Slow P-point Ultrafilters

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Abstract

We answer a question of Blass, Di Nasso, and Forti [2, 7] by proving, assuming Continuum Hypothesis or Martin's Axiom, that (1) there exists a P-point which is not interval-to-one and (2) there exists an interval-to-one P-point which is neither quasi-selective nor weakly Ramsey.

1 Introduction

The notion of quasi-selective ultrafilter is first introduced in [4] where it is actually called smooth ultrafilter. Quasi-selective ultrafilter is interesting because the existence of such ultrafilters is equivalent to the existence of coherent fine densities [4, Theorem 3.2] and to the existence of asymptotic numerosities [2, Corollary 5.3]. Since a selective ultrafilter is quasi-selective and a quasi-selective ultrafilter is a P-point, it is natural to study whether these three classes of ultrafilters are really distinct. It is shown in [2] that, under CH, these three classes of ultrafilters are distinct. In fact, the quasi-selective non-selective ultrafilter $\mathcal{F}$ constructed in [2] is also weakly Ramsey. In [7] it is pointed out that both quasi-selective ultrafilters and weakly Ramsey ultrafilters are a special kind of P-points, called interval-to-one P-points. The question whether the class of P-points, the class of interval-to-one P-points, and the class of quasi-selective or weakly Ramsey ultrafilters are distinct is asked in both [2, page 1484] and [7, page 11]. Since by a celebrated result

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of S. Shelah [10] it is consistent that there exist no P-points, we can only discuss the differences between these ultrafilters under some set theoretical assumptions beyond ZFC such as, for example, CH (Continuum Hypothesis), MA (Martin's Axiom), etc. In this paper we show that these classes of ultrafilters are distinct assuming either CH or MA.

Rapid ultrafilters are another class of ultrafilters closely related to selective ultrafilters and P-points. It is shown in [2, Corollary 1.6] that an ultrafilter $\mathcal{F}$ is selective if and only if it is rapid and quasi-selective. In [12] it is shown that an ultrafilter $\mathcal{F}$ is rapid if and only if the intersection of $\mathcal{F}$ with any tall summable ideal is non-empty. Therefore, a non-selective quasi-selective ultrafilter $\mathcal{F}$ must have empty intersection with a tall summable ideal $\mathcal{I}_f$ determined by some non-increasing function $f : \mathbb{N} \to [0,1]$. As an antonym for the word *rapid* we call an ultrafilter *f-slow* if it is disjoint from $\mathcal{I}_f$.

In this paper we want to find P-point ultrafilters which are not interval-to-one and interval-to-one P-point ultrafilters which are neither quasi-selective nor weakly Ramsey. We look for these ultrafilters among all f-slow ultrafilters for some f. Indeed, we do find them by carefully controlling the speeds of their slowness.

In Section 2 we state the definitions of various involved ideals and filters. Although many of these definitions are well-known to a set theorist, we include the definitions here to make the paper somewhat self-contained. In Section 3 we prove our results under CH and in Section 4 we do the same under MA[1]. Some open problems are in Section 5.

2 Summable ideals and slow ultrafilters

By an interval $[a, b]$ we mean an interval of *integers* $\{x \in \mathbb{Z} \mid a \leq x \leq b\}$. Occasionally, we use the interval notation $[0, 1]$ for the unit interval of reals. We hope that won't cause confusion. Let $\mathbb{N}$ be the set of all positive integers.

Definition 2.1 *A family of sets $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is a free ideal if*

1. $\mathbb{N} \notin \mathcal{I}$,
2. *$F \in \mathcal{I}$ for every finite subset F of* $\mathbb{N}$,
3. *$A \in \mathcal{I}$ and $B \subseteq A$ imply $B \in \mathcal{I}$, and*

[1] In fact, we need only MA for σ-centered partial orders.

4. $A \cup B \in \mathcal{I}$ *for any* $A, B \in \mathcal{I}$.

Definition 2.2 *A family of sets* $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ *is a free filter if*

1. $\emptyset \notin \mathcal{F}$,
2. $\mathbb{N} \setminus F \in \mathcal{F}$ *for every finite subset* F *of* $\mathbb{N}$,
3. $A \in \mathcal{F}$ *and* $B \supseteq A$ *imply* $B \in \mathcal{F}$, *and*
4. $A \cap B \in \mathcal{F}$ *for any* $A, B \in \mathcal{F}$.

A free filter $\mathcal{F}$ *is called an ultrafilter if* $A \in \mathcal{F}$ *or* $\mathbb{N} \setminus A \in \mathcal{F}$ *for each* $A \subseteq \mathbb{N}$.

Since all ideals and filters considered in this paper are free, we will omit the word "free" from now on.

Clearly, $\mathcal{I}$ is an ideal if and only if $\mathcal{F} = \{\mathbb{N} \setminus A \mid A \in \mathcal{I}\}$ is a filter. In that case, $\mathcal{I}$ and $\mathcal{F}$ are *dual* of each other.

Definition 2.3 *Let* $f : \mathbb{N} \to [0, 1]$ *be a non-increasing function such that* $\sum_{n=1}^{\infty} f(n) = \infty$. *The following set* $\mathcal{I}_f$ *is called the summable ideal determined by* f *where*

$$\mathcal{I}_f := \left\{ A \subseteq \mathbb{N} \;\middle|\; \sum_{n \in A} f(n) < \infty \right\}.$$

The summable ideal $\mathcal{I}_f$ *is tall if* $\lim_{n \to \infty} f(n) = 0$.

Notice that all non-tall summable ideals are the same which is the ideal of all finite subsets of $\mathbb{N}$.

For notational convenience we will frequently use the following expression.

$$S(f, A) := \sum_{n \in A} f(n). \tag{1}$$

Definition 2.4 *Let* $\mathcal{I}_f$ *be a summable ideal determined by* f. *A set* $A \subseteq \mathbb{N}$ *is called* f*-slow if* $A \notin \mathcal{I}_f$. *An ultrafilter* $\mathcal{F}$ *is called* f*-slow if* $\mathcal{F}$ *contains the dual filter of* $\mathcal{I}_f$.

Notice that an ultrafilter $\mathcal{F}$ is f-slow if and only if $S(f, A) = \infty$ for all $A \in \mathcal{F}$.

Definition 2.5 *Let $A \subseteq \mathbb{N}$ be an infinite set and $g : A \to \mathbb{N}$. The function g is called finite-to-one if $g^{-1}(n)$ is a finite set for every $n \in \mathbb{N}$. Here $g^{-1}(n)$ is an informal version of $g^{-1}(\{n\})$. This common convention will be used throughout this paper.*

An ultrafilter $\mathcal{F}$ is called a P-point if for every function $g : \mathbb{N} \to \mathbb{N}$ there exists an $A \in \mathcal{F}$ such that $g \restriction A$ is either a constant function or a finite-to-one function.

Notice that a function $g : \mathbb{N} \to \mathbb{N}$ can be viewed as a partition $\{g^{-1}(n) \mid n \in \mathbb{N}\}$ of $\mathbb{N}$ and a partition $P = \{B_n \mid n \in \mathbb{N}\}$ of $\mathbb{N}$ can be viewed as a function g_P with $g_P(x) = n$ if and only if $x \in B_n$.

Definition 2.6 *Let $A \subseteq \mathbb{N}$ be an infinite set and $g : A \to \mathbb{N}$. The function g is called interval-to-one if the intervals in*

$$\mathcal{C} = \{[\min(g^{-1}(n)), \sup(g^{-1}(n))] \mid n \in \mathbb{N} \ \textit{ and } \ g^{-1}(n) \neq \emptyset\}$$

are pairwise disjoint.

A P-point $\mathcal{F}$ is said to be interval-to-one if for every function $g : \mathbb{N} \to \mathbb{N}$, there is an $A \in \mathcal{F}$ such that $g \restriction A$ is interval-to-one.

Notice that the collection $\mathcal{C}$ above can contain an infinite interval when $\{n \mid g^{-1}(n) \neq \emptyset\}$ is a finite set. If $\sup S = \infty$, the interval $[a, \sup S]$ means $[a, \infty)$. Otherwise, $\sup(g^{-1}(n)) = \max(g^{-1}(n))$.

Definition 2.7 *An ultrafilter $\mathcal{F}$ is selective if for every function $g : \mathbb{N} \to \mathbb{N}$ there is an $A \in \mathcal{F}$ such that $g \restriction A$ is either a constant function or a one-to-one function.*

Definition 2.8 *An ultrafilter $\mathcal{F}$ is said to be quasi-selective if for every function $g : \mathbb{N} \to \mathbb{N}$ with $g(n) \leq n$ for all $n \in \mathbb{N}$, there exists an $A \in \mathcal{F}$ such that $g \restriction A$ is a non-decreasing function.*

Let X be a set and $[X]^2 := \{\{a, b\} \mid a, b \in X \text{ and } a \neq b\}$.

Definition 2.9 *An ultrafilter $\mathcal{F}$ is Ramsey if for every function $c : [\mathbb{N}]^2 \to \{0, 1\}$ there is an $A \in \mathcal{F}$ such that $|range(c \restriction [A]^2)| = 1$.*

The set A above is called a *c-homogeneous* set.

Definition 2.10 *An ultrafilter $\mathcal{F}$ is said to be weakly Ramsey if for every function $c : [\mathbb{N}]^2 \to \{0, 1, 2\}$ there is an $A \in \mathcal{F}$ such that $|range(c \restriction [A]^2)| \leq 2$.*

It is a well-known fact that an ultrafilter is selective if and only if it is Ramsey.

A selective ultrafilter is quasi-selective as well as weakly Ramsey, and a quasi-selective ultrafilter or a weakly Ramsey ultrafilter is an interval-to-one P-point [2, 7].

Definition 2.11 *An ultrafilter $\mathcal{F}$ is rapid if for every finite-to-one function $g : \mathbb{N} \to \mathbb{N}$, there exists an $A \in \mathcal{F}$ such that $|g^{-1}(n) \cap A| \leq n$ for all $n \in \mathbb{N}$.*

It is shown in [12] that if $\mathcal{F}$ is a rapid ultrafilter and $\mathcal{I}_f$ is a tall summable ideal determined by f, then $\mathcal{F} \cap \mathcal{I}_f \neq \emptyset$.

We say that A is almost a subset of B, denoted by $A \subseteq^* B$, if $A \setminus B$ is a finite set. Notice that an ultrafilter $\mathcal{F}$ is a P-point if and only if for every countable collection of sets $\{B_n \mid n \in \mathbb{N}\} \subseteq \mathcal{F}$, there is a set $B \in \mathcal{F}$ such that $B \subseteq^* B_n$ for every $n \in \mathbb{N}$.

The reader may find other basic information on ultrafilters in [1, 3] and basic information on set theory, especially Continuum Hypothesis and Martin's Axiom in [8, 9].

3 Assuming CH

Theorem 3.1 *Assume that* CH *holds. For each summable ideal $\mathcal{I}_f$ determined by f, there exists an f-slow P-point.*

Proof: By CH we can list all subsets of $\mathbb{N}$ as $\{B_\alpha \mid \alpha < \aleph_1\}$. We define sets A_α inductively on all ordinals $\alpha < \aleph_1$ such that for any $\alpha < \beta < \aleph_1$

1. $A_\alpha \notin \mathcal{I}_f$,
2. $A_\beta \subseteq^* A_\alpha$, and
3. either $A_{\alpha+1} \subseteq B_\alpha$ or $A_{\alpha+1} \cap B_\alpha = \emptyset$.

The theorem follows from the construction because the ultrafilter $\mathcal{F}$ generated by $\{A_\alpha \mid \alpha < \aleph_1\}$ is an f-slow P-point. Indeed, 1. guarantees that $\mathcal{F}$ is f-slow, 2. and 3. guarantees that $\mathcal{F}$ is an ultrafilter, and 2. guarantees that $\mathcal{F}$ is a P-point.

We start with $A_0 = \mathbb{N}$. Suppose that $\{A_\alpha \mid \alpha < \delta\}$ have been constructed for some $\delta < \aleph_1$ to satisfy 1., 2., and 3. when $\aleph_1$ is replaced by δ.

Case 1: δ is a limit ordinal.

Let $\{\alpha_n < \delta \mid n \in \mathbb{N}\}$ be an increasing sequence of ordinals cofinal in δ. For each $m \in \mathbb{N}$, we have $\displaystyle\bigcap_{n=1}^{m} A_{\alpha_n} \notin \mathcal{I}_f$ by 1. and 2. For each $m \in \mathbb{N}$ let $B_m = \{a_i \mid n_m \leq i < n_{m+1}\}$ be a finite increasing sequence of elements in $\displaystyle\bigcap_{n=1}^{m} A_{\alpha_n}$ such that $S(f, B_m) \geq 1$ ($S(f, B_m)$ is defined in (1)). The sequence B_m can be chosen such that a_{n_m-1}, the last element in B_{m-1}, is less than a_{n_m}, the first element in B_m, for every $m \in \mathbb{N}$. Let

$$A_\delta := \bigcup_{m=1}^{\infty} B_m.$$

The existence of B_m is guaranteed due to the fact that $S(f, \bigcap_{n=1}^{m} A_{\alpha_n}) = \infty$. Clearly, $\{A_\alpha \mid \alpha < \delta + 1\}$ satisfies 1. and 2. when $\aleph_1$ is replaced by $\delta + 1$. The condition 3. is vacuous.

Case 2: $\delta = \beta + 1$.

If $A_\beta \cap B_\beta$ is f-slow, let $A_\delta = A_\beta \cap B_\beta$. If $A_\beta \cap B_\beta$ is not f-slow, then $A_\beta \setminus B_\beta$ must be f-slow because A_β is an f-slow set. In this case let $A_\delta = A_\beta \setminus B_\beta$. Clearly, $\{A_\alpha \mid \alpha < \delta + 1\}$ satisfies Conditions 1., 2., and 3. when $\aleph_1$ is replaced by $\delta + 1$. This completes the construction. □

Notice that if $\lim_{n\to\infty} f(n) > 0$, then f-slow P-point is just the ordinary P-point. Hence Theorem 3.1 is only interesting when $\mathcal{I}_f$ is tall.

Theorem 3.2 *Let $f(n) = 1/n$. There does not exist interval-to-one f-slow P-point.*

Proof: Assume to the contrary that $\mathcal{F}$ is an interval-to-one f-slow P-point. Let $a_0 = 0$, $a_{n+1} = a_n + n^2$ for any $n \geq 0$. Notice that $[a_n, a_{n+1} - 1]$ is an interval of length n^2. Let

$$B_m := \bigcup_{n=m}^{\infty} \{a_n + m + in \mid i = 0, 1, \ldots, n-1\}.$$

Notice that the set $\{a_n + m + in \mid i = 0, 1, \ldots, n-1\}$ is the m-th arithmetic progression of length n and difference n in $[a_n, a_{n+1} - 1]$. Therefore, $P = \{B_m \mid m \in \mathbb{N}\}$ is a partition of $\mathbb{N}$. Since

$$a_{n+1} = n^2 + (n-1)^2 + \cdots + 1 = \frac{1}{6} n(n+1)(2n+1),$$

we have that

$$\begin{aligned} S(f, B_m) &= \sum_{n=m}^{\infty} S(f, B_m \cap [a_n, a_{n+1} - 1]) \\ &\leq \sum_{n=m}^{\infty} \frac{n}{a_n} \leq 1 + \sum_{n=m+1}^{\infty} \frac{3}{(n-1)^2} < \infty. \end{aligned}$$

Hence $B_m \in \mathcal{I}_f$ for every $m \in \mathbb{N}$. Since $\mathcal{F}$ is an interval-to-one f-slow P-point, there exists a set $A \in \mathcal{F}$ such that intervals $[\min(A \cap B), \max(A \cap B)]$ for all $B \in P$ are pairwise disjoint.

Claim $|A \cap [a_n, a_{n+1} - 1]| \leq 2n$.

Proof of Claim: Notice that $B_m \cap [a_n, a_{n+1} - 1] = \emptyset$ for all $m > n$. Suppose there are k sets $B_{m_1}, B_{m_2}, \ldots, B_{m_k} \in P$ for some $k \leq n$ such that $B_{m_i} \cap A \cap [a_n, a_{n+1} - 1] \neq \emptyset$ for $i = 1, 2, \ldots, k$. Let $l_i = \min(B_{m_i} \cap A \cap [a_n, a_{n+1} - 1])$ and $u_i = \max(B_{m_i} \cap A \cap [a_n, a_{n+1} - 1])$. Notice that $[l_i, u_i] \cap A$ contains at most $e_i := 1 + (u_i - l_i)/n$ elements from B_{m_i} and contains no elements from any other B_{m_j} for $j \neq i$. So

$$n^2 \geq \sum_{i=1}^{k} (u_i - l_i + 1) = \sum_{i=1}^{k} (n(e_i - 1) + 1) = n \sum_{i=1}^{k} e_i - kn + k,$$

which implies that

$$|A \cap [a_n, a_{n+1} - 1]| \leq \sum_{i=1}^{k} e_i \leq \frac{1}{n}(n^2 + kn - k) \leq \frac{1}{n}(2n^2 - k) \leq 2n.$$

This completes the proof of the claim.

Now the theorem follows from the claim because

$$S(f, A) = \sum_{n=1}^{\infty} S(f, A \cap [a_n, a_{n+1} - 1]) \leq \sum_{n=1}^{\infty} \frac{2n}{a_n} < \infty.$$

This contradicts the assumption that $A \notin \mathcal{I}_f$. $\square$

Notice that Theorem 3.2 is a consequence of ZFC.

Corollary 3.3 *Assume that* CH *holds. There exists a non-interval-to-one P-point.*

Proof: Let $f(n) = 1/n$. By Theorem 3.1 there is an f-slow P-point $\mathcal{F}$. By Theorem 3.2 the P-point $\mathcal{F}$ cannot be interval-to-one. $\square$

In the proof of the next theorem we need to use three existing results. The first one is a result in [2, Proposition 1.7] which guarantees that if $\mathcal{F}$ is a quasi-selective ultrafilter and $h(n) = 2^n$, then for every function $h' : \mathbb{N} \to \mathbb{N}$ with $h'(n) \leq h(n)$ for all $n \in \mathbb{N}$, there exists a set $A \in \mathcal{F}$ such that $h' \restriction A$ is non-decreasing.

The second result we need is so called Erdős-Szekeres Theorem [6], which says that every finite sequence of real numbers of length $n^2 + 1$ contains a monotonic subsequence of length at least $n+1$. In the proof of next theorem we need only a consequence for notational convenience that every sequence of length n^2 contains a monotonic subsequence of length at least n.

The third result we need is a lower bound for Ramsey numbers by P. Erdős [5]. We use an improved version in [11] although Erdős' original version is sufficient for our purpose. Recall that $[X]^2 := \{\{a,b\} \mid a,b \in X \text{ and } a \neq b\}$. For any $r \in \mathbb{N}$ let $R(r)$ be the smallest positive integer n such that for any $c : [X]^2 \to \{0,1\}$ where $|X| = n$ there is a set $X' \subseteq X$ such that $|X'| \geq r$ and $c \restriction [X]^2$ is a constant function. It is shown in [11] that

$$R(r) > Cr2^{\frac{r}{2}} \text{ where } C = \frac{\sqrt{2}}{e} \tag{2}$$

for all sufficiently large r.

Theorem 3.4 *Assume that* CH *holds. There exists an interval-to-one P-point, which is neither quasi-selective nor weakly Ramsey.*

Proof: Let $a_0 = 1$ and $a_n = 2^{n!}$ for $n \geq 1$. Let $p : \mathbb{N} \to \mathbb{N}$ be such that $p(x) = n$ if and only if $a_n \leq x < a_{n+1}$.

Let $f : \mathbb{N} \to [0,1]$ be such that $f(x) = 1/2^{(n-1)!}$ if $x \in p^{-1}(n)$. Notice that f is non-increasing, $\lim_{n\to\infty} f(n) = 0$, and $S(f,\mathbb{N}) \geq \sum_{n=1}^{\infty} \frac{a_{n+1} - a_n}{2^{(n-1)!}} = \infty$. Let $\mathcal{I}_f$ be the tall summmable ideal determined by f. Notice also that if $|A \cap p^{-1}(n)| \geq 2^{(n-1)!}$ for infinitely many n, then $A \notin \mathcal{I}_f$. The ultrafilter $\mathcal{F}$ that we construct will be f-slow.

By CH we can list all functions from $\mathbb{N}$ to $\mathbb{N}$ as a sequence $\{g_\alpha \mid \alpha < \aleph_1\}$. We construct a sequence $\{A_\alpha \mid \alpha < \aleph_1\}$ of sets by induction such that for any $\alpha < \beta < \aleph_1$

1. $A_\alpha \notin \mathcal{I}_f$,
2. $A_\beta \subseteq^* A_\alpha$,
3. $\{n \in \mathbb{N} \mid |A_\alpha \cap p^{-1}(n)| > a^k_{n-1}\}$ is infinite for each $k \in \mathbb{N}$,
4. either $A_{\alpha+1} \subseteq g_\alpha^{-1}(m)$ for some $m \in \mathbb{N}$ or intervals in

$$\{[\min(A_{\alpha+1} \cap g_\alpha^{-1}(m)), \max(A_{\alpha+1} \cap g_\alpha^{-1}(m))] \mid m \in \mathbb{N}\}$$

are pairwise disjoint.

Condition 1 is listed for convenience only. It is actually a consequence of Condition 3.

We start with $A_0 = \mathbb{N}$. Notice that Condition 3 is true for A_0 because if $k \geq 1$ is fixed, then $a_{n+1} - a_n > a^k_{n-1} \geq 2^{(n-1)!}$ for all sufficiently large $n \in \mathbb{N}$. Conditions 2. and 4. are vacuous for A_0.

Assume that we have obtained all sets A_α for $\alpha < \delta$ for some $\delta < \aleph_1$ such that the four properties above are satisfied when $\aleph_1$ is replaced by δ.

Case 1: δ is a limit ordinal.

Let $\{\alpha_m \mid m \in \mathbb{N}\}$ be an increasing sequence of ordinals below δ and cofinal in δ. Without loss of generality we can assume that $A_{\alpha_{m+1}} \subseteq A_{\alpha_m}$ for every $m \in \mathbb{N}$.

Let n_1 be the smallest positive integer such that $|A_{\alpha_1} \cap p^{-1}(n_1)| > a^1_{n_1-1}$. For each $m > 1$ let $n_m > n_{m-1}$ be the smallest integer such that $|A_{\alpha_m} \cap p^{-1}(n_m)| > a^m_{n_m-1}$. The existence of n_m is guaranteed by Condition 3. Now let

$$A_\delta := \bigcup_{m=1}^{\infty} (A_{\alpha_m} \cap p^{-1}(n_m)).$$

Clearly, Condition 2 is true because for a fixed m_0,

$$\bigcup_{m=m_0}^{\infty} (A_{\alpha_m} \cap p^{-1}(n_m)) \subseteq A_{\alpha_{m_0}}.$$

Hence $A_\delta \subseteq^* A_\alpha$ for any $\alpha < \delta$. Condition 3 is true because for each fixed k,

$$|A_\delta \cap p^{-1}(n_m)| > a^m_{n_m-1} \geq a^k_{n_m-1}$$

for all $m > k$. Condition 4 is vacuous.

Case 2: δ is a successor ordinal $\beta + 1$.

By the Erdős–Szekeres Theorem we have that every sequence of real numbers with n^2 terms contains a monotonic subsequence with n terms.

First we assume that there exists an $m \in \mathbb{N}$ such that for each $k \in \mathbb{N}$ there are infinitely many n with $|A_\beta \cap g_\beta^{-1}(m) \cap p^{-1}(n)| > a_{n-1}^k$. Then we set $A_\delta := A_\beta \cap g_\beta^{-1}(m)$. It is easy to check that $\{A_\alpha \mid \alpha < \delta + 1\}$ satisfies Conditions 1 – 4 above when $\aleph_1$ is replaced by $\delta + 1$. Hence we can now assume that for every m, there is a k_m and n_m such that

$$|A_\beta \cap g_\beta^{-1}(m) \cap p^{-1}(n)| \leq a_{n-1}^{k_m} \tag{3}$$

for every $n \geq n_m$.

Let $S = \{n \mid |A_\beta \cap p^{-1}(n)| > a_{n-1}\}$. We construct an increasing sequence $n_1 < n_2 < \cdots$ in S and sets $D_{n_i} \subseteq p^{-1}(n_i) \cap A_\beta$ inductively such that

a. $|D_{n_i}| > a_{n_i-1}^i$,

b. $g_\beta \restriction D_{n_i}$ is monotonic, and

c. $g_\beta(x) \neq g_\beta(y)$ for any $x \in D_{n_i}$ and $y \in D_{n_j}$ when $i \neq j$.

Let $n_1 = \min\{n \in S \mid |A_\beta \cap p^{-1}(n)| > a_{n-1}^2\}$. By the Erdős–Szekeres Theorem the sequence $\{g_\beta(x) \mid x \in A_\beta \cap p^{-1}(n_1)\}$ contains a monotonic subsequence $\{g_\beta(x) \mid x \in D_{n_1}\}$ with $|D_{n_1}| > a_{n-1}$. Notice that $g_\beta \restriction D_{n_1}$ is interval-to-one because it is monotonic. Suppose we have constructed $n_1 < n_2 < \cdots < n_t$ and $D_{n_i} \subseteq A_\beta \cap p^{-1}(n_i)$ for $i = 1, 2, \ldots, t$ such that $|D_{n_i}| > a_{n_i-1}^i$, g_β is monotonic on D_{n_i}, and $g_\beta(x) \neq g_\beta(y)$ for any $x \in D_{n_i}$ and $y \in D_{n_j}$ when $i \neq j$ for all $1 \leq i, j \leq t$.

For constructing n_{t+1} and $D_{n_{t+1}}$ let

$$X := \{m \mid D_{n_i} \cap g^{-1}(m) \neq \emptyset \text{ for some } i \leq t\}.$$

Notice that $|X| \leq a_{n_t+1}$. Let $\bar{k} = \max\{k_m \mid m \in X\}$ and $\bar{n} = \max\{n_m \mid m \in X\}$ where k_m and n_m are defined in (3). Recall that $|A_\beta \cap g_\beta^{-1}(m) \cap p^{-1}(n)| \leq a_{n-1}^{k_m} \leq a_{n-1}^{\bar{k}}$ for all $m \in X$ and all $n \geq \bar{n}$. Let $k = \max\{2p+3, \bar{k}+2\}$. By Condition 3 we can find an $n_{t+1} \geq \max\{n_t + 1, \bar{n}\} + 1$ in S such that $|A_\beta \cap p^{-1}(n_{t+1})| > a_{n_{t+1}-1}^k$. Let

$$A' = (A_\beta \setminus \bigcup \{g_\beta^{-1}(m) \mid m \in X\}) \cap p^{-1}(n_{t+1}).$$

Then we have

$$|A'| > a_{n_{t+1}-1}^k - a_{n_t+1} \cdot a_{n_{t+1}-1}^{\bar{k}} \geq a_{n_{t+1}-1}^k - a_{n_{t+1}-1}^{\bar{k}+1} \geq a_{n_{t+1}-1}^{k-1} \geq a_{n_{t+1}-1}^{2(t+1)}.$$

By Erdős–Szekeres Theorem we can find an $D_{n_{t+1}} \subseteq A'$ such that $|D_{n_{t+1}}| > a^{t+1}_{n_{t+1}-1}$ and g_β is monotonic on $D_{n_{t+1}}$. Clearly, a. and b. are satisfied. The condition c. is true because all function values of g_β on $D_{n_{t+1}}$ are not in X. Now set

$$A_\delta := \bigcup_{i=1}^{\infty} D_{n_i}.$$

Clearly, $A_\delta \subseteq A_\beta$. Hence Condition 2 is true. Since for a fixed k,

$$|A_\delta \cap p^{-1}(n_i)| = |D_i| > a^i_{n_i-1} \geq a^k_{n_i-1}$$

for all $i \geq k$, Condition 3 is true. Condition 4 is true because g_β is monotonic on each D_{n_i}, and the range of $g_\beta \restriction D_i$ and the range of $g_\beta \restriction D_j$ are pairwise disjoint for any $i \neq j$.

Let $\mathcal{F} = \{U \subseteq \mathbb{N} \mid A_\alpha \subseteq U \text{ for some } \alpha < \aleph_1\}$. Clearly, $\mathcal{F}$ is a filter. For any $B \subseteq \mathbb{N}$, there must be a function g_α such that $g_\alpha(x) = 1$ if $x \in B$ and $g_\alpha(x) = 2$ if $x \notin B$. By Condition 4 we have that $A_{\alpha+1} \subseteq g_\alpha^{-1}(1)$, which implies $B \in \mathcal{F}$, or $A_{\alpha+1} \subseteq g_\alpha^{-1}(2)$, which implies $\mathbb{N} \setminus B \in \mathcal{F}$. Hence $\mathcal{F}$ is an ultrafilter. Also by Condition 4 $\mathcal{F}$ is an interval-to-one P-point. Since $A_\alpha \notin \mathcal{I}_f$ by Condition 1, the ultrafilter $\mathcal{F}$ is f-slow.

We now show that $\mathcal{F}$ is neither quasi-selective nor weakly Ramsey.

Suppose that $\mathcal{F}$ is a quasi-selective ultrafilter. Let $h(n) = 2^n$. By [2, Proposition 1.7] any function $h' : \mathbb{N} \to \mathbb{N}$ with $h'(n) \leq h(n)$ for all $n \in \mathbb{N}$ is $\mathcal{F}$-equivalent to a non-decreasing function. For each $x \in [a_n, a_{n+1} - 1]$ let $h'(x) = a_{n+1} - (x - a_n)$. Then $h'(x) \leq a_{n+1} = 2^{(n+1)!} \leq 2^{2^{n!}} = h(a_n) \leq h(x)$. Hence there is an $A \in \mathcal{F}$ such that h' is non-decreasing on A, which implies that $A \cap [a_n, a_{n+1} - 1]$ contains at most one element. So we have that

$$S(f, A) \leq \sum_{n=1}^{\infty} \frac{1}{2^{(n-1)!}} < \infty,$$

which contradicts that $\mathcal{F}$ is f-slow.

Now we show that $\mathcal{F}$ is not weakly Ramsey.

By (2) we have that

$$R(2^{(n-2)!}) > C 2^{(n-2)!} 2^{2^{(n-2)!/2}}$$

for all sufficiently large n. Let $X_n = [a_n, a_{n+1} - 1]$. Since

$$\lim_{n\to\infty} \frac{C 2^{(n-2)!} 2^{2^{(n-2)!/2}}}{2^{(n+1)!} - 2^{n!}} = \infty,$$

there is a function $c_n : [X_n]^2 \to \{0,1\}$ for all sufficiently large n such that

$$\max\{|X| \mid X \subseteq X_n \text{ is } c_n\text{-homogeneous}\} < 2^{(n-2)!}.$$

Assume to the contrary that $\mathcal{F}$ is weakly Ramsey. For every $\{a,b\} \in [\mathbb{N}]^2$, define

$$c(\{a,b\}) = \begin{cases} 0 & \text{if } p(a) = p(b) = n \text{ and } c_n(\{a,b\}) = 0 \\ 1 & \text{if } p(a) = p(b) = n \text{ and } c_n(\{a,b\}) = 1 \\ 2 & \text{if } p(a) \neq p(b). \end{cases}$$

Then there exists a set $A \in \mathcal{F}$ such that $c \restriction [A]^2$ takes only two values among $\{0,1,2\}$. If the range of c on $[A]^2$ is $\{0,1\}$, then $A \subseteq p^{-1}(n)$ for some n. This is impossible because A is infinite. If the range of c on $[A]^2$ is $\{0,2\}$ or $\{1,2\}$, then $A \cap p^{-1}(n)$ is c_n-homogeneous, which implies that there is an $n_0 \in \mathbb{N}$ such that $|A \cap p^{-1}(n)| < 2^{(n-2)!}$ for all $n \geq n_0$. Hence

$$S(f,A) \leq S(f, A \cap [1, a_{n_0} - 1]) + \sum_{n=n_0}^{\infty} \frac{2^{(n-2)!}}{2^{(n-1)!}} < \infty,$$

which again contradicts that $A \notin \mathcal{I}_f$. This completes the proof of the theorem. □

4 Under MA

Theorem 4.1 *Assume that* MA *holds. For each summable ideal $\mathcal{I}_f$ determined by f, there exists an f-slow P-point.*

Proof: The steps of the proof for this theorem are similar to those of Theorem 3.1. So we borrow notation from there. The main difference is that the list $\{g_\alpha \mid \alpha < 2^{\aleph_0}\}$ of all functions from $\mathbb{N}$ to $\mathbb{N}$ has a length of $2^{\aleph_0}$, which may be greater than $\aleph_1$ and therefore, our construction of the sequence of sets $\{A_\alpha \mid \alpha < 2^{\aleph_0}\}$ must also have a length of $2^{\aleph_0}$. More specifically, we want to construct sets A_α inductively on all ordinals $\alpha < 2^{\aleph_0}$ such that for any $\alpha < \beta < 2^{\aleph_0}$

1. $A_\alpha \notin \mathcal{I}_f$, where $\mathcal{I}_f$ is a fixed summable ideal determined by f,
2. $A_\beta \subseteq^* A_\alpha$, and
3. either $A_{\alpha+1} \subseteq B_\alpha$ or $A_{\alpha+1} \cap B_\alpha = \emptyset$.

The steps of the construction of A_δ for successor ordinals $\delta = \beta + 1$ are identical to the correspondent part in the proof of Theorem 3.1. In the case of δ being a limit ordinal we find A_δ by MA. Notice that in this case Condition 3 is vacuous.

Suppose that $\delta < 2^{\aleph_0}$ is a limit ordinal and $\mathcal{A}_\delta = \{A_\alpha \mid \alpha < \delta\}$ has been constructed so that it satisfies Condition 1, 2, and 3 when $2^{\aleph_0}$ is replaced by δ. In order to find A_δ we define a partial order P such that

$$P = \{(n, s, F) \mid n \in \mathbb{N}, s \subseteq [1, n], F \subseteq \mathcal{A}_\delta, \text{ and } F \text{ is finite}\}.$$

For any $p = (n_p, s_p, F_p), q = (n_q, s_q, F_q) \in P$, define $p \leq q$ (or p is stronger than q) if

1. $n_p \geq n_q$,
2. $s_p \cap [1, n_q] = s_q$,
3. $F_q \subseteq F_p$,
4. $s_p \cap [n_q + 1, n_p] \subseteq A$ for every $A \in F_q$,
5. $n_p = n_q$ or $S(f, s_p \cap [n_q + 1, n_p]) \geq 1$.

If $n_p = n_q$ and $s_p = s_q$, then $(n_p, s_p, F_p \cup F_q)$ is a common lower bound of p and q. Hence $(P, \leq)$ is σ-centered. So $(P, \leq)$ has *c.c.c.* Let $D_{A,n} = \{p \in P \mid n_p \geq n \text{ and } A \in F_p\}$ for each $n \in \mathbb{N}$ and $A \in \mathcal{A}_\delta$. It is routine to check that $D_{A,n}$ is dense in P. Since $\mathbb{D} = \{D_{A,n} \mid A \in \mathcal{A}_\delta \text{ and } n \in \mathbb{N}\}$ has less than $2^{\aleph_0}$ members, there exists, by MA, a $\mathbb{D}$–generic filter $G \subseteq P$, i.e., a filter $G \subseteq P$ such that $G \cap D \neq \emptyset$ for all $D \in \mathbb{D}$. Let

$$A_\delta := \bigcup \{s_p \mid p \in G\}.$$

Then $A_\delta \subseteq^* A_\alpha$ for any $\alpha < \delta$ because $G \cap D_{A_\alpha,n} \neq \emptyset$, and $A_\delta \notin \mathcal{I}_f$ because there are increasing sequence $p_1 > p_2 > \ldots$ in G with $n_{p_1} < n_{p_2} < \cdots$ and thus $S(f, A_\delta \cap [n_{p_i} + 1, n_{p_{j+1}}]) \geq 1$ for $i = 1, 2, \ldots$. □

Corollary 4.2 *Assume* MA. *There exists a non-interval-to-one P-point.*

Proof: Notice that Theorem 3.2 is a result of ZFC. Therefore, the existence of a $1/n$-slow non-interval-to-one P-point follows from Theorem 4.1 and Theorem 3.2. □

Theorem 4.3 *Assume that* MA *holds. There exists an interval-to-one P-point, which is neither quasi-selective nor weakly Ramsey.*

Proof: Similar to the proof of Theorem 4.1 the proof of this theorem is parallel to the proof of Theorem 3.4. So we borrow also notation from there. The only step different from that of the proof of Theorem 3.4 is the construction of the set A_δ when δ is a limit ordinal. Suppose that $\delta < 2^{\aleph_0}$ is a limit ordinal and $\mathcal{A}_\delta = \{A_\alpha \mid \alpha < \delta\}$ has been constructed.

Notice that the letter p is already used for a function defined in the first line of the proof of Theorem 3.4. For obtaining the set A_δ let

$$Q = \{(s, D, F, k) \mid \\ s \subseteq \mathbb{N} \text{ is finite, } D \subseteq \bigcup_{n \in s} p^{-1}(n), \ F \subseteq \mathcal{A}_\delta \text{ is finite, and } k \in \mathbb{N}\}.$$

For any $q = (s_q, D_q, F_q, k_q), q' = (s_{q'}, D_{q'}, F_{q'}, k_{q'}) \in Q$ define that $q \le q'$ if

1. $s_q \cap [1, \max s_{q'}] = s_{q'}$,
2. $D_q \cap p^{-1}(n) = D_{q'} \cap p^{-1}(n)$ for each $n \in s_{q'}$,
3. $D_q \cap p^{-1}(n) \subseteq A$ for each $n \in s_q \setminus s_{q'}$ and $A \in F_{q'}$,
4. $k_q \ge k_{q'}$,
5. $|D_q \cap p^{-1}(n)| > a_{n-1}^{k_{q'}}$ for each $n \in s_q \setminus s_{q'}$.

It is routine to check that $(Q, \le)$ is σ-centered. Let

$$\mathbb{D} = \{\mathcal{D}_{A,n,k} \mid A \in \mathcal{A}_\delta \text{ and } n, k \in \mathbb{N}\}$$

where $\mathcal{D}_{A,n,k} = \{q \in Q \mid \max s_q > n, \ A \in F_q, \text{ and } k_q > k\}$. It is easy to see that $\mathcal{D}_{A,n,k}$ is dense in Q and $\mathbb{D}$ has less than $2^{\aleph_0}$ members. By MA there is a $\mathbb{D}$–generic filter $G \subseteq Q$. Let

$$A_\delta := \bigcup \{D_q \mid q \in G\}.$$

This completes the construction of A_δ. It is now routine to check that A_δ is the set we want. $\square$

Notice that the P-points obtained in Theorem 4.1, Corollary 4.2, and Theorem 4.3 are all $P_{\mathfrak{c}}$-points, i.e., P–points generated by $\mathfrak{c}$ sets.

5 Open questions

An interval-to-one P-point can never be $1/n$-slow by Theorem 3.2. The interval-to-one P-point which is neither quasi-selective nor weakly Ramsey in Theorem 3.4 is f-slow for a function f with a very slow speed of approaching zero. It may be interesting to characterize the slowness of f which allows an f-slow interval-to-one P-point to exist. In particular we don't know the answer to the follwing question.

Question 5.1 *Assume that* CH *holds. Can* $1/\log(n)$*-slow interval-to-one* P*-point exist?*

It is interesting to explore the relative consistency strength among these ultrafilters.

Question 5.2 *Is it consistent, relative to* ZFC*, that there exist* P*-points but not interval-to-one* P*-points?*

Is it consistent, relative to ZFC*, that there exist interval-to-one* P*-points but not quasi-selective ultrafilters?*

Is it consistent, relative to ZFC*, that there exist interval-to-one* P*-points but not weakly Ramsey ultrafilters?*

References

[1] A. Blass, *Combinatorial cardinal characteristics of the continuum*, in: M. Foreman, A. Kanamori (Eds.), Handbook of Set Theory, Springer-Verlag, 2010, 395 – 489.

[2] A. Blass, M. Di Nasso, and M. Forti, *Quasi-selective ultrafilters and asymptotic numerosities*, Advances in Mathematics, vol. 231 (2012), 1462 – 1486.

[3] D. Booth, *Ultrafilters on a countable set*, Ann. Math. Log. 2 (1970/1971), 1 – 24.

[4] M. Di Nasso, *Fine asymptotic densities for sets of natural numbers*, Proc. Amer. Math. Soc. 138 (2010) 2657 – 2665.

[5] P. Erdős, *Some remarks on the theory of graphs*, Bull. Amer. Math. Soc. 53 (1947), 292 – 294.

[6] P. Erdős and G. Szekeres, *A combinatorial problem in geometry*, Compositio Mathematica, 2: 463 – 470, (1935)

[7] M. Forti, *Quasi-selective and weakly Ramsey ultrafilters*, arXiv:1012.4338.

[8] K. Kunen, *Set Theory–an introduction to independence proofs*, North–Holland, Amsterdam, 1980.

[9] K. Kunen, *Set Theory*, Mathematical logic and foundations, vol. 34, College Publications, 2011.

[10] S. Shelah, *Proper and Improper Forcing*, second ed., Springer-Verlag, 1998.

[11] J. Spencer, (1975), *Ramsey's theorem—a new lower bound*, Journal of Combinatorial Theory Ser. A, 18, 108 – 115

[12] P. Vojtáš, *On* ω^* *and absolutely divergent series*, Topology Proceedings 19, 335 – 348, 1994.