Rapid Interval P-points

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Abstract

In this paper a rapid non-interval P-point and a rapid non-weakly Ramsey interval P-point are constructed assuming Continuum Hypothesis or Martin's Axiom. These constructions, with the help of geometric interpretations for some needed configurations, provide the examples of rapid ultrafilters for an affirmative answer to a question of Blass, Di Nasso, and Forti.

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Selective ultrafilter, P-point, interval P-point, weakly Ramsey ultrafilter, quasi-selective ultrafilter, rapid ultrafilter
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1. Introduction

Let ω be the set of all nonnegative integers. By a partition of ω we mean a family $\mathcal{P} = \{P_n \subseteq \omega \mid n \in \omega\}$ such that $\omega = \bigcup_{n\in\omega} P_n$ and $P_m \cap P_n = \emptyset$ for any distinct $m, n \in \omega$.

A family $\mathcal{F}$ of subsets of ω is a non-principal ultrafilter on ω if $\omega \in \mathcal{F}$, $F \notin \mathcal{F}$ for every finite subset F of ω, $A \cap B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$, $B \in \mathcal{F}$ whenever $B \supseteq A$ for some $A \in \mathcal{F}$, and $A \in \mathcal{F}$ or $\omega \setminus A \in \mathcal{F}$ for every $A \subseteq \omega$.

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♦ All ultrafilters considered in this paper are non-principal ultrafilters on ω.

Definition 1.1. *Let $\mathcal{F}$ be an ultrafilter. Then*

- *$\mathcal{F}$ is selective if for any partition $\mathcal{P} = \{P_n \mid n \in \omega\}$ of ω, there is an $A \in \mathcal{F}$ such that either $A = P_n$ for some $n \in \omega$ or $|A \cap P_n| \leq 1$ for every $n \in \omega$;*

- *$\mathcal{F}$ is a P-point if for any partition $\mathcal{P} = \{P_n \mid n \in \omega\}$ of ω, there is an $A \in \mathcal{F}$ such that either $A = P_n$ for some $n \in \omega$ or $|A \cap P_n| < \omega$ for every $n \in \omega$;*

- *$\mathcal{F}$ is a Q-point if for any partition $\mathcal{P} = \{P_n \mid n \in \omega\}$ of ω with each P_n being a finite set, there is an $A \in \mathcal{F}$ such that $|A \cap P_n| \leq 1$ for every $n \in \omega$;*

- *$\mathcal{F}$ is rapid if for any partition $\mathcal{P} = \{P_n \mid n \in \omega\}$ of ω with each P_n being finite, there is an $A \in \mathcal{F}$ such that $|A \cap P_n| \leq n$ for every $n \in \omega$.*

Selective ultrafilter, P-point ultrafilter, Q-point ultrafilter, rapid ultrafilter, etc. are important points in the compact topological space $\beta\omega \setminus \omega$ of all non-principal ultrafilters.

The conditions for their existence and their properties have been intensively studied (cf. [1, 3, 13]). Clearly, a selective ultrafilter is a Q-point and a Q-point is rapid. It is also clear that an ultrafilter $\mathcal{F}$ is selective if and only if $\mathcal{F}$ is both a P-point and a Q-point.

Selective ultrafilter, P-point ultrafilter, Q-point ultrafilter, and rapid ultrafilter can also be characterized by the functions from ω to ω. The following proposition is well-known.

Proposition 1.2. *Let $\mathcal{F}$ be an ultrafilter. Then*

- *$\mathcal{F}$ is selective if and only if (i) for any function $f : \omega \to \omega$ there is an $A \in \mathcal{F}$ such that $f \restriction A$ is nondecreasing, if and only if (ii) for any function $f : \omega \to \omega$ there is an $A \in \mathcal{F}$ such that $f \restriction A$ is either constant or increasing;*

- *$\mathcal{F}$ is a P-point if and only if for any function $f : \omega \to \omega$ there is an $A \in \mathcal{F}$ such that $f \restriction A$ is either constant or $\lim_{n \in A, n \to \infty} f(n) = \infty$;*

- $\mathcal{F}$ *is a Q-point if and only if for any function* $f : \omega \to \omega$ *with* $\lim_{n \in A, n \to \infty} f(n) = \infty$*, there is an* $A \in \mathcal{F}$ *such that* $f \restriction A$ *is one-to-one;*

- $\mathcal{F}$ *is rapid if and only if for any function* $f : \omega \to \omega$ *there is an* $A = \{a_0 < a_1 < a_2 < \cdots\} \in \mathcal{F}$ *such that* $f(n) \leq a_n$ *for all* $n \in \omega$*.*

Quasi-selective ultrafilters are introduced in [2] where it is used to construct an asymptotic numerosity [2, Corollary 5.3]. For any set X let

$$[X]^2 = \{\{a, b\} \mid a, b \in X,\, a \neq b\}.$$

Definition 1.3. *Let* $\mathcal{F}$ *be an ultrafilter on* ω*. Then*

- $\mathcal{F}$ *is quasi-selective if for any* $f : \omega \to \omega$ *satisfying* $f(n) \leq n$ *for every* $n \in \omega$*, there exists an* $A \in \mathcal{F}$ *such that* $f \restriction A$ *is nondecreasing.*

- $\mathcal{F}$ *is weakly Ramsey if for any coloring* $c : [\omega]^2 \to \{0, 1, 2\}$ *with three colors, there exists an* $A \in \mathcal{F}$ *such that* $|c([A]^2)| \leq 2$*.*

A weakly Ramsey ultrafilter $\mathcal{F}$ becomes a Ramsey ultrafilter if the conclusion $|c([A]^2)| \leq 2$ in the second part of Definition 1.3 is replaced by $|c([A]^2)| \leq 1$. An ultrafilter is Ramsey if and only if it is selective by a result of K. Kunen (see [3]).

Clearly, a selective ultrafilter is quasi-selective and a quasi-selective ultrafilter is a P-point. A non-selective quasi-selective ultrafilter and a non-quasi-selecitve P-point are constructed in [2] assuming CH (Continuum Hypothesis) or MA (Martin's Axiom). The non-selective quasi-selective ultrafilter constructed in [2] is in fact weakly Ramsey. It is also proven in [2] that an ultrafilter $\mathcal{F}$ is selective if and only if $\mathcal{F}$ is quasi-selective and rapid.

In [2, 5] it is pointed out that both quasi-selective ultrafilters and weakly Ramsey ultrafilters are a special kind of P-points, called interval-to-one P-points. Let's call an interval-to-one P-point simply an interval P-point.

Definition 1.4. *Let* $\mathcal{F}$ *be an ultrafilter on* ω*. Then*

- $\mathcal{F}$ *is an interval P-point if every function* $f : \omega \to \omega$ *is interval-to-one on some* $A \in \mathcal{F}$*, i.e., for every* $k \in f(A)$ *there exists an interval* I *of integers (possibly infinite) such that* $f^{-1}(k) \cap A = I \cap A$*;*

- $\mathcal{F}$ *is an interval Q-point if every interval-to-one function on* ω *is either constant or one-to-one on some* $A \in \mathcal{F}$*.*

The question whether the classes of P-points, interval P-points, and quasi-selective or weakly Ramsey ultrafilters are distinct is asked in both [2, page 1484] and [5, page 11]. Since in [12] Shelah produces a model of ZFC in which there exist no P-points, the question above is non-vacuous only when some set theoretical assumption beyond ZFC such as, for example, CH or MA, is assumed. In [6] the second author produces a non-rapid P-point[4] which is not an interval P-point and a non-rapid interval P-point which is neither quasi-selective nor weakly Ramsey assuming CH or MA. The question whether these ultrafilters can be made rapid has remained open until the current research. We will indeed construct rapid versions of these ultrafilters in this paper. Notice that Shelah constructed a model of ZFC (cf. [12]) in which there is a selective ultrafilter which is, up to isomorphism, unique among all P-points. Hence it is consistent that rapid P-point exists while non-rapid P-point does not. On the other hand, Miller proved (cf. [9]) that in Laver's model of proving the consistency of Borel conjecture rapid ultrafilter does not exist while non-rapid P-point does. Hence the existences of non-rapid P-points and rapid P-points are consistently different. Notice also that if $\mathcal{F}$ is a non-selective rapid ultrafilter, then $\mathcal{F}$ is automatically non-quasi-selective.

In §2 a rapid non-interval P-point is constructed assuming CH. In §3 a rapid non-weakly Ramsey interval P-point is constructed assuming CH. An interesting part of the constructions is the use of geometric interpretations of some needed configurations of integers. The ideas of the proofs become transparent through these geometric interpretations. These constructions can also be carried out under MA (or even weaker axioms such as $\mathfrak{p} = \mathfrak{c}$) instead of CH. The only change of the proofs one needs to make is to define some *c.c.c.* partial order which will add a set in the stage of a limit ordinal in the inductive construction of a sequence of length $\mathfrak{c}$ where a standard diagonal argument is used when CH is assumed (see Lemma 2.9 and Lemma 3.8). Since the constructions of these *c.c.c* partial orders are fairly routine, we will not include the proofs of the results under MA in this paper.

It is easy to see that an ultrafilter $\mathcal{F}$ is selective if and only if $\mathcal{F}$ is both an interval P-point and an interval Q-point. Hence a selective ultrafilter can be characterized by the following pairs.

Proposition 1.5. *Let $\mathcal{F}$ be an ultrafilter. The following are equivalent:*

[4]Non-rapid P-point is called slow P-point in [6]

1. $\mathcal{F}$ *is a selective ultrafilter;*
2. $\mathcal{F}$ *is both a P-point and a Q-point;*
3. $\mathcal{F}$ *is both rapid and quasi-selective;*
4. $\mathcal{F}$ *is both an interval P-point and an interval Q-point.*

The equivalences between (1) and (2) and between (1) and (4) are s-traightforward. The equivalence between (1) and (3) is due to Blass, Di Nasso, and Forti in [2].

Since the interval P-points form a class of ultrafilters strictly between the class of all quasi-selective or Ramsey ultrafilters and the class of all P-point ultrafilters, it is natural to think that the interval Q-points should also form a class of ultrafilters strictly between the class of all rapid ultrafilters and the class of all Q-point ultrafilters. However, this is not the case. In fact the definition of an interval Q-point does not yield new concept. It is a known fact that an ultrafilter is an interval Q-point if and only if it is a Q-point. In §4 we pose a few questions related to interval P-points.

In this paper a nonnegative integer n can also be viewed as a set $\{0, 1, 2, \ldots, n-1\}$. By an interval $[a, b]$ we mean exclusively an interval of integers.

2. Rapid non-interval P-point

In this section we construct, assuming CH, a rapid P-point which is not an interval P-point.

Lemma 2.1. *For positive integers $k, l \in \omega$, let*

$$W = W(k, l) \geq \binom{k(l-1)+1}{l} k(l-1) + 1.$$

For any k-coloring $c : W \times W \to k$ there exist $S, T \subseteq W$ such that $|S| = |T| = l$ and $c \restriction S \times T$ is a constant function.

Proof The argument is just an iterated applications of the pigeonhole principle. For each $i \in W$, by the pigeonhole principle, we can find $X_i \subseteq k(l-1)+1$ such that $|X_i| = l$ and $c \restriction \{i\} \times X_i \equiv c_i$. Notice that $k(l-1)+1 \subseteq W$. Since there are $\binom{k(l-1)+1}{l}$ possible choices of X_i and k possible choices of c_i, one can find, by the pigeonhole principle again, a set $S \subseteq W$ with $|S| = l$ such that $X_i = X_j$ and $c_i = c_j$ for any $i, j \in S$. Let $T = X_i$ and $c_0 = c_i$ for any $i \in S$. Then $c \restriction S \times T \equiv c_0$. □

Remark 2.2. *In the lemma above, $W \times W$ can be replaced by any set $W_1 \times W_2 \subseteq \omega^2$ provided that $|W_1| = |W_2| \geq \binom{k(l-1)+1}{l} k(l-1)+1$. Notice that we do not need $|W_1| = |W_2|$ in the proofs later. The requirement that $|W_1| = |W_2|$ is only for notational convenience.*

Definition 2.3. *Let $a_0 = 0$ and $a_n - a_{n-1} = n$ for all $n > 0$. Define*

$$\Delta_n = [a_{n-1}, a_n - 1] \times n \text{ for } n > 0 \text{ and } \Delta = \bigcup_{n=1}^{\infty} \Delta_n.$$

Clearly, $|\Delta_n| = n^2$.

Remark 2.4. *The set Δ can be identified as ω via the map ϕ where*

$$\phi((i,j)) = \sum_{k=1}^{n-1} |\Delta_k| + (i - a_{n-1})n + j$$

for every $(i,j) \in \Delta_n$ for some $n > 0$. Notice that the order on Δ induced by ϕ is lexicographical, i.e., $\phi((i,j)) < \phi((i',j'))$ if and only if either $i < i'$ or $i = i'$ and $j < j'$. For a function $f : \Delta \to \omega$ and $R \subseteq \Delta$, we say that $f \restriction R$ is interval-to-one if $f \circ \phi^{-1} \restriction \phi(R)$ is interval-to-one in the usual sense.

For any k-tuple $\bar{a} = (a_1, a_2, \ldots, a_k)$ and $i \in [1,k]$ let $\pi_i(\bar{a}) = a_i$ be the projection of $\bar{a}$ to the i-th coordinate. For example,

$$\pi_1((i,j)) = i \text{ and } \pi_2((i,j)) = j.$$

Lemma 2.5. *Let $A \subseteq \Delta$ be such that there exist $n > 0$ and $S_n \times T_n \subseteq \Delta_n \cap A$ with $|S_n| = |T_n| > 1$. Then $\pi_2 \restriction A$ is not interval-to-one.*

Proof Let $i_1, i_2 \in S_n$ and $j_1, j_2 \in T_n$ be such that $i_1 < i_2$ and $j_1 < j_2$. Then $(i_1, j_1), (i_2, j_1) \in \pi_2^{-1}(j_1)$ and $(i_1, j_2), (i_2, j_2) \in \pi_2^{-1}(j_2)$. Clearly, π_2 is not interval-to-one since $\phi((i_1,j_1)) < \phi((i_1,j_2)) < \phi((i_2,j_1)) < \phi((i_2,j_2))$. $\square$

Remark 2.6. *We will construct a non-interval P-point $\mathcal{F}$ on Δ instead of on ω because it is easier to visualize the box-like configuration in some Δ_n which will be used to prevent $\mathcal{F}$ from becoming an interval P-point.*

Definition 2.7. *We call that a set $A \subseteq \Delta$ is* nice *if there are $S_n \times T_n \subseteq \Delta_n \cap A$ with $|S_n| = |T_n| = l_n$ such that $\{l_n \mid n \in \omega\}$ is unbounded.*

Lemma 2.8. *If $A \subseteq \Delta$ is nice and $f : \Delta \to \omega$, then there is a $B \subseteq A$ such that B is still nice and $f \restriction B$ is either constant or finite-to-one.*

Proof We construct an increasing sequence of nonnegative integers $\{n_l \mid l \in \omega\}$ and a sequence $\{S_{n_l} \times T_{n_l} \subseteq \Delta_{n_l} \cap A \mid l \in \omega\}$ inductively such that $|S_{n_l}| = |T_{n_l}| \geq l$ and either

$$f(S_{n_l} \times T_{n_l}) \cap f(S_{n_{l'}} \times T_{n_{l'}}) = \emptyset$$

for each $l' < l$ or $f \restriction S_{n_l} \times T_{n_l}$ is constant for every $l \in \omega$.

Take $n_0 = 0$ and $S_0 = T_0 = \emptyset$. Suppose we already constructed $\{n_l : l < m\}$ and $S_{n_l} \times T_{n_l} \subseteq \Delta_{n_l} \cap A$ with $|S_{n_l}| = |T_{n_l}| \geq l$ for every $l < m$. Denote

$$X_m = f\left(\bigcup_{l<m} S_{n_l} \times T_{n_l}\right).$$

Let $|X_m| + 1 = k$. Since A is nice, we can find a sufficiently large $n_m > n_{m-1}$ and $S' \times T' \subseteq \Delta_{n_m} \cap A$ such that $|S'| = |T'| \geq \binom{k(m-1)+1}{m} k(m-1) + 1$. Define $c_m : S' \times T' \to X_m \cup \{-1\}$ by

$$c_m((i,j)) = \begin{cases} f((i,j)) & \text{if } f((i,j)) \in X_m, \\ -1 & \text{otherwise.} \end{cases}$$

By Lemma 2.1 we can find $S_{n_m} \times T_{n_m} \subseteq S' \times T'$ such that $|S_{n_m}| = |T_{n_m}| \geq m$ and $c_m \restriction S_{n_m} \times T_{n_m}$ is a constant function, say $c_m \restriction S_{n_m} \times T_{n_m} \equiv \bar{c}_m$. This completes the inductive construction.

Case 1: $\{l \in \omega \mid \bar{c}_l \equiv -1\}$ is infinite.

Let $B = \bigcup\{S_{n_l} \times T_{n_l} \mid \bar{c}_l \equiv -1\}$. Then $f \restriction B$ is a finite-to-one function. So we can assume that $\bar{c}_l = -1$ for only finitely many l.

Case 2: $\{\bar{c}_l \mid l \in \omega\}$ is unbounded.

Choose a subsequence $\{n_{l_i} \mid i \in \omega\}$ such that $\{\bar{c}_{l_i} \mid i \in \omega\}$ is strictly increasing and let $B = \bigcup\{S_{n_{l_i}} \times T_{n_{l_i}} \mid i \in \omega\}$. Then $f \restriction B$ is a finite-to-one function.

Case 3: $\{\bar{c}_l \mid l \in \omega\}$ is bounded.

Then there is a nonnegative integer $\bar{c}$ such that $\{l \in \omega \mid \bar{c}_l = \bar{c}\}$ is infinite. Let $B = \bigcup\{S_{n_l} \times T_{n_l} \mid l \in \omega,\ \bar{c}_l = \bar{c}\}$. Clearly, $f \restriction B \equiv \bar{c}$. $\square$

Lemma 2.9. *If $\{A_n \subseteq \Delta \mid n \in \omega\}$ is a sequence of nice subsets of Δ such that $A_{n+1} \subseteq^* A_n$ for every $n \in \omega$, i.e., $A_{n+1} \setminus A_n$ is a finite set, then there is a nice subset $B \subseteq \Delta$ such that $B \subseteq^* A_n$ for all $n \in \omega$.*

Proof This can be proven by a standard diagonal argument. □

For each set $S \subseteq \Delta$ let e_S be the enumeration function of the set S in the sense of lexicographical ordering. Notice that $\phi \circ e_S = e_{\phi(S)}$ is the enumeration function of $\phi(S)$ with the usual meaning.

Lemma 2.10. *If A is nice and $g : \omega \to \omega$, then there is a $B \subseteq A$ such that B is nice and $\phi(e_B(n)) \geq g(n)$ for all $n \in \omega$.*

Proof Without loss of generality we can assume that g is strictly increasing. We construct an increasing sequence of nonnegative integers $\{n_l \mid l \in \omega\}$ and a sequence $\{S_{n_l} \times T_{n_l} \subseteq \Delta_{n_l} \cap A \mid l \in \omega\}$ such that $|S_{n_l}| = |T_{n_l}| = l$ and $\phi((e_B)(n)) \geq g(n)$ for all $n \in \omega$ where $B = \bigcup\{S_{n_l} \times T_{n_l} \mid l \in \omega\}$.

Let $n_0 = 0$ and $S_0 = T_0 = \emptyset$. Suppose that we have found $\{S_{n_l} \times T_{n_l} \mid l < m\}$ such that $\phi(e_{B_{m-1}}(n)) \geq g(n)$ for every $n \leq \sum_{l=1}^{m-1} l^2$ where

$$B_{m-1} = \bigcup\{S_{n_l} \times T_{n_l} \mid l < m\}.$$

Choose $n_m \in \omega$ such that $n_m > n_{m-1}$, $\phi(a_{n_m-1}) \geq g\left(\sum_{l=0}^{m} l^2\right)$, and there exists $S_{n_m} \times T_{n_m} \subseteq \Delta_{n_m} \cap A$ with $|S_{n_m}| = |T_{n_m}| \geq m$. The existence of $S_{n_m} \times T_{n_m}$ is guaranteed by the fact that A is nice. By trimming down the size, we can assume that $|S_{n_m}| = |T_{n_m}| = m$. Let $B_{n_m} = \bigcup\{S_{n_l} \times T_{n_l} \mid l \leq m\}$. Then $|B_{n_m} \setminus B_{n_{m-1}}| = m^2$. For each $n \in \omega$ with $\sum_{l=1}^{m-1} l^2 < n \leq \sum_{l=1}^{m} l^2$ we have that

$$\phi(e_{B_m}(n)) \geq \phi(a_{n_m-1}) \geq g\left(\sum_{l=0}^{m} l^2\right) \geq g(n).$$

This completes the construction.

Let $B = \bigcup\{S_{n_l} \times T_{n_l} \mid l \in \omega\}$. Then B is a nice subset of A and $\phi(e_B(n)) \geq g(n)$ for every $n \in \omega$. □

Theorem 2.11 (CH)**.** *There is a rapid non-interval P-point.*

Proof Let ${}^{\Delta}\omega = \{f_\alpha : \alpha < \omega_1\}$ and ${}^{\omega}\omega = \{g_\alpha : \alpha < \omega_1\}$.

We will construct $\{A_\alpha \subseteq \Delta \mid \alpha < \omega_1\}$ inductively on α satisfying the following conditions:

(1) $\forall \alpha < \omega_1\, (A_\alpha \textit{ is nice})$,

(2) $\forall \alpha < \beta < \omega_1\, (A_\beta \subseteq^* A_\alpha)$,

(3) $\forall \alpha < \omega_1$, $f_\alpha \restriction A_\alpha$ is constant or $f_\alpha \restriction A_\alpha$ is finite-to-one,

(4) $\forall \alpha < \omega_1\, \forall n \in \omega\, (\phi(e_{A_\alpha}(n)) \geq g_\alpha(n))$.

Suppose that $\{A_\alpha \mid \alpha < \delta\}$ for some $\delta < \omega_1$ has been obtained. We find A_δ in the following three steps.

Firstly, let $B_1 \subseteq \Delta$ be a nice set such that $B_1 = \Delta$ if $\delta = 0$ or $B_1 = A_{\delta-1}$ if δ is a successor ordinal or $B_1 \subseteq^* A_\alpha$ for every $\alpha < \delta$ by Lemma 2.9 if δ is a limit ordinal. Secondly, let $B_2 \subseteq B_1$ be nice such that $f_\delta \restriction B_2$ is either constant or finite-to-one by Lemma 2.8. Finally, let $A_\delta = B_3 \subseteq B_2$ be nice such that $\phi(e_{B_3}(n)) \geq g_\delta(n)$ for every $n \in \omega$.

By condition (2) the sets in $\{A_\alpha \mid \alpha < \omega_1\}$ generate a non-trivial filter $\mathcal{F}$. Since every nice set is infinite, $\mathcal{F}$ is non-principal. Notice that $\mathcal{F}$ is an ultrafilter because for any $A \subseteq \Delta$ the characteristic function χ_A is f_α for some $\alpha < \omega_1$. So $\chi_A \restriction A_\alpha \equiv 1$ implies $A_\alpha \subseteq A$ and $\chi_A \restriction A_\alpha \equiv 0$ implies $A_\alpha \subseteq \omega \setminus A$. Hence $A \in \mathcal{F}$ or $\omega \setminus A \in \mathcal{F}$. The ultrafilter $\mathcal{F}$ is rapid by condition (4), a P-point by condition (2), and not an interval P-point by condition (1) and Lemma 2.5 because the function π_2 is not interval-to-one on any $A \in \mathcal{F}$. $\square$

3. Rapid non-weakly Ramsey interval P-point

In this section we construct, assuming CH, a non-weakly Ramsey rapid interval P-point.

Definition 3.1. *We will use the following notation frequently in this section.*

$$\Gamma = \{(x, y, z) \in \omega^3 \mid x \in \omega \text{ and } y, z < x\}$$

$$\Gamma_x = \Gamma \cap \{(x, y, z) \mid y, z < x\} \text{ for each } x \in \omega$$

$$\Gamma_{x,y} = \Gamma_x \cap \{(x, y, z) \mid z < x\} \text{ for any } x \in \omega \text{ and } y < x\,.$$

The set Γ_x will be called x-th plane and $\Gamma_{x,y}$ will be called $\{x, y\}$-th vertical line.

Remark 3.2. *We can identify* Γ *as* ω *via the map*

$$\psi((x,y,z)) = \sum_{j=0}^{x-1} j^2 + \sum_{j=0}^{y-1} jx + z,$$

i.e., the order among the elements in Γ *is lexicographical, i.e.,*

$$\psi((x_1,y_1,z_1)) < \psi((x_2,y_2,z_2))$$

if and only if $x_1 < x_2$*, or* $x_1 = x_2$ *but* $y_2 < y_2$*, or* $x_1 = x_2$ *and* $y_1 = y_2$ *but* $z_1 < z_2$*. We call a function* $f \in {}^{\Gamma}\omega$ *interval-to-one on* $T \subseteq \Gamma$ *if* $f \upharpoonright T$ *is interval-to-one in the sense of lexicographical ordering of* Γ*.*

We now define a coloring c of $[\Gamma]^2$ with three colors. Our goal is to make sure the weakly Ramsey-ness fails for c.

Definition 3.3. *For any* $\{(x_1,y_1,z_1),(x_2,y_2,z_2)\} \in [\Gamma]^2$ *let*

$$c((x_1,y_1,z_1),(x_2,y_2,z_2)) = \begin{cases} 0, & \text{if } x_1 = x_2 \text{ and } y_1 = y_2 \\ 1, & \text{if } x_1 = x_2 \text{ and } y_1 \neq y_2 \\ 2, & \text{if } x_1 \neq x_2 \end{cases}$$

A set $A \subseteq \Gamma$ *is called* $\{i,j\}$*-homogeneous of* c *if* $c([A]^2) = \{i,j\}$*.*

Remark 3.4. *Notice that*

1. *if* A *is a* $\{0\}$*-homogeneous set of* c*, then* A *is a subset of an* $\{x,y\}$*-vertical line for some* $x, y \in \omega$*;*
2. *if* A *is a* $\{1\}$*-homogeneous set of* c*, then* A *is a subset of the graph of a function* $g : \{x\} \times [0, x-1] \to [0, x-1]$ *in* x*-th plane;*
3. *if* A *is a* $\{2\}$*-homogeneous set of* c*, then* A *is a subset of the graph of a function* $g : \omega \to \omega^2$*;*
4. *if* A *is a* $\{0,1\}$*-homogeneous set of* c*, then* A *is a subset of the* x*-th plane for some* $x \in \omega$*;*
5. *if* A *is a* $\{0,2\}$*-homogeneous set of* c*, then* $A \cap \Gamma_x$ *is a subset of one vertical line for every* $x \in \omega$*;*

6. *if A is a $\{1,2\}$-homogeneous set of c, then $|A \cap \Gamma_{x,y}| \leq 1$ for any $y < x$ in ω.*

We employ again some geometric interpretations of configurations. It is easier to visualize in Γ instead of in ω how we can construct an ultrafilter $\mathcal{F}$ so that no sets in $\mathcal{F}$ are $\{i,j\}$-homogeneous. This will prevent $\mathcal{F}$ from becoming weakly Ramsey.

Definition 3.5. *We say that a subset B_x of Γ_x is k-large if there exist at least k vertical lines Γ_{x,y_i} such that $|\Gamma_{x,y_i} \cap B_x| \geq k$ for $i = 1, 2, \ldots, k$. Notice that a k-large set contains l-large subset for any $l \leq k$. We say that B_x is exactly k-large if $B_x \subseteq \bigcup_{i=1}^{k} \Gamma_{x,y_i}$ and $|B_x \cap \Gamma_{x,y_i}| = k$ for distinct elements $\{y_i < x \mid i = 1, 2, \ldots, k\}$. Notice that every k-large set contains an exactly k-large subset and if B_x is exactly k-large, then $|B_x| = k^2$.*

Definition 3.6. *We call a set $A \subseteq \Gamma$ pretty if for every $k \in \omega$, there exists an $x \geq k$ such that $A \cap \Gamma_x$ is k-large.*

Remark 3.7. *If A is a pretty subset of Γ, then $|c([A]^2)| = 3$ because a pretty subset of Γ cannot be a subset of $\{i,j\}$-homogeneous set of c for any $\{i,j\} \subseteq 3$.*

Lemma 3.8. *If $\{A_n \subseteq \Gamma \mid n \in \omega\}$ is a $\subseteq^*$-decreasing sequence of pretty subsets of Γ, then there is a pretty set $A \subseteq \Gamma$ such that $A \subseteq^* A_n$ for all $n \in \omega$.*

Proof Without loss of generality we can assume that $A_{n+1} \subseteq A_n$ for every $n \in \omega$. We construct an increasing sequence $\{x_n \mid n \in \omega\}$ and a sequence

$$\{B_{x_n} \subseteq \Gamma_{x_n} \cap A_n \mid n \in \omega\}$$

such that B_{x_n} is n-large.

Suppose we have found $\{B_{x_l} \mid l < n\}$. Then the existence of B_{x_n} is guaranteed by the fact that A_n is pretty so that one can find a large x_n such that $A \cap \Gamma_{x_n}$ is n-large. Now let $B = \bigcup_{n \in \omega} B_{x_n}$. Clearly, B is a pretty set and $B \subseteq^* A_n$ for every $n \in \omega$. □

Lemma 3.9. *If A is a pretty subset of Γ and $g : \omega \to \omega$, then there is a $B \subseteq A$ such that B is pretty and $\psi(e_B(n)) \geq g(n)$ for all $n \in \omega$.*

Proof Without loss of generality we can assume that g is increasing. We again construct an increasing sequence of positive integers $\{x_m \mid m \in \omega\}$ and a sequence $\{B_{x_m} \subseteq A \cap \Gamma_{x_m} \mid m \in \omega\}$ inductively such that (1) B_{x_m} is m-large and (2) for any $n \leq |\bigcup_{l\leq m} B_{x_l}|$ we have $\psi(e_{\bigcup_{l\leq m} B_{x_l}}(n)) \geq g(n)$.

Suppose that we have found $\{B_{x_l} \subseteq A \cap \Gamma_{x_l} \mid l < m\}$ such that B_{x_l} is l-large and $\psi(e_{\bigcup_{l<m} B_{x_l}}(n)) \geq g(n)$ for every $n \leq |\bigcup_{l<m} B_{x_l}|$. Because A is pretty we can choose x_m sufficiently large such that $\psi((x_m, 0, 0)) \geq g(N)$ where

$$N = |\bigcup_{l<m} B_{x_l}| + m^2$$

and $A \cap \Gamma_{x_m}$ is m-large. Let $B_{x_m} \subseteq A \cap \Gamma_{x_n}$ be exactly m-large. For each $|\bigcup_{l<m} B_{x_l}| < n \leq |\bigcup_{l\leq m} B_{x_l}| = |\bigcup_{l<m} B_{x_l}| + m^2$ we have

$$\psi(e_{\bigcup_{l\leq m} B_{x_l}}(n)) \geq \psi((x_m, 0, 0)) \geq g(N) \geq g(n).$$

This completes the inductive construction and hence the proof of the lemma.
□

Lemma 3.10. *Let $X \subseteq \omega$ be such that $|X| \geq n^2$ and $f : X \to \omega$. There exists a $Y \subseteq X$ with $|Y| \geq n$ such that $f \restriction Y$ is constant or one-to-one.*

Proof If $|f(X)| \geq n$ we can form Y by selecting one element from each $f^{-1}(m)$ for every $m \in f(X)$. If $|f(X)| < n$, then there exists one $m \in f(X)$ such that $Y = f^{-1}(m)$ contains more than n elements. □

Lemma 3.11. *If $A_x \subseteq \Gamma_x$ is k^4-large and $f : A_x \to \omega$, then there exists a set $B_x \subseteq A_x$ such that B_x is k-large and $f \restriction B_x$ is either a constant function, or a one-to-one function, or a constant function with distinct value on each vertical line, i.e., $f \restriction (B_x \cap \Gamma_{x,y}) \equiv v_y$ for each $y \in \pi_2(B_x)$ and v_y are distinct for all $y \in \pi_2(B_x)$.*

Proof Without loss of generality we can assume that A_x is exactly k^4-large and $A_x = \bigcup_{y\in Y} A_x \cap \Gamma_{x,y}$ for some $Y \subseteq [0, x-1]$ with $|Y| = k^4$. By Lemma 3.10, for each $y \in Y$ we fix a subset $B'_{x,y} \subseteq A_x \cap \Gamma_{x,y}$ such that $|B'_{x,y}| = k^2$ and $f \restriction B'_{x,y}$ is constant with value v_y or one-to-one. By the pigeonhole principle we can find a $Y' \subseteq Y$ with $|Y'| = k^2$ such that f is constant on $B'_{x,y}$ for all $y \in Y'$ or one-to-one on $B'_{x,y}$ for all $y \in Y'$.

Suppose f is constant with the value v_y on $B'_{x,y}$ for all $y \in Y'$. Then by Lemma 3.10 again we can find $Y'' \subseteq Y'$ with $|Y''| = k$ such that v_y for $y \in Y''$ are all the same or all different. Let $B'_x = \bigcup_{y\in Y''} B'_{x,y}$.

If v_y are all the same, then $f \restriction B'_x$ is a constant function. If v_y for $y \in Y''$ are all different, then $f \restriction B'_x$ is a constant function with distinct value on each vertical line $B'_{x,y}$. Clearly, $B_x = B'_x$ is k-large.

Suppose that f is one-to-one on $B'_{x,y}$ for all $y \in Y'$. We construct a sequence of sets $\{C_{x,y_i} \subseteq B'_{x,y_i} \mid i = 1, 2, \ldots, k\}$ inductively such that y_i are distinct in Y', $|C_{x,y_i}| = k$, and $\{f(C_{x,y_i}) \mid i = 1, 2, \ldots, k\}$ contains pairwise disjoint sets. Let y_0 be any element in Y' and C_{x,y_1} be any subset of B'_{x,y_1} with $|C_{x,y_1}| = k$. Suppose we have found $\{C_{x,y_j} \mid j < i\}$ for some $i \leq k$. Let $y_i \in Y' \smallsetminus \{y_1, y_2, \ldots, y_{i-1}\}$ and

$$D = \left\{ a \in B'_{x,y_i} \mid f(a) \in f\left(\bigcup_{j<i} C_{x,y_j} \right) \right\}.$$

Since f is one-to-one on each $B'_{x,y}$, we have

$$|D| \leq (i-1)k \leq (k-1)k.$$

Hence there is a $C_{x,y_i} \subseteq B'_{x,y_i} \smallsetminus D$ with $|C_{x,y_i}| = k$ such that

$$f(C_{x,y_i}) \cap f\left(\bigcup_{j<i} C_{x,y_j} \right) = \emptyset.$$

This completes the construction. Now let $B_x = \bigcup_{i=1}^k C_{x,y_i}$. Clearly, we have that f is one-to-one on B_x and B_x is k-large. $\square$

Lemma 3.12. *For each pretty set $A \subseteq \Gamma$ and $f : \Gamma \to \omega$, there exists a pretty set $B \subseteq A$ such that $f \restriction B$ is either constant or interval-to-one.*

Proof Since A is pretty, we can find an infinite set $J \subseteq \omega$ such that $A \cap \Gamma_x$ is k^4-large when x is the k-th element of J. Let $A'_x \subseteq A \cap \Gamma_x$ be such that A'_x is exactly k^4-large. By Lemma 3.11 we can find $A''_x \subseteq A'_x$ such that A''_x is exactly k-large and $f \restriction A''_x$ is either one-to-one, or constant, or constant on each vertical line with distinct values. Clearly, $A' = \bigcup_{x \in J} A''_x$ is a pretty subset of A.

According to the type of $f \restriction (A' \cap \Gamma_x)$ we partition J into three sets:

- $J_1 = \{x \in J \mid f \restriction (A' \cap \Gamma_x) \equiv V_x\}$,
- $J_2 = \{x \in J \mid f \restriction (A' \cap \Gamma_{x,y}) \equiv v_{x,y}$ and $v_{x,y}$ are different for all $y \in \pi_2(A' \cap \Gamma_x)\}$,

- $J_3 = \{x \in J \mid f \restriction (A' \cap \Gamma_x) \text{ is a one-to-one function.}\}$.

Case 1: J_1 is infinite.

If $\{V_x \mid x \in J_1\}$ is bounded, then there is an infinite set $J_1' \subseteq J_1$ such that $V_x = V$ for all $x \in J_1'$. Then $B = \bigcup_{x \in J_1'} (A' \cap \Gamma_x)$ is a pretty set and $f \restriction B \equiv V$. If $\{V_x \mid x \in J_1\}$ is unbounded, we can find an infinite subset $J_1' \subseteq J_1$ such that $\{V_x \mid x \in J_1'\}$ is strictly increasing. Let $B = \bigcup_{x \in J_1'} (A' \cap \Gamma_x)$. Then $f \restriction B$ is a nondecreasing function, hence interval-to-one.

Case 2: J_2 is infinite.

We construct an increasing sequence $\{x_n \in J_2 \mid n \in \omega\}$ and a sequence

$$\{B_{x_n} \subseteq A' \cap \Gamma_{x_n} \mid n \in \omega\}$$

by induction so that B_{x_n} is exactly n-large and $f\left(\bigcup_{k<n} B_{x_k}\right) \cap f(B_{x_n}) = \emptyset$ for all $n \in \omega$. Suppose we have found $\{B_{x_k} \mid k < n\}$. Let $X = f\left(\bigcup_{k<n} B_{x_k}\right)$. Then

$$|X| \leq \sum_{k=0}^{n-1} |f(B_{x_k})| = \sum_{k=1}^{n-1} k = \frac{1}{2}n(n-1).$$

Since A' is pretty, we can find $x_n > x_{n-1}$ sufficiently large in J_2 such that $A' \cap \Gamma_{x_n}$ is $\frac{1}{2}n(n+1)$-large. Let

$$D = \left\{ y \in \pi_2(A' \cap \Gamma_{x_n}) \mid v_{x_n,y} \in f\left(\bigcup_{k<n} B_{x_k}\right) \right\}.$$

Then $|D| \leq \frac{1}{2}n(n-1)$. Hence there is a set $Y \subseteq \pi_2(A' \cap \Gamma_{x_n})$ such that $|Y| = n$ and $Y \cap D = \emptyset$. We can now choose $B_{x_n} \subseteq \bigcup\{A' \cap \Gamma_{x_n,y} \mid y \in Y\}$ such that it is exactly n-large by deleting some elements from each vertical line $A' \cap \Gamma_{x_n,y}$ for $y \in Y$. Clearly, $f(B_{x_n}) \cap f\left(\bigcup_{k<n} B_{x_k}\right) = \emptyset$. Let $B = \bigcup_{n \in \omega} B_{x_n}$. Then B is pretty and $f \restriction B$ is interval-to-one because for each $n \in f(B)$, the set $f^{-1}(n) \cap B$ is in one vertical line.

Case 3: J_3 is infinite.

We again construct an increasing sequence $\{x_n \in J_3 \mid n \in \omega\}$ and a sequence $\{B_{x_n} \subseteq A \cap \Gamma_{x_n} \mid n \in \omega\}$ by induction so that B_{x_n} is exactly n-large and $f(\bigcup_{k<n} B_{x_k}) \cap f(B_{x_n}) = \emptyset$ for all $n \in \omega$. Suppose we have found

$\{B_{x_k} \mid k < n\}$. Let $X = f(\bigcup_{k<n} B_{x_k})$. Then

$$|X| \leq \sum_{k=0}^{n-1} |f(B_{x_k})| = \sum_{k=1}^{n-1} k^2.$$

Since A' is pretty, we can find $x_n > x_{n-1}$ sufficiently large in J_3 such that $A' \cap \Gamma_{x_n}$ is $(n + |X|)$-large. Since $f \restriction (A' \cap \Gamma_{x_n,y})$ is one-to-one for each $y \in \pi_2(A' \cap \Gamma_{x_n})$, we can find $B_{x_n,y} \subseteq A' \cap \Gamma_{x_n,y}$ such that $|B_{x_n,y}| = n$ and $f(B_{x_n,y}) \cap f\left(\bigcup_{k<n} B_{x_k}\right) = \emptyset$. Let $Y' \subseteq \pi_2(A' \cap \Gamma_{x_n})$ be such that $|Y'| = n$ and $B_{x_n} = \bigcup\{B_{x_n,y} \mid y \in Y'\}$. Then B_{x_n} is exactly n-large and $f(B_{x_n}) \cap f\left(\bigcup_{k<n} B_{x_k}\right) = \emptyset$. This completes the construction. Now let $B = \bigcup_{n\in\omega} B_{x_n}$. Clearly, B is pretty and $f \restriction B$ is one-to-one, hence interval-to-one.

The lemma is true because one of J_i for $i = 1, 2, 3$ must be infinite. □

Theorem 3.13 (CH)**.** *There is a rapid non-weakly Ramsey interval P-point.*

Proof Enumerate ${}^{\Gamma}\omega = \{f_\alpha : \alpha < \omega_1\}$ and ${}^{\omega}\omega = \{g_\alpha : \alpha < \omega_1\}$. We will construct a sequence $\{A_\alpha \subseteq \Gamma : \alpha < \omega_1\}$ inductively on α satisfying the following conditions:

(1) $\forall \alpha < \omega_1 \, (A_\alpha$ is a pretty set),

(2) $\forall \alpha < \beta < \omega_1 \, (A_\beta \subseteq^* A_\alpha)$

(3) $\forall \alpha < \omega_1$, $f_\alpha \restriction A_\alpha$ is either constant or interval-to-one.

(4) $\psi(e_{A_\alpha}(n)) \geq g_\alpha(n)$ for every $n \in \omega$.

Suppose that $\{A_\alpha \mid \alpha < \delta\}$ for some $\delta < \omega_1$ has been obtained. We find A_δ in the following three steps.

Firstly, let $B_1 \subseteq \Gamma$ be a pretty set such that $B_1 = \Gamma$ if $\delta = 0$, $B_1 = A_{\delta-1}$ if δ is a successor ordinal, or $B_1 \subseteq^* A_\alpha$ for every $\alpha < \delta$ by Lemma 3.8 if δ is a limit ordinal. Secondly, let $B_2 \subseteq B_1$ be pretty such that $f_\delta \restriction B_2$ is either constant or interval-to-one by Lemma 3.12. Finally, let $A_\delta = B_3 \subseteq B_2$ be pretty such that $\psi(e_{B_3}(n)) \geq g_\delta(n)$ for every $n \in \omega$ by Lemma 3.9.

By condition (2) the sequence $\{A_\alpha \mid \alpha < \omega_1\}$ generates a non-trivial filter $\mathcal{F}$. Since every pretty set is infinite, $\mathcal{F}$ is non-principal. Notice that $\mathcal{F}$ is an ultrafilter because for any $A \subseteq \Gamma$ the characteristic function χ_A is f_α for some $\alpha < \omega_1$. So $\chi_A \restriction A_\alpha \equiv 1$ implies $A_\alpha \subseteq A$ and $\chi_A \restriction A_\alpha \equiv 0$ implies

$A_\alpha \subseteq \omega \setminus A$. Hence we have $A \in \mathcal{F}$ or $\omega \setminus A \in \mathcal{F}$. The ultrafilter $\mathcal{F}$ is rapid by condition (4), an interval P-point by condition (3), and not weakly Ramsey by Remark 3.7. □

Notice that an rapid non-weakly interval P-point is automatically non-quasi-selective because rapidity plus quasi-selectivity is equivalent to selectivity.

Notice also that the idea of using sequences of sufficiently large products of finite sets to construct P-points with special properties as we did in this paper was used in other papers in literature such as, for example, [10] before our research.

4. Questions

Since the interval Q-point is not an properly intermediate concept between Q-point and rapid ultrafilter, we would like to ask whether it is possible to have some interesting concept strictly between rapid ultrafilter and Q-point parallel to the role of interval P-point as a properly intermediate concept between P-point and quasi-selective ultrafilter.

Question 4.1. *Is there a reasonably defined class $\mathcal{C}$ of ultrafilters strictly between the class of all Q-points and the class of all rapid ultrafilters such that an ultrafilter $\mathcal{F}$ is selective if and only if $\mathcal{F} \in \mathcal{C}$ and $\mathcal{F}$ is an interval P-point?*

Remark 4.2. *Miller [9] establishes the equivalence among three descriptions of rapid ultrafilters $\mathcal{F}$, i.e., (1) every function $f : \omega \to \omega$ is dominated by an enumeration function of some set in $\mathcal{F}$ $\iff$ (2) for any collection $\{P_n \mid n \in \omega\}$ of finite sets of nonnegative integers there exists an $X \in \mathcal{F}$ such that $|P_n \cap X| \leq n$ $\iff$ (3) there is a function $h : \omega \to \omega$ such that for any collection $\{P_n \mid n \in \omega\}$ of finite sets of nonnegative integers there exists an $X \in \mathcal{F}$ with $|P_n \cap X| \leq h(n)$.*

Interesting part of Miller's proof is that even if $h(n)$ is taken to be much larger than n, (3) still implies that $\mathcal{F}$ is rapid. What can we say if $h(n)$ is much smaller than n? Is it possible to make $h(n)$ so small that the statement in (3) becomes strictly stronger than rapidity? The answer is negative by [11, Observation 3.1]. Notice that if h is a constant function in (3), then $\mathcal{F}$ becomes a Q-point.

Observation 4.3 (D. Raghavan & J. Verner, 2018). *An ultrafilter $\mathcal{F}$ is rapid if and only if the following is true: Let $h : \omega \to \omega$ be a given function with $\lim_{n\to\infty} h(n) = \infty$. Then for any collection $\{P_n \mid n \in \omega\}$ of finite subsets of ω, there exists an $X \in \mathcal{F}$ such that $|P_n \cap X| \leq h(n)$ for all $n \in \omega$.*

Remark 4.4. *Notice that if h does not approach ∞, then there exists an infinite set $I \subseteq \omega$ such that $h \restriction I \equiv c$ for some $c \in \omega$. Now let $\{P_n \mid n \in I\}$ be any partition of ω into finite sets and let $P_n = \emptyset$ when $n \notin I$. If $X \in \mathcal{F}$ such that $|P_n \cap X| \leq h(n)$, then $|P_n \cap X| \leq c$ for every $n \in \omega$. Hence there exists an $X' \subseteq X$ in $\mathcal{F}$ such that $|P_n \cap X'| \leq 1$ for any $n \in I$, which implies that $\mathcal{F}$ is a Q-point. Therefore, controlling the values of h does not yield a proper definition of some class of ultrafilters strictly between Q-points and rapid ultrafilters.*

The following two questions ask whether the existence of the rapid P-points in this paper is consistent with the non-existence of non-rapid P-points.

Question 4.5. *Is it consistent with* ZFC *that rapid non-interval P-points exist but non-rapid P-points do not?*

Question 4.6. *Is it consistent with* ZFC *that rapid interval non-weakly Ramsey P-points exist but non-rapid P-points do not?*

Question 4.7. *Is it true that $\mathfrak{d} = \mathfrak{c}$ if and only if every $< \mathfrak{c}$-generated filter can be extended to an interval P-point?*

It is well known that P-point is isomorphic invariant. It is also shown in [2] that quasi-selective ultrafilter is not isomorphic invariant. An interval P-point is between a P-point and a quasi-selective ultrafilter. Hence it is interesting to know whether interval P-point is isomorphic invariant. We believe that it is not and would like to make a conjecture below. We hope to prove this conjecture soon.

Conjecture 4.8 (CH). *There exists a one-to-one function $f : \omega \to \omega$ and an interval P-point $\mathcal{F}$ such that $f(\mathcal{F})$ is not an interval P-point.*

Based on this conjecture, it makes no sense to discuss the Rudin-Keisler order or Rudin-Blass order among interval P-points. However, it may be interesting to explore the structure of the class of interval P-points with respect to the strong Rudin-Blass order $\leq^+_{RB}$ (see [8]).

Question 4.9 (CH)**.** *What kind of structure does the class of interval P-points have with respect to* $\leq^+_{RB}$*? How similar can this structure be to the structure of the class of P-points with respect to Rudin-Blass order?*

For example, we would like to ask, inspired by the results in [7, 11], the following question.

Question 4.10 (CH)**.** *Can we construct a sequence of length* $\mathfrak{c}^+$ *of (rapid/non-rapid) interval P-points with respect to* $\leq^+_{RB}$*? Is it true that any* $\leq^+_{RB}$*-increasing sequence of (rapid/non-rapid) interval P-points with length* $\mathfrak{c}$ *is bounded above by a (rapid/non-rapid) interval P-point?*

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