|2A_1| \succcurlyeq 3|A_1|$, then $|A_1| \geqslant |A_2|$ implies

$$|2A| \succcurlyeq 3|A_1| + |A_1| + |A_2| + 2|A_2| \sim 3|A| + |A_1| \succ \frac{10}{3}|A|$$

and $|A_1| < |A_2|$ implies

$$|2A| \succcurlyeq 3|A_1| + 2|A_2| + 2|A_2| \sim 3|A| + |A_2| \succ \frac{10}{3}|A|.$$

Hence we can assume that $|2A_1| \sim 2|A_1|+b_1 \prec 3|A_1|$. If $|2(A_0 \cup A_2)| \succcurlyeq 3|A_2|$, then $|A_1| \geqslant |A_2|$ implies

$$|2A| \succcurlyeq 2|A_1| + 2|A_1| + 3|A_2| \sim 3|A| + |A_1| \succ \frac{10}{3}|A|$$

and $|A_1| < |A_2|$ implies, because of $u_2 - l_2 \leqslant u_1 < l_0 - u_1$,

$$|2A| \succcurlyeq 2|A_1| + |A_1| + |A_2| + 3|A_2| \sim 3|A| + |A_2| \succ \frac{10}{3}|A|.$$

Hence we can assume that $|2(A_0 \cup A_2)| \sim 2|A_2| + b_{0,2} \prec 3|A_2|$. Now we have that

$$|2A| \succcurlyeq 2|A_1| + b_1 + |A_1 + (A_0 \cup A_2)| + 2|A_2| + b_{0,2} \succcurlyeq 3|A| + b_1 + b_{0,2}.$$

Hence

$$\begin{aligned} H + 1 &\sim 2u_2 - 2u_0 + 2u_0 - 2u_1 + 2u_1 - u_2 \\ &\preccurlyeq 2(|A_2| + b_{0,2}) + 2(|A_1| + b_1) + 2u_0 - 2u_1 - u_2 \\ &\preccurlyeq 2|A| + 2b + 2(u_0 - u_1) - u_2 \prec 2|A| + 2b \end{aligned}$$

because $u_2 = u_2 - u_0 + u_0 - u_1 + u_1 \succcurlyeq 2(u_0 - u_1) + u_1 \succ 2(u_0 - u_1)$. We now have a contradiction by the failure of (7). This completes the proof of the claim.

□ (Claim 7.1.7)

Claim 7.1.8 $g_2 = 1$.

Proof of Claim 7.1.8 Assume $g_2 > 1$. Then $|A_2| \preccurlyeq \frac{1}{3}|A|$, $|A_1| \succcurlyeq \frac{2}{3}|A|$, $s_1 \preccurlyeq \frac{1}{4}u_1$, $|A_1| \succcurlyeq \frac{3}{4}u_1$, and $R_1 - L_1 \succcurlyeq \frac{1}{2}u_1$.

We first prove the claim under the assumption that $u_2 - l_2 \succcurlyeq l_2 - 2u_1$. If $u_2 - l_2 \preccurlyeq R_1 - L_1$, then

$$\begin{aligned} |2A| &\succcurlyeq 2|A_1| + b_1 + |A_1 + A_2| \\ &\quad + |(l_0 + A_2) \cap [u_2 + u_1, 2l_2]| + 2|A_2| + b_2 \\ &\succcurlyeq 3|A| + b_1 + b_2 + A_1(L_1, L_1 + (u_2 - l_2)) - |A_2| \\ &\quad + |(l_0 + A_2) \cap [u_2 + u_1, 2l_2]|. \end{aligned}$$

Hence

$$\begin{aligned}
H+1 &\sim u_2 - 2u_1 + 2u_1 \\
&\preccurlyeq 2|A_1| + 2b_1 - 2p_1 + u_2 - l_2 + l_2 - 2u_1 \\
&\preccurlyeq 2|A| + 2b - 2b_2 - 2A_1(L_1, L_1 + (u_2 - l_2)) - 2p_1 \\
&\quad -2|(l_0 + A_2) \cap [u_2 + u_1, 2l_2]| + 2(u_2 - l_2) \\
&\preccurlyeq 2|A| + 2b - 2b_2 - 2|(l_0 + A_2) \cap [u_2 + u_1, 2l_2]|.
\end{aligned}$$

By the failure of (7), we have $b_2 \sim 0$, which implies that A_2 is full in $I_2 \cap (l_2 + g_2\, {}^*\mathbb{N})$, $u_2 - l_2 \sim l_2 - 2u_1$ and $|(l_0 + A_2) \cap [u_2 + u_1, 2l_2]| \sim 0$, which in turn imply that g_2 is hyperfinite because $l_0 + l_2 \prec 2l_2$, $l_0 + u_2 \succ u_1 + u_2$, and $2l_2 \succ u_1 + u_2$. So $|A_2| \sim 0$ and $|A_1| \sim |A|$. Let $I_J = [0, u_0]$ and $A_J = A_1 \cup A_0$. If $l_2 - l_0 \succcurlyeq u_1$, then

$$\begin{aligned}
|2A| &\succcurlyeq 2|A_J| + b_J + |A_J + A_2| \\
&\succcurlyeq 3|A| + b_J + A_J(L_J, L_J + (u_2 - l_2)).
\end{aligned}$$

Notice that $b_J \preccurlyeq \frac{1}{3}|A| \sim \frac{1}{3}|A_J|$ implies that $s_J \preccurlyeq \frac{1}{4}|I_J|$, $A_J \succcurlyeq \frac{3}{4}|I_J|$, and $R_J - L_J \succcurlyeq \frac{1}{2}|I_J|$. Notice also that $|I_J| \preccurlyeq |A_J| + b_J - p_J$. If $u_2 - l_2 \succcurlyeq R_J - L_J$, then $b_J + A_J(L_J, L_J + (u_2 - l_2)) \succcurlyeq R_J - L_J \succcurlyeq \frac{1}{2}|A|$, which implies that $|2A| \succ \frac{10}{3}|A|$, a contradiction to (6). If $u_2 - l_2 \prec R_J - L_J$, then

$$\begin{aligned}
H+1 &\preccurlyeq u_2 - 2u_0 + 2|A_J| + 2b_J - 2p_J \\
&\preccurlyeq 2|A| + 2b - 2A_J(L_J, L_J + (u_2 - l_2)) - 2p_J + u_2 - 2u_0 \\
&\preccurlyeq 2|A| + 2b - 2u_2 + 2l_2 + u_2 - 2u_0 \\
&= 2|A| + 2b - (u_2 - l_2) + (l_2 - 2u_1) - 2(u_0 - u_1) \prec 2|A| + 2b.
\end{aligned}$$

So we can now assume that $l_2 - l_0 \prec u_1$, which implies that

$$2l_0 \succ 2l_2 - 2u_1 \succcurlyeq 2(l_2 - 2u_1) + 2u_1 \sim l_2 - 2u_1 + u_2 - l_2 + 2u_1 = u_2.$$

If $l_0 \prec 2u_1$, then by Lemma 3.2 we can delete elements one by one from the left of A_2 (note that here we use the facts that (i) $|A_2| \sim 0$ to keep the truth of (6) and (ii) $2l_0 \succ u_2$ so that the remaining set will not become a subset of a *b.p.* of difference 1) so that we can assume that $A_2 = \{u_2\}$. But this contradicts Claim 7.1.5. So we can assume that $l_0 \sim 2u_1$. Hence

$$|2A| \succcurlyeq 2|A_1| + b_1 + |(A_0 \cup A_2) + A_1| \succcurlyeq 3|A| + b_1 + A_1(L_1, L_1 + u_2 - l_0).$$

If $u_2 - l_0 \succcurlyeq R_1 - L_1$, then $|2A| \succcurlyeq 3|A| + R_1 - L_1 \succ \frac{10}{3}|A|$. If $u_2 - l_0 \prec R_1 - L_1$, then

$$\begin{aligned} H + 1 &\sim u_2 - l_0 + 2u_1 \preccurlyeq u_2 - l_0 + 2(|A_1| + b_1 - p_1) \\ &\preccurlyeq 2|A| + 2b - 2p_1 - 2A_1(L_1, L_1 + u_2 - l_0) + u_2 - l_0 \prec 2|A| + 2b. \end{aligned}$$

So we can now assume that $u_2 - l_2 \succ R_1 - L_1$. If $g_2 > 2$, then

$$|2A| \succcurlyeq 2|A_1| + b_1 + |A_1| + 3|A_2| + 2|A_2| \sim 3|A| + b_1 + 2|A_2|,$$

which implies that $b_1 + 2|A_2| \preccurlyeq \frac{1}{3}|A|$, $|A_2| \preccurlyeq \frac{1}{6}|A|$, $|A_1| \succcurlyeq \frac{5}{6}|A|$, $s_1 \preccurlyeq \frac{1}{4}|I_1|$, and $R_1 - L_1 \succcurlyeq \frac{1}{2}|I_1|$. If $b_1 \prec \frac{1}{5}|I_1|$, then $R_1 - L_1 \succ \frac{3}{5}|I_1|$. Hence

$$\begin{aligned} |2A| &\succcurlyeq 2|A_1| + b_1 + A_1 + A_1(L_1, L_1 + (u_2 - l_2)) + 2|A_2| \\ &\succcurlyeq 3|A| + R_1 - L_1 - |A_2| \succ 3|A| + \frac{3}{5}|A_1| - \frac{1}{6}|A| \\ &\succcurlyeq 3|A| + \frac{1}{2}|A| - \frac{1}{6}|A| = \frac{10}{3}|A|, \end{aligned}$$

which contradicts (6). If $b_1 \succcurlyeq \frac{1}{5}|I_1|$, then

$$2|A_2| \preccurlyeq \frac{1}{3}|A| - b_1 \preccurlyeq \frac{1}{3}|A| - \frac{1}{5}|I_1| \preccurlyeq \frac{1}{3}|A| - \frac{1}{6}|A| = \frac{1}{6}|A|.$$

Hence $|A_2| \preccurlyeq \frac{1}{12}|A|$ and $|A_1| \succcurlyeq \frac{11}{12}|A|$. This implies that

$$\begin{aligned} |2A| &\succcurlyeq 2|A_1| + b_1 + |A_1| + A_1(L_1, L_1 + (u_2 - l_2)) + 2|A_2| \\ &\succcurlyeq 3|A| + R_1 - L_1 - |A_2| \succcurlyeq 3|A| + \frac{1}{2}|A_1| - |A_2| \\ &\succcurlyeq 3|A| + \frac{11}{24}|A| - \frac{2}{24}|A| \succ \frac{10}{3}|A|, \end{aligned}$$

which contradicts (6). We now assume that $g_2 = 2$. If $|2A_2| \succcurlyeq 3|A_2|$, then

$$\begin{aligned} |2A| &\succcurlyeq 2|A_1| + b_1 + |A_1| + A_1(0, u_2 - l_2) + 3|A_2| \\ &\succcurlyeq 3|A| + u_2 - l_2 \succ 3|A| + \frac{1}{2}|A_1| \succcurlyeq 3|A| + \frac{1}{3}|A|, \end{aligned}$$

which contradicts (6). Suppose that $|2A_2| \sim 2|A_2| + b_2 \prec 3|A_2|$. Let

$$A_1^0 = A_1[0, u_2 - l_2] \cap (2\,{}^*\mathbb{N}) \text{ and } A_1^1 = A_1[0, u_2 - l_2] \cap (1 + 2\,{}^*\mathbb{N}).$$

If $|A_2 + A_1^i| \succcurlyeq \frac{1}{2}|I_2| + |A_1^i|$ for $i = 0$ or 1, then $|A_1 + A_2| \succcurlyeq \frac{1}{2}|I_2| + |A_1| + |A_2|$. If $|A_2 + A_1^i| \succcurlyeq |A_2| + 2|A_1^i|$ for $i = 0, 1$, then

$$|A_2 + A_1| \succcurlyeq |A_1| + 2|A_2| + A_1(0, u_2 - l_2) \succ |A_1| + |A_2| + \frac{1}{2}|I_2|$$

because $u_2 - l_2 \succ R_1 - L_1 \succcurlyeq L_1$. Hence $|A_1 + A_2| \succcurlyeq |A_1| + |A_2| + \frac{1}{2}|I_2|$ is always true. Let $x \leqslant u_1$ be such that $A_1(x, u_1) \sim 0$ but $A_1(y, u_1) \succ 0$ for any $y \prec x$. Let $A_1' = A_1 \smallsetminus [x, u_1 - 1]$. Then $|A_1| \sim |A_1'|$. Let $|2A_1'| \sim 2|A_1'| + b_1' \sim 2|A_1| + b_1' \preccurlyeq |2A_1|$. Then $b_1' \preccurlyeq b_1$. This now implies that

$$\begin{aligned}|2A| \succcurlyeq\; & 2|A_1'| + b_1' + |A_1| + |A_2| + \frac{1}{2}|I_2| \\ & +2|A_2| + b_2 + |l_0 + A_1[u_1 + x - l_0, l_2 - l_0]| \\ \succcurlyeq\; & 3|A| + b_1' + b_2 + \frac{1}{2}|I_2| + A_1(u_1 + x - l_0, l_2 - l_0).\end{aligned}$$

Hence

$$\begin{aligned}H + 1 &\sim 2\left(\frac{u_2 - l_2}{2}\right) + l_2 - 2u_1 + 2u_1 \\ &\preccurlyeq 2(|A_2| + b_2) + l_2 - 2u_1 + 2(|A_1'| + b_1') \\ &\preccurlyeq 2|A| + 2b - |I_2| + l_2 - 2u_1 - 2A_1(u_1 + x - l_0, l_2 - l_0) \prec 2|A| + 2b\end{aligned}$$

because $u_2 - l_2 \succcurlyeq l_2 - 2u_1$ and $u_2 - l_2 \sim l_2 - 2u_1$ implies that $A_1(u_1 + x - l_0, l_2 - l_0) \succ 0$.

We now prove the claim under the assumption that $u_2 - l_2 \prec l_2 - 2u_1$. Since $l_0 \preccurlyeq 2u_1$, we have that $u_2 - l_2 \prec l_2 - l_0$. Let $I_J = [0, u_0]$ and $A_J = A_1 \cup A_0$.

Assume $l_0 + u_1 \preccurlyeq l_2$. Then

$$|2A| \succcurlyeq |2A_J| + |A_J + A_2| + |2A_2| \succcurlyeq 3|A| + b_J + |A_2|,$$

which implies that $b_J \preccurlyeq \frac{1}{3}|A_J|$, $|I_J| \sim l_0 \preccurlyeq |A_J| + b_J \preccurlyeq \frac{4}{3}|A_1| \preccurlyeq \frac{4}{3}u_1$, $s_J \preccurlyeq \frac{1}{4}|I_J|$, and $R_J - L_J \succcurlyeq \frac{1}{2}|I_J|$. If $u_2 - l_2 \preccurlyeq R_J - L_J$, then

$$\begin{aligned}|2A| &\succcurlyeq 2|A_J| + b_J + A_J + A_J(L_J, L_J + (u_2 - l_2)) + 2|A_2| \\ &\sim 3|A| + b_J + A_J(L_J, L_J + (u_2 - l_2)) - |A_2|.\end{aligned}$$

Hence

$$\begin{aligned}H+1 &\sim u_2-2l_0+2l_0 \preccurlyeq u_2-2l_0+2|A_J|+2b_J-2p_J\\ &\preccurlyeq 2|A|+2b-2p_J-2A_J(L_J,L_J+(u_2-l_2))+u_2-2l_0\\ &\preccurlyeq 2|A|+2b-2(u_2-l_2)+u_2-2l_0=2|A|+2b-u_2+2l_2-2l_0\\ &\preccurlyeq 2|A|+2b-(u_2-l_2)-(2l_0-l_2)\prec 2|A|+2b.\end{aligned}$$

Notice that the last step uses a fact that $2l_0 \sim 2u_0 \geqslant l_2$. If $u_2-l_2 \succ R_J-L_J$, then $l_2-2u_1 \succ u_2-l_2 \succ R_J-L_J \succcurlyeq \frac{1}{2}l_0 \succcurlyeq \frac{1}{4}l_2$, which implies that $l_2 \succ \frac{8}{3}u_1$. On the other hand, we have $l_0 \preccurlyeq \frac{4}{3}u_1$, which implies that $l_2 \preccurlyeq 2l_0 \preccurlyeq \frac{8}{3}u_1$, a contradiction.

Assume $l_0+u_1 \succ l_2$. This implies that

$$u_1+(l_2-2u_1)-(u_2-l_2) \succcurlyeq u_1 \succ l_2-l_0,$$

which implies $l_2+l_0 \succ u_1+u_2$. Hence

$$|2A| \succcurlyeq |(2A_J)\cup(A_1+A_2)|+|A_0+A_2|+|2A_2|.$$

Suppose $l_0 \preccurlyeq \frac{3}{2}u_1$. Then $l_2-l_0 \succcurlyeq 2u_1-\frac{3}{2}u_1=\frac{1}{2}u_1$. If $|2A_J| \succcurlyeq 3|A_J|$, then

$$\begin{aligned}|2A| &\succcurlyeq 3|A_J|+|A_2+A_1[u_1-(l_2-l_0),u_1]|+3|A_2|\\ &\succcurlyeq 3|A|+|A_2|+A_1\left(\frac{1}{2}u_1,u_1\right)\succcurlyeq 3|A|+\frac{2}{3}|A_2|+\frac{1}{3}|A|\end{aligned}$$

because $|A_1| \succcurlyeq \frac{3}{4}|I_1|$ implies that $A_1\left(\frac{1}{2}u_1,u_1\right) \succcurlyeq \frac{1}{3}|A_1|$. If $|A_2| \succ 0$, then $|2A| \succ \frac{10}{3}|A|$. If $|A_2| \sim 0$, we can delete elements one by one from the left of A_2 by Lemma 3.2 so that A_2 can be assumed to be $\{u_2\}$. But this contradicts Claim 7.1.5 where $|I_2|$ is showed to be $\succ 0$. If $|2A_J| \sim 2|A_J|+b_J \prec 3|A_J|$, then

$$\begin{aligned}|2A| \succcurlyeq\ & 2|A_J|+b_J+|A_2+A_1[u_1-(l_2-l_0),u_1]|\\ &+3|A_2| \sim 3|A|+b_J-A_1(0,u_1-(l_2-l_0)).\end{aligned}$$

Hence

$$\begin{aligned}H+1 &\preccurlyeq u_2-2l_0+2|A_J|+2b_J\\ &\preccurlyeq 2|A|+2b+2A_1(0,u_1-(l_2-l_0))+u_2-2l_0\\ &\preccurlyeq 2|A|+2b+2u_1-2l_2+2l_0+u_2-2l_0\\ &=2|A|+2b-2(l_2-2u_1)+(u_2-2u_1)\prec 2|A|+2b\end{aligned}$$

because $u_2 - l_2 \prec l_2 - 2u_1$. Suppose $l_0 \succ \frac{3}{2}u_1$. If $u_2 \succcurlyeq l_0 + u_1$, then $|2A_J| \succcurlyeq 3|A_J|$ implies that

$$|2A| \succcurlyeq 3|A_J| + |u_2 + A_1| + 3|A_2| \sim 3|A| + |A_1| \succ \frac{10}{3}|A|,$$

and $|2A_J| \sim 2|A_J| + b_J \prec 3|A_J|$ implies that

$$|2A| \succcurlyeq 2|A_J| + s_1 + (l_0 - u_1) + |A_1| + 3|A_2| \succ 3|A| + \frac{1}{2}u_1 \succcurlyeq \frac{10}{3}|A|.$$

So we can assume that $u_2 \prec l_0 + u_1$. If $u_2 - 2u_1 \succ \frac{1}{2}u_1$, then

$$\begin{aligned} |2A| &\succcurlyeq 2|A_1| + b_1 + |l_0 + A_1[u_1 - (l_0 - u_1), u_2 - l_0]| \\ &\quad + |u_2 + A_1| + 3|A_2| \succcurlyeq 3|A| + u_2 - 2u_1 \succ 3|A| + \frac{1}{2}u_1 \succcurlyeq \frac{10}{3}|A|. \end{aligned}$$

So we can assume that $u_2 - 2u_1 \preccurlyeq \frac{1}{2}u_1$. If $l_0 + L_1 \prec 2u_1 + \frac{1}{2}(l_2 - 2u_1) \prec l_0 + R_1$, then, because of $l_0 - u_1 \succ l_2 - 2u_1 \succ u_2 - l_2$, we have

$$\begin{aligned} |2A| &\succcurlyeq 2|A_1| + b_1 + |l_0 + A_1[2u_1 - l_0, l_2 - l_0]| + |A_2 + A_1| + 3|A_2| \\ &\sim 3|A| + b_1 + A_1(2u_1 - l_0, l_2 - l_0) + A_1(L_1, L_1 + (u_2 - l_2)). \end{aligned}$$

Hence, because of $L_1 \prec (2u_1 - l_0 + l_2 - l_0)/2 \prec R_1$, we have

$$\begin{aligned} H &= u_2 - 2u_1 + 2u_1 \\ &\preccurlyeq 2|A| + 2b - 2p_1 - 2A_1(2u_1 - l_0, l_2 - l_0) \\ &\quad -2A_1(L_1, L_1 + (u_2 - l_2)) + u_2 - 2u_1 \\ &\prec 2|A| + 2b - (l_2 - 2u_1) - (u_2 - l_2) + u_2 - 2u_1 = 2|A| + 2b. \end{aligned}$$

Hence we can assume either $l_0 + L_1 \succcurlyeq 2u_1 + \frac{1}{2}(l_2 - 2u_1)$ or $l_0 + R_1 \preccurlyeq 2u_1 + \frac{1}{2}(l_2 - 2u_1)$. Notice also that we are assuming that $\frac{3}{2}u_1 \prec l_0 \preccurlyeq 2u_1$, $u_2 \prec l_0 + u_1$, and $u_2 - 2u_1 \preccurlyeq \frac{1}{2}u_1$.

Suppose $l_0 + L_1 \succcurlyeq 2u_1 + \frac{1}{2}(l_2 - 2u_1)$. Let $s_1' = |[0, L_1 - 1] \setminus A_1|$ and $s_1'' = s_1 - s_1'$. If $R_1 - (u_2 - l_0) \preccurlyeq L_1$, then

$$\begin{aligned} |2A| &\succcurlyeq 2|A_1| + b_1 + |l_0 + A_1[2u_1 - l_0, R_1]| \\ &\quad + |u_2 + A_1[R_1 - (u_2 - l_0), u_1]| + 3|A_2| \\ &\succcurlyeq 3|A| + s_1' - A_1(0, 2u_1 - l_0) + s_1'' + A_1(R_1 - (u_2 - l_0), u_1) \\ &\succ 3|A| + \frac{1}{2}(u_1 - L_1) \succcurlyeq 3|A| + \frac{1}{2}u_1 \succcurlyeq \frac{10}{3}|A|. \end{aligned}$$

If $R_1 - (u_2 - l_0) \succ L_1$, then

$$|2A| \succcurlyeq 3|A| + b_1 - A_1(0, 2u_1 - l_0) + A_1(R_1 - (u_2 - l_0), u_1),$$

which implies that

$$\begin{aligned} H + 1 &\preccurlyeq u_2 - 2u_1 + 2(A_1 + b_1 - p_1) \\ &\preccurlyeq 2|A| + 2b + 2A_1(0, 2u_1 - l_0) - 2p_1 \\ &\quad -2A_1(R_1 - (u_2 - l_0), R_1) - 2A_1(R_1, u_1) + u_2 - 2u_1 \\ &\preccurlyeq 2|A| + 2b + 4u_1 - 2l_0 - 2(u_2 - l_0) - (u_1 - R_1) + u_2 - 2u_1 \\ &= 2|A| + 2b + u_1 + R_1 - u_2 \\ &\preccurlyeq 2|A| + 2b + 2u_1 - u_2 \prec 2|A| + 2b. \end{aligned}$$

Suppose $l_0 + R_1 \preccurlyeq 2u_1 + \frac{1}{2}(l_2 - 2u_1)$. This implies $R_1 \prec l_2 - l_0$. Notice also that $\frac{1}{2}u_1 \preccurlyeq R_1 - L_1 \preccurlyeq R_1$. If $|2A_J| \succcurlyeq 3|A_J|$, then

$$\begin{aligned} |2A| &\succcurlyeq 3|A_J| + |A_2 + A_1[u_1 - (l_2 - l_0), u_1]| + 3|A_2| \\ &\succ 3|A| + A_1\left(\frac{1}{2}u_1, u_1\right) + 2|A_2| \succcurlyeq \frac{10}{3}|A|. \end{aligned}$$

If $|2A_J| \sim 2|A_J| + b_J \prec 3|A_J|$, then

$$\begin{aligned} |2A| &\succcurlyeq 2|A_J| + b_J + |u_2 + A_1[u_1 - (u_2 - l_0), u_1]| + 3|A_2| \\ &\succcurlyeq 3|A| + b_J - A_1(0, u_1 - (u_2 - l_0)). \end{aligned}$$

Hence

$$\begin{aligned} H + 1 &\preccurlyeq u_2 - 2l_0 + 2A_J + 2b_J \\ &\preccurlyeq 2|A| + 2b + 2A_1(0, u_1 - (u_2 - l_0)) + u_2 - 2l_0 \\ &\preccurlyeq 2|A| + 2b + 2(u_1 - u_2 + l_0) + u_2 - 2l_0 \\ &= 2|A| + 2b + 2u_1 - u_2 \prec 2|A| + 2b. \end{aligned}$$

This completes the proof of the claim.

□ (Claim 7.1.8)

We are left to prove the theorem when $g_1 = g_2 = 1$. Notice that we have assumed, without loss of generality, that $|I_1| \geqslant |I_2|$ and $2u_0 \geqslant l_2$, which implies that $l_2 - l_0 \precsim l_0$ and $2(l_2 - l_0) \precsim l_2 - l_0 + l_0 = l_2 \prec u_2$. We also assume that $l_0 \succ u_1$ and $u_0 \prec l_2$ by Case 7.1.1.

If $|2A_1| \succsim 3|A_1|$, then $|2A| \succsim 3|A_1| + |A_1 + A_2| + 2|A_2| + b_2$. If $|A_1| \succ \frac{1}{3}|A|$, then $|2A| \succsim 3|A| + |A_1| \succ \frac{10}{3}|A|$. If $|A_1| \precsim \frac{1}{3}|A|$, then $|A_2| \succsim \frac{2}{3}|A|$, which implies that

$$|2A| \succsim 3|A_1| + |\{0, u_1\} + A_2| + 2|A_2| \sim 3|A| + |A_2| \succ \frac{10}{3}|A|.$$

Therefore, we can assume that $|2A_1| = 2|A_1| - 1 + b_1 \prec 3|A_1|$.

If $2u_1 < l_0$, then $[0, u_1] \cup [l_0, u_2]$ is a *b.p.* of difference 1 unless $u_2 - l_0 \geqslant l_0 - u_1$. Suppose $2u_1 < l_0$ and $u_2 - l_0 \geqslant l_0 - u_1 > u_1$. If $|(A_0 \cup A_2) + A_1| \succsim |A_2| + 2|A_1|$, then $|2A| \succsim 3|A| + |A_1|$. Hence $|A_1| \precsim \frac{1}{3}|A|$ and $|A_2| \succsim \frac{2}{3}|A|$, which imply that

$$\begin{aligned}|2A| &\succsim 2|A_1| + b_1 + |\{0, u_1\} + A_2| + 2|A_2| + b_2 \\ &\succsim 3|A| + |A_2| - |A_1| + b_1 + b_2 \succsim \frac{10}{3}|A| + b_1 + b_2.\end{aligned}$$

By (6) we have $b_1 \sim b_2 \sim 0$ and $|A_1| \sim \frac{1}{3}|A| \sim 2|A_2|$. Hence A_1 is full in I_1 and A_2 is full in I_2. Now we have $2|I_1| \sim 2|A_1| \sim |A_2| \sim |I_2|$, which contradicts the assumption that $|I_1| \geqslant |I_2|$. Thus we can assume, by Theorem A.4, that $|(A_0 \cup A_2) + A_1| \succsim u_2 - l_0 + |A_1|$. Since $|2A| \succsim 3|A| + b_1 + b_2 + u_2 - l_0 - |A_2|$, then

$$\begin{aligned}H + 1 &\sim u_2 - 2u_1 + 2u_1 \\ &\precsim 2|A_1| + 2b_1 + u_2 - 2u_1 \\ &\precsim 2|A| + 2b - 2(u_2 - l_0) - 2b_2 + u_2 - 2u_1 \\ &\precsim 2|A| + 2b - (u_2 - l_0) + (l_0 - u_1) - u_1 \\ &\precsim 2|A| + 2b - u_1 \prec 2|A| + 2b.\end{aligned}$$

Therefore, we can now assume that $2u_1 \geqslant l_0$.

Suppose $|2A_2| \succsim 3|A_2|$. Then $|2A| \succsim 3|A| + b_1 + |A_2|$, which implies $|A_2| \precsim \frac{1}{3}|A|$, $|A_1| \succsim \frac{2}{3}|A|$, $b_1 \precsim \frac{1}{3}|A_1|$, and $R_1 - L_1 \succsim \frac{1}{2}u_1$. Since $|A_1 + A_2| \succsim$

$|A_1| + A_1(x, x + (u_2 - l_2))$ for any $x \in [0, u_1 - (u_2 - l_2)]$, we have that

$$\begin{aligned}|2A| &\succcurlyeq 2|A_1| + b_1 + |A_1| + A_1(x, x + (u_2 - l_2)) + 3|A_2| \\ &\succcurlyeq 3|A| + b_1 + A_1(x, x + (u_2 - l_2)) \succcurlyeq 3|A| + u_2 - l_2.\end{aligned}$$

If $u_2 - l_2 \succ \frac{1}{2}u_1$, then $|2A| \succ 3|A| + \frac{1}{2}u_1 \succcurlyeq \frac{10}{3}|A|$. So we can assume that $u_2 - l_2 \preccurlyeq \frac{1}{2}u_1$. If $2(l_2 - u_1) \preccurlyeq u_2$, which implies that $u_2 - 2u_1 \preccurlyeq 2(u_2 - l_2)$, then

$$\begin{aligned}H + 1 &\sim u_2 - 2u_1 + 2|A_1| + 2b_1 - 2p_1 \\ &\preccurlyeq 2|A| + 2b - 2|A_2| - 2p_1 - 2A_1(L_1, L_1 + (u_2 - l_2)) + u_2 - 2u_1 \\ &\preccurlyeq 2|A| + 2b - 2|A_2| - 2(u_2 - l_2) + u_2 - 2u_1.\end{aligned}$$

By the failure of (7) we have that $|A_2| \sim 0$ and $2(u_2 - l_2) \sim u_2 - 2u_1$, which implies that $u_2 - l_2 \sim l_2 - 2u_1$. If $l_0 \prec 2u_1$, then we can delete elements one by one from the left of A_2 by Lemma 3.2 so that we can assume $A_2 = \{u_2\}$. This contradicts Claim 7.1.5. Thus we can assume that $l_0 \sim 2u_1$. If $u_2 - l_0 \succcurlyeq \frac{1}{2}u_1$, then

$$|2A| \succcurlyeq 2|A_1| + b_1 + |\{l_0, u_2\} + A_1| \succcurlyeq 3|A| + \frac{1}{2}u_1 \succ \frac{10}{3}|A|.$$

If $u_2 - l_0 \prec \frac{1}{2}u_1$, then

$$\begin{aligned}|2A| &\succcurlyeq 2|A_1| + b_1 + |A_1| + A_1(L_1, L_1 + (u_2 - l_0)) \\ &\sim 3|A| + b_1 + A_1(L_1, L_1 + (u_2 - l_0)).\end{aligned}$$

Hence

$$\begin{aligned}H + 1 &\preccurlyeq u_2 - 2u_1 + 2|A_1| + 2b_1 - 2p_1 \\ &\preccurlyeq 2|A| + 2b - 2p_1 - 2A_1(L_1, L_1 + (u_2 - l_0)) + u_2 - 2u_1 \\ &\preccurlyeq 2|A| + 2b - (u_2 - l_0) \prec 2|A| + 2b.\end{aligned}$$

Now we can assume that $2(l_2 - u_1) \succ u_2$, which implies that $u_2 - l_2 \prec l_2 - 2u_1$. We can assume that $|A_2| \succ 0$ because otherwise we can, by Lemma 3.2, delete elements one by one from the left of A_2 so that we can eventually assume that $A_2 = \{u_2\}$, which contradicts Claim 7.1.5. Suppose that $u_2 - l_2 \succcurlyeq l_0 - u_1$. Then

$$l_0 + u_1 = l_0 - u_1 + 2u_1 \preccurlyeq u_2 - l_2 + 2u_1 \prec l_2 - 2u_1 + 2u_1 = l_2,$$

which implies that $|2A| \succcurlyeq |2A_J| + |A_1 + A_2| + |2A_2| \succcurlyeq 3|A| + b_J + |A_2|$, $b_J \preccurlyeq \frac{1}{3}|A_J|$, and $R_J - L_J \succcurlyeq \frac{1}{2}l_0$. Since

$$\begin{aligned}|2A| &\succcurlyeq |2A_J| + |A_1 + A_2| + |2A_2| \\ &\succcurlyeq 2|A_J| + b_J + |A_1| + A_1(L_J, L_J + (u_2 - l_2)) + 3|A_2| \\ &\succcurlyeq 3|A| + b_J + A_1(L_J, L_J + (u_2 - l_2)),\end{aligned}$$

we have that

$$\begin{aligned}H + 1 &\preccurlyeq u_2 - 2l_0 + 2|A_J| + 2b_J - 2p_J \\ &\preccurlyeq 2|A| + 2b - 2p_J - 2A_1(L_J, L_J + (u_2 - l_2)) + u_2 - 2l_0 \\ &\preccurlyeq 2|A| + 2b - u_2 + 2(l_2 - l_0) \\ &\preccurlyeq 2|A| + 2b - u_2 + (l_2 - l_0) + l_0 \prec 2|A| + 2b.\end{aligned}$$

We can now assume that $u_2 - l_2 \prec l_0 - u_1$. Then $l_0 + A_2$ is disjoint from $A_1 + A_2$ and from $2A_2$. Suppose $l_0 + u_1 \preccurlyeq l_2$. If $|2A_J| \succcurlyeq 3|A_J|$, then

$$|2A| \succcurlyeq 3|A_J| + |A_2 + A_1| + |l_0 + A_2| + |2A_2| \succcurlyeq 4|A| \succ \frac{10}{3}|A|.$$

So we can assume $|2A_J| \sim 2|A_J| + b_J \prec 3|A_J|$. Hence

$$|2A| \succcurlyeq 2|A_J| + b_J + |A_1| + |A_2| + 3|A_2| \succcurlyeq 3|A| + b_J + |A_2|,$$

which implies that $|A_2| \preccurlyeq \frac{1}{3}|A|$, $|A_J| \succcurlyeq \frac{2}{3}|A|$, $b_J \preccurlyeq \frac{1}{3}|A_J|$, and $R_J - L_J \succcurlyeq \frac{1}{2}l_0$. Thus $|2A| \succcurlyeq 3|A| + b_J + A_1(L_J, L_J + (u_2 - l_2))$, which implies that

$$\begin{aligned}H + 1 \preccurlyeq{} & u_2 - 2l_0 + 2|A| + 2b_J - 2p_J \\ & -2A_1(L_J, L_J + (u_2 - l_2)) \preccurlyeq 2|A| + 2b - u_2 + 2(l_2 - l_0) \prec 2|A| + 2b.\end{aligned}$$

Suppose $l_2 \prec l_0 + u_1 \preccurlyeq u_2$. If $|2A_J| \succcurlyeq 3|A_J|$, then

$$|2A| \succcurlyeq 3|A_J| + |u_2 + A_1| + |l_0 + A_2| + |2A_2| \succcurlyeq 4|A|.$$

If $|2A_J| \sim 2|A_J| + b_J \prec 3|A_J|$, then

$$|2A| \succcurlyeq 2|A_J| + b_J + |u_2 + A_1| + 4|A_2| \sim 3|A| + b_J + |A_2|,$$

which implies that $R_J - L_J \succcurlyeq \frac{1}{2}l_0$. Hence

$$\begin{aligned}|2A| &\succcurlyeq 2|A_J| + b_J + |A_1 + A_2| - (u_1 + l_0 - l_2) + 4|A_2| \\ &\succcurlyeq 3|A| + b_J + A_1(L_J, L_J + (u_2 - l_2)) - (u_1 + l_0 - l_2) + |A_2|,\end{aligned}$$

which implies that

$$\begin{aligned}H + 1 &\preccurlyeq u_2 - 2l_0 + 2A_J + 2b_J - 2p_J \\ &\quad -2A_1(L_J, L_J + (u_2 - l_2)) + 2(u_1 + l_0 - l_2) \\ &\preccurlyeq 2|A| + 2b + u_2 - 2l_0 - 2u_2 + 2l_2 + 2u_1 + 2l_0 - 2l_2 \\ &= 2|A| + 2b - u_2 + 2u_1 \prec 2|A| + 2b.\end{aligned}$$

Suppose $l_0 + u_1 \succ u_2$.

Assume $l_0 \preccurlyeq \frac{3}{2}u_1$. Then $u_2 - l_2 \prec \frac{1}{2}(u_2 - 2u_1) \prec \frac{1}{2}(l_0 - u_1) \preccurlyeq \frac{1}{4}u_1$ and $l_2 - l_0 \succ 2u_1 - \frac{3}{2}u_1 = \frac{1}{2}u_1$. If $|2A_J| \succcurlyeq 3|A_J|$, then

$$\begin{aligned}|2A| &\succcurlyeq 3|A_J| + |A_2 + A_1[u_1 - (l_2 - l_0), u_1]| + 4|A_2| \\ &\succ 3|A| + |A_2| + A_1\left(\frac{1}{2}u_1, u_1\right) \succcurlyeq \frac{10}{3}|A|.\end{aligned}$$

If $|2A_J| \sim 2|A_J| + b_J \prec 3|A_J|$, then

$$\begin{aligned}|2A| &\succcurlyeq 2|A_J| + b_J + |A_1[u_1 - (u_2 - l_0), u_1] + u_2| \\ &\quad +4|A_2| \succcurlyeq 3|A| + b_J - A_1(0, u_1 - u_2 + l_0).\end{aligned}$$

Hence

$$\begin{aligned}H + 1 &\preccurlyeq u_2 - 2l_0 + 2|A_J| + 2b_J \\ &\preccurlyeq 2|A| + 2b + 2A_1(0, u_1 - u_2 + l_0) + u_2 - 2l_0 \\ &\preccurlyeq 2|A| + 2b + 2u_1 - 2u_2 + 2l_0 + u_2 - 2l_0 \\ &= 2|A| + 2b - u_2 + 2u_1 \prec 2|A| + 2b.\end{aligned}$$

Assume $l_0 \succ \frac{3}{2}u_1$. If $l_0 + L_1 \preccurlyeq \frac{1}{2}(2u_1 + l_2) \preccurlyeq l_0 + R_1$, then

$$\begin{aligned}|2A| &\succcurlyeq 2|A_1| + b_1 + A_1(2u_1 - l_0, l_2 - l_0) + |A_1| \\ &\quad +A_1(L_1, L_1 + (u_2 - l_2)) + 4|A_2| \\ &\succcurlyeq 3|A| + b_1 + A_1(2u_1 - l_0, l_2 - l_0) + A_1(L_1, L_1 + (u_2 - l_2)) + |A_2|,\end{aligned}$$

which implies that

$$\begin{aligned}H+1 &\preccurlyeq u_2-2u_1+2|A_1|+2b_1-2p_1\\ &\preccurlyeq 2|A|-2|A_2|+2b+u_2-2u_1-2p_1\\ &\quad -2A_1(2u_1-l_0,l_2-l_0)-2A_1(L_1,L_1+(u_2-l_2))\\ &\prec 2|A|+2b+u_2-2u_1-(l_2-2u_1)-(u_2-l_2)=2|A|+2b.\end{aligned}$$

Thus we can now assume that either $l_0+L_1 \succ \frac{1}{2}(2u_1+l_2)$ or $l_0+R_1 \prec \frac{1}{2}(2u_1+l_2)$. Suppose $l_0+L_1 \succ \frac{1}{2}(2u_1+l_2)$. If $l_2-l_0 \succcurlyeq R_1-L_1$, then

$$\begin{aligned}|2A| &\succcurlyeq 2|A_1|+b_1+|l_0+A_1[2u_1-l_0,u_1]|\\ &\quad +|A_2+A_1[u_1-(l_2-l_0),u_1]|+4|A_2|\\ &\succcurlyeq 3|A|+b_1-A_1(0,2u_1-l_0)+|A_2|+A_1(u_1-(l_2-l_0),u_1).\end{aligned}$$

Notice that $2u_1-l_0<\frac{1}{2}(2u_1+l_2)-l_0 \prec L_1 \preccurlyeq \frac{1}{2}u_1$ and $l_2-l_0 \succcurlyeq R_1-L_1 \succcurlyeq \frac{1}{2}u_1$. Let $s_1'=|[0,L_1]\setminus A_1|$ and $s_1''=s_1-s_1'$. Then

$$\begin{aligned}|2A| &\succcurlyeq 3|A|+s_1'-A_1(0,2u_1-l_0)+|A_2|\\ &\quad +s_1''+A_1(u_1-(l_2-l_0),u_1) \succcurlyeq 3|A|+|A_2|+\frac{1}{2}u_1 \succ \frac{10}{3}|A|.\end{aligned}$$

If $l_2-l_0 \prec R_1-L_1$, then

$$\begin{aligned}|2A| &\succcurlyeq 2|A_1|+b_1+|l_0+A_1[2u_1-l_0,R_1]|\\ &\quad +|A_2+A_1[R_1-(l_2-l_0),u_1]|+4|A_2|\\ &\succcurlyeq 3|A|+b_1-A_1(0,2u_1-l_0)\\ &\quad +A_1(R_1-(u_2-l_2),R_1)+A_1(R_1-(l_2-l_0),u_1),\end{aligned}$$

which implies that

$$\begin{aligned}H+1 &\preccurlyeq u_2-2u_1+2|A_1|+2b_1-2p_1\\ &\prec 2|A|+2b+u_2-2u_1+2A_1(0,2u_1-l_0)\\ &\quad -2A_1(R_1-(u_2-l_2),R_1)-2p_1-2A_1(R_1-(l_2-l_0),u_1)\\ &\preccurlyeq 2|A|+2b+u_2-2u_1+4u_1-2l_0-(u_2-l_2)-(u_1-R_1)-2(l_2-l_0)\\ &= 2|A|+2b+u_1+R_1-l_2 \preccurlyeq 2|A|+2b+2u_1-l_2 \prec 2|A|+2b.\end{aligned}$$

Suppose $l_0 + R_1 \prec \frac{1}{2}(2u_1 + l_2)$. This implies $\frac{1}{2}u_1 \preccurlyeq R_1 \prec l_2 - l_0$. If $|2A_J| \succcurlyeq 3|A_J|$, then

$$\begin{aligned}|2A| &\succcurlyeq 3|A_J| + |A_2 + A_1[u_1 - (l_2 - l_0), u_1]| + 4|A_2| \\ &\succ 3|A| + A_1\left(\frac{1}{2}u_1, u_1\right) + 2|A_2| \succcurlyeq \frac{10}{3}|A|.\end{aligned}$$

If $|2A_J| \sim 2|A_J| + b_J \prec 3|A_J|$, then

$$\begin{aligned}|2A| &\succcurlyeq 2|A_J| + b_J + |u_2 + A_1[u_1 - (u_2 - l_0), u_1]| + 4|A_2| \\ &\succcurlyeq 3|A| + b_J - A_1(0, u_1 - (u_2 - l_0)).\end{aligned}$$

Hence

$$\begin{aligned}H + 1 &\preccurlyeq u_2 - 2l_0 + 2A_J + 2b_J \\ &\prec 2|A| + 2b + 2A_1(0, u_1 - (u_2 - l_0)) + u_2 - 2l_0 \\ &\preccurlyeq 2|A| + 2b + 2u_1 - 2u_2 + 2l_0 + u_2 - 2l_0 \\ &= 2|A| + 2b + 2u_1 - u_2 \prec 2|A| + 2b.\end{aligned}$$

Now we can assume that $|2A_2| \sim 2|A_2| + b_2 \prec 3|A_2|$. If $2(l_2 - u_1) \preccurlyeq u_2$, then

$$\begin{aligned}|2A| &\succcurlyeq 2|A_1| + b_1 + |A_1| + |A_2| + b_{1,2} + 2|A_2| + b_2 \\ &\quad +|l_0 + A_1[2u_1 - l_0, l_2 - l_0]| \\ &\succcurlyeq 3|A| + b_1 + b_2 + b_{1,2} + A_1(2u_1 - l_0, l_2 - l_0),\end{aligned}$$

which implies that

$$\begin{aligned}H + 1 &\sim 2(u_2 - l_2) + 2u_1 - u_2 + 2l_2 - 2u_1 \\ &\preccurlyeq 2|A_1| + 2b_1 + 2|A_2| + 2b_2 - u_2 + 2(l_2 - u_1) \\ &\preccurlyeq 2|A| + 2b - 2b_{1,2} - 2A_1(2u_1 - l_0, l_2 - l_0) - u_2 + 2(l_2 - u_1).\end{aligned}$$

By the failure of (7) we conclude that $b_{1,2} \sim 0$, which implies that A_1 is full in I_1 and A_2 is full in I_2, and $u_2 \sim 2(l_2 - u_1)$, which implies that $l_2 \succ 2u_1$. Hence $A_1(2u_1 - l_0, l_2 - l_0) \succ 0$, which implies that $H + 1 \prec 2|A| + 2b$.

We can now assume that $2(l_2 - u_1) \succ u_2$.

Suppose that $u_2 - l_2 \succcurlyeq l_0 - u_1$. Then $l_2 - 2u_1 \succ u_2 - l_2 \succcurlyeq l_0 - u_1$ and hence $l_0 + u_1 \prec l_2$. If $|2A_J| \succcurlyeq 3|A_J|$, then

$$\begin{aligned} |2A| &\succcurlyeq 3|A_J| + \max\{|A_1| + |A_2|, 2|A_2|\} + 2|A_2| \\ &\sim 3|A| + \max\{|A_1|, |A_2|\} \succ \frac{10}{3}|A|. \end{aligned}$$

Hence we can assume that $|2A_J| \sim 2|A_J| + b_J \prec 3|A_J|$. So

$$|2A| \succcurlyeq 2|A_J| + b_J + |A_1| + |A_2| + 2|A_2| + b_2 \sim 3|A| + b_J + b_2,$$

which implies

$$\begin{aligned} H + 1 &\sim 2(u_2 - l_2) + 2l_0 - u_2 + 2l_2 - 2l_0 \\ &\preccurlyeq 2|A| + 2b - (u_2 - l_2) - (2l_0 - l_2) \prec 2|A| + 2b. \end{aligned}$$

Suppose that $u_2 - l_2 \prec l_0 - u_1$. Then $l_0 + A_2$ is disjoint from $A_1 + A_2$ and from $2A_2$. Notice that

$$|2A| \succcurlyeq 2|A_1| + b_1 + |A_1| + |A_2| + |l_0 + A_2| + 2|A_2| \succcurlyeq 3|A| + |A_2|$$

implies that $|A_2| \preccurlyeq \frac{1}{3}|A|$, $|A_1| \succcurlyeq \frac{2}{3}|A|$, $s_1 \preccurlyeq \frac{1}{4}|I_1|$, and $R_1 - L_1 \succcurlyeq \frac{1}{2}u_1$. Notice also that

$$|2A| \succcurlyeq 2|A_1| + b_1 + |A_1| + A_1(0, u_2 - l_2) + |l_0 + A_2| + 2|A_2| \succcurlyeq 3|A| + u_2 - l_2,$$

which implies that

$$u_2 - l_2 \preccurlyeq \frac{1}{3}|A| \sim \frac{1}{3}|A_1| + \frac{1}{3}|A_2| \preccurlyeq \frac{1}{3}|A_1| + \frac{1}{6}|A_1| = \frac{1}{2}|A_1| \preccurlyeq \frac{1}{2}u_1.$$

Suppose $l_0 + u_1 \preccurlyeq l_2$. If $|2A_J| \succcurlyeq 3|A_J|$, then

$$\begin{aligned} |2A| &\succcurlyeq 3|A_J| + |A_1 + A_2| + |l_0 + A_2| + |2A_2| \\ &\succcurlyeq 3|A_J| + \max\{|A_1| + |A_2|, 2|A_2|\} + 2|A_2| \\ &\succcurlyeq 3|A| + \max\{|A_1|, |A_2|\} \succ \frac{10}{3}|A|. \end{aligned}$$

So we can assume that $|2A_J| \sim 2|A_J| + b_J \prec 3|A_J|$. Hence

$$|2A| \succcurlyeq 2|A_J| + b_J + |A_1| + |A_2| + 2|A_2| + b_2 \sim 3|A| + b_J + b_2,$$

which implies that

$$\begin{aligned} H+1 &\sim 2(u_2-l_2)+2l_0-u_2+2l_2-2l_0 \\ &\preccurlyeq 2|A|+2b_J+2b_2-(u_2-l_2)-(2l_0-l_2) \prec 2|A|+2b. \end{aligned}$$

Suppose $l_2 \prec l_0+u_1 \preccurlyeq u_2$. If $|2A_J| \succcurlyeq 3|A_J|$, then

$$|2A| \succcurlyeq 3|A_J|+|u_2+A_1|+|l_0+A_2|+|2A_2| \succcurlyeq 3|A|+|A_1|,$$

which implies that $|A_1| \preccurlyeq \frac{1}{3}|A|$ and $|A_2| \succcurlyeq \frac{2}{3}|A|$. Hence

$$|2A| \succcurlyeq 2|A_1|+|\{0,u_1\}+A_2|+|l_0+A_2|+2|A_2| \succcurlyeq 3|A|+2|A_2|-|A_1| \succ \frac{10}{3}|A|.$$

If $|2A_J| \sim 2|A_J|+b_J \prec 3|A_J|$, then

$$|2A| \succcurlyeq 2|A_J|+b_J+|u_2+A_1|+3|A_2|+b_1 \sim 3|A|+b_J+b_2.$$

Hence

$$H+1 \sim 2|A|+2b-(u_2-l_2)-(2l_0-l_2) \prec 2|A|+2b.$$

So we can now assume that $l_0+u_1 \succ u_2$.

Assume $l_0 \preccurlyeq \frac{3}{2}u_1$. Then $u_2-l_2 \prec \frac{1}{2}(u_2-2u_1) \prec \frac{1}{2}(l_0-u_1) \preccurlyeq \frac{1}{4}u_1$ and $l_2-l_0 \succ 2u_1-l_0 \succcurlyeq \frac{1}{2}u_1$. If $|2A_J| \succcurlyeq 3|A_J|$, then

$$\begin{aligned} |2A| &\succcurlyeq 3|A_J|+|A_2+A_1[u_1-(l_2-l_0),u_1]|+3|A_2| \\ &\succcurlyeq 3|A|+|A_2|+A_1\left(\frac{1}{2}u_1,u_1\right) \succ \frac{10}{3}|A|. \end{aligned}$$

If $|2A_J| \sim 2|A_J|+b_J \prec 3|A_J|$, then

$$\begin{aligned} |2A| &\succcurlyeq 2|A_J|+b_J+|u_2+A_1[u_1-(u_2-l_0),u_1]|+3|A_2|+b_2 \\ &\succcurlyeq 3|A|+b_J+b_2-A_1(u_1-(u_2-l_0),u_1). \end{aligned}$$

Hence

$$\begin{aligned} H+1 &\sim 2(u_2-l_2)+2l_0-u_2+2l_2-2l_0 \\ &\preccurlyeq 2|A_2|+2b_2+2|A_J|+2b_J-u_2+2l_2-2l_0 \\ &\preccurlyeq 2|A|+2b+2A_1(0,u_1-u_2+l_0)-u_2+2l_2-2l_0 \\ &\preccurlyeq 2|A|+2b+2u_1-2u_2+2l_0-u_2+2l_2-2l_0 \\ &\preccurlyeq 2|A|+2b-2(u_2-l_2)-(u_2-2u_1) \prec 2|A|+2b. \end{aligned}$$

Assume $l_0 \succ \frac{3}{2}u_1$. If $l_0 + L_1 \preccurlyeq \frac{1}{2}(2u_1 + l_2) \preccurlyeq l_0 + R_1$, then

$$\begin{aligned} |2A| &\succcurlyeq 2|A_1| + b_1 + |l_0 + A_1[2u_1 - l_0, l_2 - l_0]| + |A_1 + A_2| + 3|A_2| + b_2 \\ &\succcurlyeq 3|A| + b_1 + b_2 + A_1(2u_1 - l_0, l_2 - l_0) + A_1(L_1, L_1 + (u_2 - l_2)), \end{aligned}$$

which implies that

$$\begin{aligned} H + 1 &\sim 2(u_2 - l_2) + 2u_1 - u_2 + 2l_2 - 2u_1 \\ &\preccurlyeq 2|A| + 2b - 2p_1 - 2A_1(2u_1 - l_0, l_2 - l_0) - 2A_1(L_1, L_1 + (u_2 - l_2)) \\ &\quad -u_2 + 2l_2 - 2u_1 \\ &\preccurlyeq 2|A| + 2b - (l_2 - 2u_1) - (u_2 - l_2) - u_2 + 2l_2 - 2u_1 \\ &= 2|A| + 2b - 2u_2 + 2l_2 \prec 2|A| + 2b. \end{aligned}$$

Thus we can now assume that either $l_0 + L_1 \succ \frac{1}{2}(2u_1 + l_2)$ or $l_0 + R_1 \prec \frac{1}{2}(2u_1 + l_2)$. Suppose $l_0 + L_1 \succ \frac{1}{2}(2u_1 + l_2)$. If $l_2 - l_0 \succcurlyeq R_1 - L_1$, then

$$\begin{aligned} |2A| &\succcurlyeq 2|A_1| + b_1 + |l_0 + A_1[2u_1 - l_0, u_1]| \\ &\quad +|A_2 + A_1[u_1 - (l_2 - l_0), u_1]| + 3|A_2| \\ &\succcurlyeq 3|A| + b_1 - A_1(0, 2u_1 - l_0) + |A_2| + A_1(u_1 - (l_2 - l_0), u_1). \end{aligned}$$

Notice that $2u_1 - l_0 < \frac{1}{2}(2u_1 + l_2) - l_0 \prec L_1$ and $l_2 - l_0 \succcurlyeq R_1 - L_1 \succcurlyeq \frac{1}{2}u_1$. Let $s_1' = |[0, L_1] \setminus A_1|$ and $s_1'' = s_1 - s_1'$. Then $s_1' - A_1(0, 2u_1 - l_0) \succcurlyeq s_1' - A_1(0, L_1) \sim 0$ and $s_1'' + A_1(u_1 - (l_2 - l_0), u_1) \succcurlyeq \frac{1}{2}u_1$. Hence

$$|2A| \succcurlyeq 3|A| + |A_2| + \frac{1}{2}u_1 \succ \frac{10}{3}|A|.$$

If $l_2 - l_0 \prec R_1 - L_1$, then

$$\begin{aligned} |2A| &\succcurlyeq 2|A_1| + b_1 + |l_0 + A_1[2u_1 - l_0, R_1]| \\ &\quad +|A_2 + A_1[R_1 - (l_2 - l_0), u_1]| + 3|A_2| + b_2 \\ &\succcurlyeq 3|A| + b_1 + b_2 - A_1(0, 2u_1 - l_0) \\ &\quad +A_1(R_1 - (u_2 - l_2), R_1) + A_1(R_1 - (l_2 - l_0), R_1), \end{aligned}$$

which implies that

$$H + 1 \preccurlyeq 2u_1 + 2(u_2 - l_2) - u_2 + 2l_2 - 2u_1$$

$$
\begin{aligned}
&\preccurlyeq 2|A| + 2b + 2A_1(0, 2u_1 - l_0) - 2A_1(R_1 - (u_2 - l_2), R_1) \\
&\quad -2p_1 - 2A_1(R_1 - (l_2 - l_0), R_1) - u_2 + 2l_2 - 2u_1 \\
&\preccurlyeq 2|A| + 2b + 4u_1 - 2l_0 - u_2 + l_2 - 2l_2 + 2l_0 - u_2 + 2l_2 - 2u_1 \\
&\preccurlyeq 2|A| + 2b - (u_2 - 2u_1) - (u_2 - l_2) \prec 2|A| + 2b.
\end{aligned}
$$

Suppose $l_0 + R_1 \prec \frac{1}{2}(2u_1 + l_2)$. This implies $\frac{1}{2}u_1 \preccurlyeq R_1 \prec l_2 - l_0$. If $|2A_J| \succcurlyeq 3|A_J|$, then

$$
\begin{aligned}
|2A| &\succcurlyeq 3|A_J| + |A_2 + A_1[u_1 - (l_2 - l_0), u_1]| + 3|A_2| \\
&\succ 3|A| + A_1\left(\frac{1}{2}u_1, u_1\right) + |A_2| \succcurlyeq \frac{10}{3}|A|.
\end{aligned}
$$

If $|2A_J| \sim 2|A_J| + b_J \prec 3|A_1|$, then

$$
\begin{aligned}
|2A| &\succcurlyeq 2|A_J| + b_J + |u_2 + A_1[u_1 - (u_2 - l_0), u_1]| \\
&\quad +3|A_2| + b_1 \\
&\succcurlyeq 3|A| + b_J + b_2 - A_1(0, u_1 - (u_2 - l_0)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
H + 1 &\preccurlyeq 2l_0 + 2(u_2 - l_2) - u_2 + 2l_2 - 2l_0 \\
&\preccurlyeq 2|A| + 2b_J + 2b_2 - u_2 + 2l_2 - 2l_0 \\
&\preccurlyeq 2|A| + 2b + 2A_1(0, u_1 - (u_2 - l_0)) - u_2 + 2l_2 - 2l_0 \\
&\preccurlyeq 2|A| + 2b + 2u_1 - 2u_2 + 2l_0 - u_2 + 2l_2 - 2l_0 \\
&= 2|A| + 2b - 2(u_2 - l_2) - (u_2 - 2u_1) \prec 2|A| + 2b.
\end{aligned}
$$

This completes the proof of Theorem 7.1.

□

A Appendix

Theorem A.1 (G. A. Freiman, [2]) *Let A be a finite set of integers and $|A| = k > 2$. If $|2A| = 2k - 1 + b < 3k - 3$, then A is a subset of an* a.p. *of length at most $k + b$.*

The proof of Theorem A.1 can also be found in [12, p.28]

Theorem A.2 (G. A. Freiman, [2, 1]) *Let A be a finite set of integers and $|A| = k$. If $|2A| = 3k - 3$, then A is either a subset of an* a.p. *of length at most $2k - 1$ or a* b.p.

Theorem A.3 (G. A. Freiman, [2]) *Let $A \subseteq \mathbb{Z}^2$ be such that $|A| = k > 10$. If $|2A| = 3k - 3 + b$ for $0 \leqslant b < k - 3$ and A is not a subset of a straight line, then A is F_2-isomorphic to a subset of $\{(0,0),(1,0),\ldots,(l_1 - 1, 0)\} \cup \{(0,1),(1,1),\ldots,(l_2 - 1, 1)\}$ where $l_1 + l_2 \leqslant k + b$.*

The proof of Theorem A.3 can also be found in [14].

Theorem A.4 (V. Lev & P. Y. Smeliansky, [10]) *Let A and B be two finite set of non-negative integers such that $0 \in A \cap B$, $|A|, |B| > 1$, $\gcd(A) = 1$, $m = \max A$, and $n = \max B \leqslant m$. If $m = n$, then $|A + B| \geqslant \min\{m + |B|, |A| + 2|B| - 3\}$. If $m > n$, then $|A + B| \geqslant \min\{m + |B|, |A| + 2|B| - 2\}$.*

The proof of Theorem A.4 can also be found in [12, p.118]. There is a generalization of Theorem A.4 by Stanchescu in the following form.

Theorem A.5 (Y. V. Stanchescu, [13]) *Let A and B be two finite set of non-negative integers such that $0 \in A \cap B$, $|A|, |B| > 1$, $m = \max A$, $n = \max B \leqslant m$, $h_A = m + 1 - |A|$, and $h_B = n + 1 - |B|$. Define*

$$\delta = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m > n. \end{cases}$$

If $m \leqslant |A| + |B| - 2 - \delta$, then

$$|A + B| \geqslant |A| + |B| - 1 + \max(h_A, h_B).$$

Theorem A.6 (Y. V. Stanchescu, [14, 15]) *Let $A \subseteq \mathbb{Z}^2$ be a finite set of lattice points with the property that*

$$|2A| = 3|A| - 3 + b \text{ for } 0 \leqslant b < \frac{1}{2}|A| - 4.$$

(a) If $|A| \geqslant 1344$, then A lies on no more than three parallel lines.

(b) If $|A| \geqslant 1344$ and A is not contained in any two parallel lines, then A can be covered by three compatible arithmetic progressions[4] *having together no more than $\frac{3}{4}(|A| + b + 2)$ terms.*

References

[1] Y. Bilu, *Structure of sets with small sumset*, Astérisque 258 (1999), 77—108.

[2] G. A. Freiman, **Foundations of a structural theory of set addition**. Translated from the Russian. Translations of Mathematical Monographs, Vol 37. American Mathematical Society, Providence, R. I., 1973

[3] G. A. Freiman, *Structure theory of set addition. II. Results and problems*, Paul Erdös and his mathematics, I (Budapest, 1999), 243–260, Bolyai Soc. Math. Stud., 11, János Bolyai Math. Soc., Budapest, 2002.

[4] G. A. Freiman, *Inverse problem of additive number theory, IV. On addition of finite sets, II.*, Ucen. Zap. Elabuz. Gos. Ped. Inst. VIII (1960) 72–116

[5] C. W. Henson, *Foundations of nonstandard analysis–A gentle introduction to nonstandard extension*, in **Nonstandard Analysis: Theory and Applications**, ed. by N. J. Cutland, C. W. Henson, and L. Arkeryd, Kluwer Academic Publishers 1997.

[6] R. Jin, *Freiman's Inverse Problem with Small Doubling Property*, Advances in Mathematics, 216 (2007), No. 2, 711–752.

[7] R. Jin, *Inverse Problem for Cuts*, Logic and Analysis, 1, (2007), No. 1, 61 — 89.

[8] R. Jin, *Nonstandard methods for upper Banach density problems*, The Journal of Number Theory, 91 (2001), 20—38.

[4] Three arithmetic progressions I_1, I_2, and I_3 in $\mathbb{Z}^2$ are compatible if they lie on three parallel lines, respectively, and have the same common difference, i.e., $u_1 - v_1 = u_2 - v_2 = u_3 - v_3$ where u_i, v_i are two consecutive elements in I_i for $i = 1, 2, 3$.

[9] V. Lev, *On the structure of sets of integers with small doubling property*, unpublished manuscripts (1995).

[10] V. Lev and P. Y. Smeliansky, *On addition of two distinct sets of integers*, Acta Arithmetica, 70 (1995), No. 1, 85–91.

[11] T. Lindstrom, *An invitation to nonstandard analysis*, in **Nonstandard Analysis and Its Application**, ed. by N. Cutland, Cambridge University Press 1988.

[12] M. B. Nathanson, **Additive Number Theory–Inverse Problems and the Geometry of Sumsets**, Springer, 1996.

[13] Y. V. Stanchescu, *On addition of two distinct sets of integers*, Acta Arith. 75 (1996), no. 2, 191–194.

[14] Y. V. Stanchescu, *On the structure of sets with small doubling property on the plane (I)*, Acta Arith. 83 (1998), no. 2, 127–141.

[15] Y. V. Stanchescu, *On the structure of sets of lattice points in the plane with a small doubling property (II)*, Astérisque 258 (1999), 217–240.