Abstract

Let $A$ be a set of $k$ integers. We study Freiman’s inverse problem with small doublings and continue the work of G. A. Freiman, I. Bardaji and D. J. Grynkiewicz by characterizing the detailed structure of $A$ in Theorem 2.2 below when the sumset $A + A$ contains exactly $3k - 3$ integers. Besides some familiar structures, such a set $A$ can have a configuration composed of “additively minimal triangles.”

1 Introduction and Propositions

The letters $A, B$ always represent finite sets of integers and $|A|$ means the cardinality of $A$. Let $A ± B := \{a ± b : a \in A, b \in B\}$ for any sets of integers $A$ and $B$. Let $2A := A + A$. If $a$ is an integer, then $A ± a := A ± \{a\}$ and $a ± A := \{a\} ± A$.

Freiman’s inverse problem for small doubling constants seeks structural information of $A$ or $2A$ when the size of $2A$ is small, say for example, less than $4|A|$.

A set $B$ is called a bi-arithmetic progression if $B = I_0 \cup I_1$ where $I_0$ and $I_1$ are arithmetic progressions with a common difference such that $2I_0$, $I_0 + I_1$, $2I_1$ are pairwise disjoint. The common difference of $I_0$ and $I_1$ is called the difference of $B$. The expression $B = I_0 \cup I_1$ gives a (bi-arithmetic

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*Mathematics Subject Classification 2010 Primary 11P70

*Keywords: Freiman’s inverse problem, arithmetic progression, bi-arithmetic progression, detailed structure, additive number theory

$^1B$ is a bi-arithmetic progression if and only if $B$ is Freiman isomorphic to two parallel line segments of the integral lattice points on the plane.
progression) decomposition of \( B \). For example, \( B = \{0, 3, 5, 6, 8\} \) is a bi-arithmetic progression of difference 3 and has a decomposition \( \{0, 3, 6\} \cup \{5, 8\} \).

Let \( G \) and \( G' \) be two abelian semi-groups, \( A \subseteq G \) and \( B \subseteq G' \). A bijection \( \varphi : A \mapsto B \) is a Freiman isomorphism (of order 2) if

\[
    a + b = c + d \quad \text{if and only if} \quad \varphi(a) + \varphi(b) = \varphi(c) + \varphi(d)
\]

for any \( a, b, c, d \in A \).

The following two classical theorems on Freiman’s inverse problem with small doublings were proven more than fifty years ago (see [2, page 11, page 15] or [8]).

**Theorem 1.1 (G. A. Freiman)** Let \( A \) be a set of \( k \) integers with \( k > 2 \). If \( |2A| = 2k - 1 + b < 3k - 3 \), then \( A \) is a subset of an arithmetic progression of length at most \( k + b \).

**Theorem 1.2 (G. A. Freiman)** Let \( A \) be a set of \( k \) integers with \( k \geq 2 \). If \( |2A| = 3k - 3 \), then one of the following is true.

1. \( A \) is a bi-arithmetic progression;
2. \( A \) is a subset of an arithmetic progression of length at most \( 2k - 1 \);
3. \( k = 6 \) and \( A \) is a Freiman isomorphism image of the set \( K_6 \) where

\[
    K_6 = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (2, 0)\} \subseteq \mathbb{Z}^2. \tag{1}
\]

Notice that the part 2 above implies that \( k > \frac{1}{2}|I| \), i.e. \( A \) is a large subset of the arithmetic progression \( I \).

Part 1 and 2 in Theorem 1.2 show the regularity of the structure of \( A \) when \( |2A| = 3k - 3 \). Part 3 is an exception. If \( A \) is the set \( \{0, 1, 2, a, a + 1, 2a\} \) for any \( a > 3 \), then \( A \) is Freiman isomorphic to \( K_6 \). Clearly, this \( A \) can be made neither a subset of an arithmetic progression nor a subset of a bi-arithmetic progression of reasonable length.

We call each element in \( V = \{(0, 2), (2, 0), (0, 0)\} \) a vertex of \( K_6 \). Notice that each permutation of \( V \) can be extended to a Freiman isomorphism from
If $\varphi : K_6 \mapsto B$ is a Freiman isomorphism, we also call the elements in $\varphi(V)$ vertices of $\varphi(K_6)$.

Theorem 1.2 is much more difficult to prove than Theorem 1.1 is. There has been a few generalizations of Theorem 1.2. In [5] it is proved that the structure of $A$ is the same as the structure of $A$ characterized in Theorem 1.2 when $|A \pm A| = 3k - 3$. In [6], a generalization of Theorem 1.2 is given, which characterizes the structure of $A$ when $k$ is sufficiently large and $|2A| = 3k - 3 + b$ for $0 \leq b \leq \epsilon k$, where $\epsilon$ is a small positive real number independent of $k$.

Recently, Freiman discovered in [3, 4] some interesting detailed structural information of $A$ when $|2A| < 3k - 3$. By saying “detailed structural information” we mean any structural information other than that of $A$ being a large subset of an arithmetic progression. The term “detailed structure” first appeared in [4]. The main result in [3, 4] is the following.

**Theorem 1.3 (A. G. Freiman, 2009)** Let $A$ be a set of $k$ integers. If

$$|2A| < 3k - 3,$$

then $2A$ contains an arithmetic progression of length $2k - 1$.

For the sum of two distinct sets, the following theorem in [1] adds extra detailed structural information to the structural information obtained in [7] and [9].

**Theorem 1.4 (I. Bardaji and D. J. Grynkiewicz, 2010)** Let $A$ and $B$ be nonempty sets of $k_1$ and $k_2$ integers, respectively, with

$$\max B - \min B \leq \max A - \min A \leq k_1 + k_2 - 3$$

and $|A + B| \leq k_1 + 2k_2 - 3 - \delta(A, B)$.

Then $A + B$ contains an arithmetic progression of length $k_1 + k_2 - 1$.

The number $\delta(A, B)$ in Theorem 1.4 is defined to be 1 if $A + t \subseteq B$ for some integer $t$ and 0 otherwise. If checking the proof in [1] carefully, the reader can find that the condition $\max A - \min A \leq k_1 + k_2 - 3$ in Theorem 1.4
can be weakened to $\max A - \min A \leq k_1 + k_2 - 2$ when $\max B - \min B < \max A - \min A$

In this paper we seek detailed structural information for $A$ when $|2A| = 3k - 3$. The most part of the structural information we have found is consistent with that in Theorem 1.3 and Theorem 1.4. But there are some significant exceptions involving a new concept of set configurations called triangles (see Definition 1.7 and Theorem 2.2).

Let $a$ and $b$ be integers. Throughout this paper we will write $[a, b]$ for the interval of integers between $a$ and $b$ including $a$ and $b$. Notice that $[a, b] = \emptyset$ if $a > b$. For any set $A$ of integers, we will use the following notation:

$$A(a, b) := |A \cap [a, b]|.$$

We now introduce a few propositions, which will be used in the proof of the main result.

**Proposition 1.5** If $A(x, y) > \frac{1}{2}(y - x + 1)$, then $y + x \in 2A$.

**Proof** The conclusion is true because $A \cap (x + y - A) \cap [x, y] \neq \emptyset$.

**Proposition 1.6** If $\varphi : K_6 \mapsto B \subseteq \mathbb{Z}$ is an Freiman isomorphism from $K_6$ in (1) to $B$, then

1. $\min B$ and $\max B$ are vertices of $\varphi(K_6)$.
2. If $x, y \in B$ are vertices, then $\frac{1}{2}(x + y) \in B$.
3. If $B \subseteq [a, b]$, then $b - a \geq 10$.
4. If $B \subseteq [0, 10]$, then $B$ is either $B_1 = \{0, 1, 2, 5, 6, 10\}$, $B_2 = \{0, 2, 4, 5, 7, 10\}$, or $B_3 = 10 - B_1$, or $B_4 = 10 - B_2$.

**Proof** Part 1, 2 follow from the definition of Freiman isomorphism.

Part 3: Suppose $\varphi(\{(0, 0), (0, 1), (0, 2)\}) = \{a, a + d, a + 2d\}$ where $a = \min B$. Then part 3 can be easily verified for $d = 1, 2, 3, \text{ or } \geq 4$.

Part 4: Suppose $\varphi(\{(0, 0), (0, 1), (0, 2)\}) = \{0, d, 2d\}$. Then part 4 can be easily verified for $10 = \varphi((0, 2))$ or $10 = \varphi((2, 0))$.

We introduce new names of some set configurations in order to be efficient and informative in describing them in part 4 of Theorem 2.2.
Definition 1.7 Let $B \subseteq [u, v]$.

- **B is anti-symmetric in** $[u, v]$ if
  
  $$B \cap (u + v - B) = \emptyset$$
  
  and
  
  $$B \cup (u + v - B) = [u, v];$$

- **B is half dense in** $[u, v]$ if $B(u, v) = \frac{1}{2}(v - u + 1);$.

- A half dense set $B$ in $[u, v]$ is a **forward triangle** in $[u, v]$ if $B(u, x) > \frac{1}{2}(x - u + 1)$ for any $x \in [u, v - 1]$. We denote $\mathcal{FT}[u, v]$ for the collection of all forward triangles in $[u, v]$;

- A half dense set $B$ in $[u, v]$ is a **backward triangle** in $[u, v]$ if $B(x, v) > \frac{1}{2}(v - x + 1)$ for any $x \in [u + 1, v]$. We denote $\mathcal{BT}[u, v]$ for the collection of all backward triangles in $[u, v]$;

- $B \in \mathcal{FT}[u, v]$ is **additively minimal** if
  
  $$|2(B \cup \{v + 1\})| = 3(|B| + 1) - 3;$$

- $B \in \mathcal{BT}[u, v]$ is **additively minimal** if
  
  $$|2(B \cup \{u - 1\})| = 3(|B| + 1) - 3;$$

- Let $\mathcal{FT}_{am}[u, v]$ and $\mathcal{BT}_{am}[u, v]$ denote the collection of all additively minimal forward triangles and the collection of all additively minimal backward triangles, respectively.

We call the interval $[u, v]$ in Definition 1.7 the **host interval** of $B$ because $B$ is half dense in $[u, v]$ even though $u$ or $v$ may not be in $B$.

The following are some consequences of Definition 1.7. For simplicity, we sometimes list only the properties of forward triangles. For backward triangles, one can easily formulate symmetric properties.

**Proposition 1.8** Let $B \subseteq [u, v]$.

1. $B$ is anti-symmetric in $[u, v]$ if and only if $B(u, v)$ is half dense in $[u, v]$ and $u + v \notin 2B$. 

2. If \( B \in \mathcal{FT}[u, v] \) and \( v - u > 1 \), then \( u, u + 1 \in B \) and \( v, v - 1 \notin B \).

3. If \( B \in \mathcal{FT}[u, v] \) and \( b > v \), then

\[
2(B \cup \{ b \}) \supseteq [2u, u + v - 1] \cup (b + (B \cup \{ b \})),
\]

which implies that \( |2(B \cup \{ b \})| \geq 3|B \cup \{ b \}| - 3 \).

4. If \( B \in \mathcal{FT}[u, v] \), then \( B \) is additively minimal if and only if \( B \) is anti-symmetric in \( [u, v] \) and

\[
(2B) \cap (v + 1 + [u, v]) \subseteq v + 1 + B.
\]

5. If \( B \in \mathcal{FT}[u, v] \), then either \( B = [u, \frac{1}{2}(u + v - 1)] \) or \( |2(B \cup \{ b \})| > 3|B \cup \{ b \}| - 3 \) for any \( b > v + 1 \).

6. If \( C \in \mathcal{BT}_{am}[1, u] \) and \( D \in \mathcal{FT}_{am}[u + 2, n - 1] \) for some \( 4 \leq u \leq n - 6 \) and \( A = \{0\} \cup C \cup D \cup \{n\} \), then \( |2A| = 3|A| - 3 \).

7. If \( P = \{0, 2, \ldots, 2(m - 1)\} \) and \( B \in \mathcal{FT}_{am}[2m, n - 1] \) for \( m \in [0, \frac{1}{2}n - 2] \), and \( A = P \cup B \cup \{n\} \), then \( |2A| = 3|A| - 3 \).

**Proof**  Part 1,2,3 are easy.

Part 4: Let \( b = v + 1 \) in part 3. Then \( B \) is additively minimal if and only if two sides of the displayed expression in part 3 are the same set, which is true if and only if \( u + v \notin 2B \) and \( (2B) \cap (v + 1 + [u, v]) \subseteq v + 1 + B \). Now \( B \) is anti-symmetric if and only if \( u + v \notin 2B \) by part 1 above. Notice that \( |2u, u + v - 1] = v - u = |2B \cup \{ v + 1 \}| - 3 \).

Part 5: For convenience we can assume, without loss of generality, that \( u = 0 \). Suppose that \( B \) is not an interval.

Let \( a = \text{max} B \). Then \( a > \frac{1}{2}(v - 1) \), which implies \( a \geq v - a \). If \( v - a \notin B \), then \( a > v - a \) because \( a \in B \). Let \( b > v + 1 \). It suffices to show that either \( v \in 2B \) or \( v + 1 \in 2B \) by part 3. Suppose that \( v \notin 2B \). Then \( B \) is anti-symmetric in \( [0, v] \) and \( v - a \notin B \). Hence

\[
\frac{1}{2}(v + 1) = B(0, v) = B(0, v - a - 1) + B(v - a + 1, a) \leq v - a + B(v - a + 1, a),
\]

6.
which implies that $B(v - a + 1, a) \geq \frac{1}{2}(2a - v + 1) > \frac{1}{2}(2a - v)$. Hence

$$v + 1 = a + (v - a + 1) \in 2(B \cap [v - a + 1, a]) \subseteq 2B$$

by Proposition 1.5.

Part 6: Let $A = \{0\} \cup C \cup D \cup \{n\}$. Then $|A| = |C| + |D| + 2 = \frac{1}{2}u + \frac{1}{2}(n - u - 2) + 2 = \frac{1}{2}(n + 2)$. By the additive minimality of the triangles $C$ and $D$ we have

$$|2A| = |0 + (\{0\} \cup C)| + ||u + 2, u + 2 + n - 2|| + |n + (D \cup \{n\})|$$

$$= |A| + n - 1 = 3|A| - 3.$$

Part 7: Let $A = P \cup B \cup \{n\}$. Then $|A| = |P| + |B| + 1 = m + \frac{1}{2}(n - 2m) + 1 = \frac{1}{2}n + 1$. By the additive minimality of the triangle $B$ and the fact that $2m, 2m + 1 \in B$, we have that

$$|2A| = |P| + ||2m, 2m + n - 2|| + |B| + 1$$

$$= m + n - 1 + \frac{1}{2}(n - 2m) + 1 = \frac{3}{2}n = 3|A| - 3.$$

Remark 1.9 Part 4 of Proposition 1.8 gives the structure of an additively minimal forward triangle. A symmetric structure can be described for an additively minimal backward triangle.

Part 5 of Proposition 1.8 justifies the definition of a forward triangle being additively minimal by looking at the cardinality of $2(B \cup \{v + 1\})$ instead of the cardinality of $2(B \cup \{b\})$ for any $b > v + 1$. Since $v + 1$ is implicitly determined by the definition of additive minimality, we can call $B$ additively minimal in its host interval $[u, v]$ without mentioning the element $v + 1$.

Blank Assumption After normalization, we can always assume, throughout this paper, that the set $A$ satisfies (2) below with letters $n$ and $k$ reserved throughout this paper.

$$0 = \min A, \gcd(A) = 1, n = \max A, \text{ and } k = |A|. \tag{2}$$

Proposition 1.10 Suppose that $0 < a < b < n$ and $A \cap [a, b] = \{a, b\}$. 

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1. Clearly,
\[(2(A \cap [0, b])) \cap (2(A \cap [a, n])) = \{2a, a + b, 2b\}.\]  

2. If \(|2A| = 3k - 3\),
\[|2(A \cap [0, b])| \geq 3A(0, b) - 3, \text{ and } |2(A \cap [a, n])| \geq 3A(a, n) - 3,\]
then \(|2(A \cap [0, b])| = 3A(0, b) - 3, |2(A \cap [a, n])| = 3A(a, n) - 3,\] and \((2A) \setminus ((2(A \cap [0, b])) \cup (2(A \cap [a, n]))) = \emptyset.\]

3. Let \(B \subseteq [u, v], u, v \in B, \text{ and } \gcd(B - u) = 1\). If \(|B| \leq \frac{1}{2}(v - u + 3)\),
then \(|2B| \geq 3|B| - 3\). If \(|2B| = 3|B| - 3\) and \(|B| \leq \frac{1}{2}(v - u + 1)\), then \(B\) is either a bi-arithmetic progression or a Freiman isomorphism image of \(K_6\) defined in (1).

**Proof** Part 1 is trivial. Part 2 follows from the inequalities
\[3k - 3 = |2A| \geq |2(A \cap [0, b])| + |2(A \cap [a, n])| - |\{2a, a + b, 2b\}| \geq 3A(0, b) - 3 + 3A(a, n) - 3 - 3 = 3A(0, a - 1) + 3 + 3A(a, n) - 6 = 3k - 3,\]
which imply (4) and \(|2A| = |2(A \cap [0, b])| + |2(A \cap [a, n])| - 3\).

Part 3 follows from Theorem 1.1 and Theorem 1.2.

## 2 Main Theorem

**Definition 2.1** For any \(m \in [0, \frac{1}{2}n - 2]\) let
\[\mathcal{TP}I_{m,n} := \{\{0, 2, \ldots, 2(m - 1)\} \cup B \cup \{n\} : B \in \mathcal{FT}_{am}[2m, n - 1]\}.\]  
For any \(u \in [4, n - 6]\), let
\[\mathcal{TP}II_{u,n} := \{\{0\} \cup C \cup D \cup \{n\} : C \in \mathcal{BT}_{am}[1, u] \text{ and } D \in \mathcal{FT}_{am}[u + 2, n - 1]\}.\]
A set in $\mathcal{TPI}_{m,n}$ is said to have a type 1 structure and a set in $\mathcal{TPII}_{u,n}$ is said to have a type 2 structure. If $A$ has a type 1 structure or a type 2 structure, then $|2A| = 3k - 3$ by Proposition 1.8.

The following is the main theorem.

**Theorem 2.2** If $|A| = k \geq 2$ and

$$|2A| = 3k - 3,$$

then one of the following must be true.

1. $A$ is a bi-arithmetic progression;
2. $2A$ contains an interval of length $2k - 1$;
3. $k = 6$ and $A$ is a Freiman isomorphism image of $K_6$ defined in (1);
4. $k = \frac{1}{2}n + 1$ and either $A$ or $n - A$ is in $\mathcal{TPI}_{m,n}$ defined in (6) for some $m \in \left[0, \frac{1}{2}n - 2\right]$ or $A$ is in $\mathcal{TPII}_{u,n}$ defined in (7) for some $u \in [4, n - 6]$.

**Remark 2.3** Notice that there are generally more than one set in each of $\mathcal{FT}_{am}[u, v]$ and $\mathcal{BT}_{am}[u, v]$. For example, \{0, 1, 2, 3, 4\}, \{0, 1, 3, 4, 7\}, \{0, 1, 2, 5, 6\}, and \{0, 1, 2, 4, 6\} are all in $\mathcal{FT}_{am}[0, 9]$.

Notice also that $2A$ contains an interval of length $2k - 3$ when $A \in \mathcal{TPI}_{m,n}$ or $A \in \mathcal{TPII}_{u,n}$.

**Proof of Theorem 2.2:** If $n + 1 > 2k - 1$, then $A$ is either a bi-arithmetic progression or a Freiman isomorphism image of $K_6$ by Theorem 1.2. Hence we can assume that $n + 1 \leq 2k - 1$ or equivalently, $k \geq \frac{1}{2}n + 1$.

Let $H = [0, n] \setminus A$ and $h = |H|$. The elements in $H$ are called the holes of $A$. Thus $h$ counts the number of holes in $H$. A non-empty interval $[x, y] \subseteq H$ is called a gap of $A$ if $x - 1, y + 1 \in A$. We now divide the proof into two parts and devote one subsection for each of them.
2.1 Proof of Theorem 2.2 when $k > \frac{1}{2}n + 1$

For each $x \in [0, n]$, $k > \frac{1}{2} (n + 2)$ implies that

$$
either A(0, x) > \frac{1}{2}(x + 1) or A(x, n) > \frac{1}{2}(n - x + 1).$$

(9)

So for any $x \in [0, n]$, either $x \in 2A$ or $x + n \in 2A$ by Proposition 1.5. Let

- $H_1 = \{x \in H : x \not\in 2A \text{ and } x + n \in 2A\}$ and $h_1 = |H_1|$,
- $H_2 = \{x \in H : x \in 2A \text{ and } x + n \not\in 2A\}$ and $h_2 = |H_2|$,  
- $H_3 = \{x \in H : x \in 2A \text{ and } x + n \in 2A\}$ and $h_3 = |H_3|$.

In [3], the elements in $H_1$ are called left stable holes, the elements in $H_2$, right stable holes, and the elements in $H_3$, unstable holes. By (9) and Proposition 1.5, we have that $H = H_1 \cup H_2 \cup H_3$ and $h = h_1 + h_2 + h_3$.

Since $|A \cup (n + A)| = 2k - 1$, the following is true:

$$k - 2 = |(2A) \setminus (A \cup (n + A))|$$

by (8). It is easy to verify that three sets $B_1 = \{x + n : x \in H_1\}$, $B_2 = \{x : x \in H_2\}$, and $B_3 = \{x, x + n : x \in H_3\}$ are pairwise disjoint and $B_1 \cup B_2 \cup B_3 = (2A) \setminus (A \cup (n + A))$. Hence $k - 2 = h_1 + h_2 + 2h_3$, which implies that

$$k - 2 - h = k - 2 - h_1 - h_2 - h_3 = h_3.$$

(10)

We now prove the following lemma which implies that $2A$ contains $2k - 1$ consecutive integers when $A$ is not a bi-arithmetic progression of difference 1 or 4.

**Lemma 2.4** Let $l, r \in [0, n]$ be such that $A(0, l) \leq \frac{1}{2}(l + 1)$ and $A(n - r, n) \leq \frac{1}{2}(r + 1)$. If $A$ is not a bi-arithmetic progression of difference 1 or 4, then $l < n - r$.

**Corollary 2.5** Let $A$, $l$, and $r$ be in Lemma 2.4. If $A$ is not a bi-arithmetic progression of difference 1 or 4, then $2A$ contains $2k - 1$ consecutive integers.
Proof  Let \( l \) and \( r \) be maximal so that for any \( x \in [l+1, n] \) we have \( A(0, x) > \frac{1}{2}(x + 1) \) and for any \( x \in [0, n-r-1] \) we have that \( A(x, n) > \frac{1}{2}(n - x + 1) \). By Lemma 2.4 we have \( l < n - r \). Let \( x \in [l + 1, 2n - r - 1] \). If \( x \leq n \), then \( x \in 2A \) because \( A(0, x) > \frac{1}{2}(x + 1) \). If \( x > n \), then \( x \in 2A \) because \( A(x - n, n) > \frac{1}{2}(2n - x + 1) \). Hence \([l + 1, 2n - r - 1] \subseteq 2A\). Note that

\[
k = A(0, l) + A(l + 1, n - r - 1) + A(n - r, n)
\leq \frac{1}{2}(l + 1) + (n - r - l - 1) + \frac{1}{2}(r + 1) = n - \frac{1}{2}(l + r),
\]

which implies \( 2k - 1 \leq 2n - l - r - 1 = |[l + 1, 2n - r - 1]| \).

Proof of Lemma 2.4  Without loss of generality, we can assume that \( A \) is not a bi-arithmetic progression of difference 1 or 4. Assume to the contrary that \( l \geq n - r \). Clearly, \( l \neq n - r \) by (9). Hence we can assume that \( l > n - r \).

Let

\[
r_0 = \min \left\{ x \in [n - l, r] : A(n - x, n) \leq \frac{1}{2}(x + 1) \right\} .
\]

(11)

By (9) we have that \( n - r_0 < l \). Let

\[
l_0 = \min \left\{ x \in [n - r_0, l] : A(0, x) \leq \frac{1}{2}(x + 1) \right\} .
\]

(12)

We have that \( n - r_0 < l_0 \) again by (9). By the minimality of \( l_0 \) and \( r_0 \), it is true that \( A(0, x) > \frac{1}{2}(x + 1) \) and \( A(x, n) > \frac{1}{2}(n - x + 1) \) for any \( x \in H \cap [n - r_0 + 1, l_0 - 1] \). So every hole in \([n - r_0 + 1, l_0 - 1]\) is an unstable hole. Thus

\[
H(n - r_0 + 1, l_0 - 1) \leq h_3.
\]

(13)

Now we have that

\[
h \geq H(0, l_0) + H(n - r_0, n) - H(n - r_0, l_0)
\geq \frac{1}{2}(l_0 + 1) + \frac{1}{2}(r_0 + 1) - H(n - r_0, l_0)
\geq \frac{1}{2}(n + 1) + \frac{1}{2}(l_0 - (n - r_0) + 1) - H(n - r_0, l_0)
\geq \frac{1}{2}(k + h) - \frac{1}{2}H(n - r_0, l_0).
\]

(14) (15) (16) (17)
By solving the inequality above, we get that $h \geq k - H(n - r_0, l_0)$, which implies that

$$0 \geq k - 2 - h - H(n - r_0, l_0) + 2 \geq h_3 - H(n - r_0 + 1, l_0 - 1) \geq 0 \tag{18}$$

by (10) and (13). Thus all inequalities in (14)–(18) become equalities. In particular, it is true that

$$H(n - r_0, l_0) = l_0 - (n - r_0) + 1 = h_3 + 2. \tag{19}$$

Notice that (19) implies that $[n - r_0, l_0] \cap A = \emptyset$ and $H_3 = [n - r_0 + 1, l_0 - 1]$. Notice also that $l_0$ is a left stable hole and $n - r_0$ is a right stable hole. These facts are important for the rest of the proof.

All arguments above this line are due to Freiman in [3]. The remaining part of the proof is new. Notice that if $|2A| < 3k - 3$, then (10) becomes $k - 2 - h > h_3$, which leads to a contradiction that $0 > 0$ in (18). So the rest of the proof is needed only because (18) does not lead to a contradiction so far. Notice also that $l_0 > n - r_0$ can happen when, for example, $A = [0, 10] \cup [22, 32]$ or $A = \{0, 4, 8, \ldots, 40\} \cup \{1, 5, 9, \ldots, 41\}$. Fortunately, in these two cases, $A$ is a bi-arithmetic progression of difference 1 or 4, respectively.

It is easy to verify that

$$A(0, l_0) = \frac{1}{2}(l_0 + 1) \quad \text{and} \quad A(n - r_0, n) = \frac{1}{2}(r_0 + 1). \tag{20}$$

Since (20), $l_0$ is left stable hole, and $n - r_0$ is a right stable hole, we have, by Proposition 1.8, that $A \cap [0, l_0]$ and and $A \cap [n - r_0, n]$ are anti-symmetric. Let

$$a = \max(A \cap [0, n - r_0]) \quad \text{and} \quad b = \min(A \cap [l_0, n]). \tag{21}$$

Then $a < n - r_0$, $b > l_0$, and $b - a \geq 2 + l_0 - (n - r_0) \geq 3$. Since

$$A(0, a) = A(0, l_0) = \frac{1}{2}(l_0 + 1) \geq \frac{1}{2}(a + 3) \geq \frac{1}{2}(a + 1) + 1,$$

we have that $a > 0$ and

$$\gcd(A \cap [0, a]) = 1. \tag{22}$$

By the same reason, we have that $b < n$ and

$$\gcd(A \cap [b, n] - b) = 1. \tag{23}$$
By part 3 of Proposition 1.10, we can assume that $|2(A \cap [0, b])| \geq 3A(0, b) - 3$ and $|2(A \cap [a, n])| \geq 3A(a, n) - 3$. Hence (3), (4), and (5) are true by Proposition 1.10. We now use these facts to derive contradictions. Let

$$a' = \max A \cap [0, a - 1] \text{ and } b' = \min A \cap [b + 1, n].$$

A contradiction will be derived under each of the following conditions:

- $b' - b < a - a'$,
- $b' - b > a - a'$,
- $b' - b = a - a' > b - a$,
- $1 < b' - b = a - a' \leq b - a$, and
- $b' - b = a - a' = 1$.

Assume that $b' - b < a - a'$. Then $a' + b' = 2a$ by (3) and (5), which implies that $2a \not\in b + A \cap [0, b]$. The fact that $2a \not\in b + A \cap [0, b]$ will be used in the next several paragraphs to show that $|2(A \cap [0, b])| > 3A(0, b) - 3$, which contradicts (4).

Let $z = \max \{x \in [-1, l_0 - 1] : A(0, x) \leq \frac{1}{2}(x + 1)\}$. Clearly, $z + 1 \in A$ and $A(0, z) = \frac{1}{2}(z + 1)$ by the maximality of $z$. Notice that $A \cap [z + 1, l_0] \in FT[z + 1, l_0]$.

If $z = -1$, then $|2(A \cap [0, b])| \geq 3A(0, b) - 2 > 3A(0, b) - 3$ by part 3 and 4 of Proposition 1.8.

Suppose that $z > -1$. Then $z > 0$ because $A(0, 0) = 1 > \frac{1}{2}$.

If gcd$(A \cap [0, z + 1]) = 1$, then $|2(A \cap [0, z + 1])| \geq 3A(0, z + 1) - 3$ by part 3 of Proposition 1.10. If $z \in A$, then $z + b \in 2A$ and

$$|2(A \cap [0, b])| \geq |2(A \cap [0, z + 1])| - 1 + |[z + 2, z + l_0]| + |(b + A \cap [z, b]) \cup \{2a\}| \geq 3A(0, z) + 3A(z + 1, b) - 2 > 3A(0, b) - 3.$$

So we can assume that $z \not\in A$. Let

$$z' = \max A \cap [0, z - 1].$$

Then $z' + z + 2 \in (2A) \setminus (2(A \cap [0, z + 1]))$. Hence

$$|2(A \cap [0, b])|$$
\[ \geq |(2A \cap [0, z + 1]) \cup \{z' + z + 2\}| - 1 \\
+ |(2z + 2, z + t_0] + |(b + A \cap [z + 1, b]) \cup \{2a\}| \\
\geq 3A(0, z) + 3A(z + 1, b) - 2 > 3A(0, b) - 3. \]

Thus we can assume that $\gcd(A \cap [0, z + 1]) = d > 1$. Clearly, $d = 2$ and $A \cap [0, z + 1]$ is an arithmetic progression of difference 2 by the fact that $A(0, z) = \frac{1}{2}(z + 1)$. Hence

\[ |2(A \cap [0, b])| \\
\geq |A \cap [0, z - 1] + A \cap [0, z + 1]| + |(z + 2) + A \cap [0, z - 1]| \\
+ |(2z + 2, z + l_0] + |(b + A[z + 1, b]) \cup \{2a\}| \\
\geq 3A(0, b) - 2 > 3A(0, b) - 3. \]

Notice that $|2(A \cap [0, b])| > 3A(0, b) - 3$ is true if $2a$, which is not in $b + A \cap [0, b]$, is replaced by any element $c \in (2A \cap [b + z + 1, 2b]) \setminus (b + A \cap [z + 1, b])$ in the argument above.

Assume that $a - a' < b' - b$. The proof is symmetric to the case for $a - a' > b' - b$.

Assume that $b' - b = a - a' = d'$.

If $d' > b - a$, then $2a \notin b + A \cap [0, b]$. Hence $|2(A \cap [0, b])| > 3A(0, b) - 3$ by the same argument as above, which contradicts (4). Thus, we can assume that $d' \leq b - a$.

Suppose that $1 < d' \leq b - a$. Let $a''$ be the greatest element in $A \cap [0, a']$ which is not congruent to $a$ modulo $d'$. The number $a''$ exists by (22).

If $b' + a'' \in 2(A \cap [a, n])$, then $b' + a'' = 2a$ by (3), which implies $2a = b + (d' + a'') \not\in b + A \cap [0, b]$ by the maximality of $a''$. Hence $|2(A \cap [0, b])| > 3A(0, b) - 3$. So by (5), we can assume that $b' + a'' \in 2(A \cap [0, b])$. Notice that $b' + a'' \geq b + z + 1$. This is true because $b' \geq b + 2$ and $a'' \geq z - 1$ due to the facts that $d' > 1$, $a + 1 < l_0$, and $A \cap [z + 1, l_0] \in \mathcal{FT}[z + 1, l_0]$. Clearly, $b' + a'' \not\in b + A \cap [0, b]$. Hence, again, $|2(A \cap [0, b])| > 3A(0, b) - 3$ by the same argument as above.

We can now assume that $d' = 1$, i.e.,

\[ a' = a - 1 \in A \text{ and } b' = b + 1 \in A. \]  

\[ (24) \]
The derivation of a contradiction under this case is much harder that the previous cases. The reason for this is perhaps that $A$ satisfies (24) when $A$ is a bi-arithmetic progression of difference 1 or 4.

Since $A \cap [0, l_0]$ and $A \cap [n - r_0, n]$ are anti-symmetric and $[n - r_0, l_0] \cap A = \emptyset$, we have that $[0, l_0 - (n - r_0)] \subseteq A$ and $[2n - l_0 - r_0, n] \subseteq A$. In particular, we have that

$$0, 1, n - 1, n \in A.$$ \hspace{1cm} (25)

Next we prove four claims for the existence of unstable holes if $A$ has a certain configuration. These claims will be used to derive a contradiction.

Claim 1 If $z \in A$, then $z - 1 \in A$ or $z + 1 \in A$.

Proof of Claim 1 Suppose that Claim 1 is not true. Then $z \in [3, n - 3] \setminus [a - 2, b + 2]$ by (24) and (25). If $A(0, z - 1) > \frac{1}{2}z$, then $z - 1 \in H_3$ because $n + z - 1 = (n - 1) + z \in 2A$. Symmetrically, if $A(z + 1, n) > \frac{1}{2}(n - z)$, then $z + 1 \in H_3$. Both contradicts (19). However, $A(0, z - 1) \leq \frac{1}{2}z$ and $A(z + 1, n) \leq \frac{1}{2}(n - z)$ contradicts the assumption $k > \frac{1}{2}n + 1$.

Claim 1 says that $A$ does not contains any isolated points in $A$.

Claim 2 If $z \in H$, then either $z - 1 \in H$ or $z + 1 \in H$.

Proof of Claim 2 If $z - 1, z + 1 \in A$ and $z \in H$, then $z \notin [n - r_0, l_0]$ and $z = (z - 1) + 1, z + n = (z + 1) + (n - 1) \in 2A$, which contradict (19).

Claim 2 says that there do not exist any isolated holes of $A$.

Claim 3 (a) If $0 < x < y < z < n$ are such that $x, z, z + 1 \in H$, $y \in A$, and $A(0, z) = \frac{1}{2}(z + 1)$, then $z + 1$ is an unstable hole.

(b) If $0 < x < y < z < n$ are such that $x - 1, x, z \in H$, $y \in A$, and $A(x, n) = \frac{1}{2}(n - x + 1)$, then $x - 1$ is an unstable hole.

Proof of Claim 3 We prove (a) only and (b) follows by symmetry. Without loss of generality, let $x = \max H \cap [0, y]$. Notice that $z \notin 2A$ because $z \notin H_3$. Now $A(0, z) = \frac{1}{2}(z + 1)$ implies that $c = z - x \in A$. Hence $z + 1 = c + (x + 1) \in 2A$. 

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Claim 3 (a) implies that if \([0, a] \not\subseteq A\), then \(b = l_0 + 1\) because \(b > l_0 + 1\) implies that \(l_0 + 1\) is an unstable hole, which contradicts (19). By symmetry, Claim 3 (b) implies that \(a = n - r_0 - 1\) if \([b, n] \not\subseteq A\).

**Claim 4** If \([x, y] \subseteq H\) is a gap of \(A\) with \(y - x \geq 2\), \(H \cap [0, x - 1] \neq \emptyset\), and \(H \cap [y + 1, n] \neq \emptyset\), then \([x, y]\) contains an unstable hole.

**Proof of Claim 4** If \(A(0, x) \leq \frac{1}{2}(x + 1)\), then \(x \in H_3\). If \(A(y, n) \leq \frac{1}{2}(n - y + 1)\), then \(y \in H_3\). Otherwise, one can find a \(t \in [x + 1, y - 1]\) such that \(t \in H_3\) by Claim 3.

Claim 4 says that if \(A\) has a gap \([x, y]\) of length at least 3, i.e., \(y - x \geq 2\), then \([x, y]\) is either the first gap or the last gap or the middle gap \([a + 1, b - 1]\) of \(A\).

We now continue the proof of Theorem 2.2 by deriving a contradiction under the assumption that \(d' = 1\), i.e., \(a - 1, a, b, b + 1 \in A\).

If \(n - b < b - a\) and \(a < b - a\), then \(A\) is a subset of the bi-arithmetic progression \([0, a] \cup [b, n]\) of difference 1. So \(|2A| = 3k - 3\) implies that \(A = [0, a] \cup [b, n]\) by Theorem 1.1. Thus we can now assume that either \(n - b \geq b - a\) or \(a \geq b - a\).

Without loss of generality, let \(a \geq b - a\). So we have \(H \cap [0, a] \neq \emptyset\). Let

\[
z = \min \left\{ x \in [0, l_0] : A(0, x) \leq \frac{1}{2}(x + 1) \right\}.
\]

Then \(z \neq a\).

**Case 1** \(z > a\).

It is easy to see that \(z > a\) implies \(z = l_0\). Let \(y = \min H \cap [0, a]\). Then \(y \not\in [n - r_0 + 1, l_0 - 1]\). Since \(y \in 2A\) by Proposition 1.5 and the minimality of \(z\), and \(y + n = (y + 1) + (n - 1) \in 2A\), we have \(y \in H_3\), which contradicts (19).

**Case 2** \(z < a\).

Notice that \(z \not\in A\) by the minimality of \(z\) and \(z > 2\). By the same argument as in Case 1, we have that \(A \cap [0, z] = [0, \frac{z - 1}{2}]\). If \(A(z - 1, n) >

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\( \frac{1}{2}(n-z+2) \), then \( z - 1 \) is an unstable hole below \( a \) by Proposition 1.5. Hence we can assume that \( A(z-1,n) \leq \frac{1}{2}(n-z+2) \). Since

\[
A(z-1,n) = k - A(0,z-2) = k - A(0,z) \geq \frac{1}{2}(n+3) - \frac{1}{2}(z+1) = \frac{1}{2}(n-z+2),
\]

we have that

\[
A(z-1,n) = \frac{1}{2}(n-z+2). \quad (27)
\]

By Claim 3 (b), we can assume that \( z - 2 \in A \) because otherwise \( z - 2 \) becomes an unstable hole below \( a \). So \( z - 2 = \frac{1}{2}(z-1) \), which implies that \( z = 3 \) and \( A \cap [0,z] = [0,1] \).

It is worth mentioning that (27) and \( A(0,z) \leq \frac{1}{2}(z+1) \) imply

\[
k = \frac{1}{2}(n+3) = \frac{1}{2}(k+h+2), \quad (28)
\]

which implies \( k - 2 = h \) and

\[
h_3 = k - 2 - h = 0. \quad (29)
\]

So \( A \) has no unstable holes and \( n - r_0 = l_0 - 1 \).

Let

\[
V = \{ x \in [0,n] : x \equiv 0,1 \mod 4 \}.
\]

We can assume that \( A \neq V \) because otherwise \( A \) is a bi-arithmetic progression of difference 4. Let

\[
z' = \min \{ x \in [0,n] : A \cap [0,x] \neq V \cap [0,x] \}.
\]

Notice that \( n \geq z' > z = 3 \) and \( A \cap [0,z'-1] = V \cap [0,z'-1] \) is the maximal bi-arithmetic progression of difference 4 inside \( A \) containing 0, 1. The rest of the proof is divided into four cases in terms of the value of \( z' \) modulo 4.

**Case 2.1** \( z' \equiv 0 \mod 4 \).

Clearly, \( z' \notin A \) because otherwise \( A \cap [0,z'] = V \cap [0,z'] \).

If \( z' > 4 \), then \( A \cap [0,z'] = \{ 0,1,4,5,\ldots,z'-4,z'-3 \} \) and \( z' \) is at least 8. Since \( A(0,z'-1) = \frac{1}{2}z' \) by the definition of \( V \), \( 3, z'-1, z' \in H \), and \( 4 \in A \), we have that \( z' \) is an unstable hole by Claim 3, which contradicts (29).
So we can now assume that \( z' = 4 \), which implies that \( A \cap [0, 4] = \{0, 1\} \).
Let \( c = \min A \cap [z', a] \).

Recall that \( l_0 \) is a left stable hole and \( A \cap [0, l_0] \) is anti-symmetric in \([0, l_0]\) by Proposition 1.8. Since \( 0, 1, c \in A \) and \([2, c - 1] \subseteq H\), we have that \( l_0, l_0 - 1, l_0 - c \not\in A \) and \([l_0 - c + 1, l_0 - 2] \subseteq A \). Consequently, \( l_0 - 2 = a \) by (21). Clearly, \( c < l_0 - c + 1 \). Since \([2, c - 1] \) is a gap of \( A \) with length at least \( 3 \), \([l_0 - c + 1, a]\) is an interval in \( A \) with length at most \( 3 \).

Suppose that \( c < l_0 - c + 1 \). Then \( c < l_0 - c \) and \( t = l_0 - c \in H \) because \( A \cap [0, l_0] \) is anti-symmetric in \([0, l_0]\). Since \( t + 1 \in A \), we have that \( t - 1 \not\in A \) by Claim 2. If \( t - 2 \in H \), then the gap \( I \) of \( A \) containing \( t \) has a length at least three. Hence \( I \) contains an unstable hole by Claim 4. But if \( t - 2 \in A \), then \( t - 1 + n \in 2A \) because \( A(t - 1, n) = A(t - 1, a) + A(n - r_0, n) > \frac{1}{2}(n - t + 2) \), and \( t - 1 = (t - 2) + 1 \in 2A \). Hence \( t - 1 \) is an unstable hole. Both contradicts (29).

We can now assume that \( c = l_0 - c + 1 \), which implies that \( A \cap [0, b] = \{0, 1\} \cup [c, a] \cup \{b\} \). So

\[
2(A \cap [0, b]) \supseteq [0, 2] \cup [c, a + 1] \cup [2c, a + b] \cup \{2b\} \quad \text{and}
\]

\[
|2(A \cap [0, b])| \geq 3 + a - c + 2 + a + b - 2c + 1 + 1 = 3a - 3c + 10 = 3A(0, b) - 12 + 10 = 3A(0, b) - 2.
\]

**Case 2.2** \( z' \equiv 1 \text{ (mod 4)} \).

We have that \( z' \not\in A \), \( z' - 1 \in A \), and \( z' - 2 \not\in A \). Hence \( z' - 1 \) is an isolated point of \( A \), which contradicts Claim 1.

**Case 2.3** \( z' \equiv 2 \text{ (mod 4)} \).

We have that \( z', z' - 1, z' - 2 \in A \) and \( z' - 3 \not\in A \).

Let \( c = \max \{x \geq z' : [z', x] \subseteq A\} \).

Notice that \([z' - 2, c] \subseteq A \). The proof of Case 2.3 is divided into four subcases for \( c = n, \ b < c < n, \ c = a, \) or \( c < a \). Notice that \( c = b \) is impossible because \( c - 1 \in A \).

**Case 2.3.1** \( c = n \).
Since $A = (V \cap [0, z' - 3]) \cup [z' - 2, n]$, we have that

$$k = A(0, z' - 3) + A(z' - 2, n) = \frac{1}{2}(z' - 2) + n - z' + 3$$

$$= \frac{1}{2}(n + 1) + \frac{1}{2}(n - z' + 3) \geq \frac{1}{2}(n + 1) + \frac{3}{2} = \frac{1}{2}(n + 4),$$

which contradicts (28).

**Case 2.3.2** $b < c < n$.

Clearly, $b \leq z' - 2$. Recall that $a + 3 = b$. Let $x = 2n - r_0 - c$ and $y = 2n - r_0 - z' + 2$. Since $z' - 3 \not\in A$, $[z' - 2, c] \subseteq A$, and $c + 1 \not\in A$, and since $n - r_0$ is a right stable hole and $A \cap [n - r_0, n]$ is anti-symmetric in $[n - r_0, n]$, we have that $[x, y]$ is a gap of $A$ with length $y - x + 1 = c - z' + 3 \geq 3$. Notice also that $n - 2 \not\in A$ because $b = n - r_0 + 2 \in A$. Notice that $c < x$ because gaps of $A$ below $c$ are also gaps of $V$ with length 2 while the length of $[x, y]$ is at least 3. By Claim 4 we can assume that $[y + 1, n] \subseteq A$. Hence $y = n - 2$.

Suppose that $c + 1 < x$. Since $c \in A$ and $c + 1 \not\in A$, we have that $c + 2 \not\in A$ by Claim 2. If $c + 3 \not\in A$, then the gap $I$ of $A$ with $c + 1 \in I$ contains an unstable hole by Claim 4. So we can assume that $c + 3 \in A$. By the fact that the interval $[z' - 2, c]$ contains at least three elements, we have that $A(0, c + 2) > \frac{1}{2}(c + 3)$, which implies that $c + 2 \in 2A$ by Proposition 1.5. Also $c + 2 + n = (c + 3) + (n - 1) \in 2A$. Hence $c + 2 \in H_3$, which contradicts (29).

Thus we can assume that $c + 1 = x$. Now we have that

$$A \cap [a, n] = \{a\} \cup [b, c] \cup \{n - 1, n\}.$$

Hence

$$2(A \cap [a, n]) = \{2a\} \cup [a + b, 2c] \cup [n - 1 + b, n + c] \cup [2n - 2, 2n]$$

and

$$|2(A \cap [a, n])| = 1 + 2c - a - b + 1 + c - b + 2 + 3$$

$$= 3c - 3b + 10 = 3A(a, n) - 12 + 10 = 3A(a, n) - 2.$$

**Case 2.3.3** $c = a$.
Since $A \cap [0, l_0]$ is anti-symmetric in $[0, l_0]$, we have that $[x, y] = l_0 - [z' - 2, a] = [2, l_0 - z' + 2] \subseteq H$ is a gap of $A$ below $z' - 2$ with length at least 3, which is impossible because all gaps of $A$ below $z' - 2$ must be the gaps of $V$ with length 2.

**Case 2.3.4** $c < a$.

Since $A \cap [0, l_0]$ is anti-symmetric in $[0, l_0]$, we have that $[x, y] = [l_0 - c, l_0 - z' + 2] \subseteq H$ is a gap of $A$ with length at least 3. Since gaps of $A$ below $z' - 2$ has length 2, we have that $c < x$. By Claim 4, $[x, y]$ contains an unstable hole, which contradicts (29).

**Case 2.4** $z' \equiv 3 \pmod{4}$.

By the definition of $z'$ we have that $z', z' - 2 \in A$ and $z' - 1 \not\in A$. Therefore, $z' - 1$ is an isolated hole, which contradicts Claim 2.

This completes the proof of Lemma 2.4 as well as Theorem 2.2 when $k > \frac{1}{2}(n + 2)$.

**Remark 2.6** Part 2 of Theorem 2.2 is a structural property for $2A$. So it is an indirect description of a structural property of $A$. Let $l'$ and $r'$ be the maximal $l$ and $r$, respectively, as defined in Lemma 2.4. Then $l' < n - r'$, which is a direct description of a structural property of $A$. The conclusion that $l' < n - r'$ in Lemma 2.4 not only implies part 2 of Theorem 2.2 (the converse is not true), but also gives some geometric information for $A$. Roughly speaking, $l' < n - r'$ indicates that $A$ is thin in $[0, l']$ and in $[n - r', n]$, and $A$ is thick in $[l' + 1, n - r' - 1]$. Therefore, we can say that Lemma 2.4 is a more detailed description of the structural property of $A$ than part 2 of Theorem 2.2 when $k \geq \frac{1}{2}(n + 3)$.

### 2.2 Proof of Theorem 2.2 when $k = \frac{1}{2}n + 1$

Throughout this subsection we fix that

$$k = \frac{1}{2}(n + 2).$$

(30)

Notice that (30) cannot occur when $n$ is an odd number.
Let $x$ be a hole in $A$. We call $x$ a balanced hole if $A(0, x) = \frac{1}{2}(x + 1)$ and $A(x, n) = \frac{1}{2}(n - x + 1)$. Notice that if $A(0, y) = \frac{1}{2}(y + 1)$ and $A(y, n) = \frac{1}{2}(n - y + 1)$ for some $y \in [0, n]$, then $y \notin A$ and if $A(0, x) = \frac{1}{2}(x + 1)$ for some hole $x$ in $A$, then $x$ is a balanced hole by (30).

We want to show that $A \in TP_{I_{m,n}}$ or $n - A \in TP_{I_{m,n}}$ or $A \in TP_{II_{u,n}}$ where $TP_{I_{m,n}}$ and $TP_{II_{u,n}}$ are defined in Definition 2.1. It is worth mentioning that if $n = 10$ and $B$ is a Freiman isomorphism image of $K_6$ in (1), then $|B| = \frac{1}{2}(n + 2)$ and $B = B_i$ for $i = 1, 2, 3, 4$ where $B_i$’s are defined in part 4 of Proposition 1.6. Notice that $B_1 \in TP_{I_{0,10}}$ and $B_2 \in TP_{I_{2,10}}$.

Case 1 $\ 0, 1 \in A$.

In this case we want to show that $A \in TP_{I_{0,n}}$ or $A$ is an arithmetic progression of difference 1 or 4. Let

$$z = \min \left\{ x \in [0, n] : A(0, x) \leq \frac{1}{2}(x + 1) \right\}. \quad (31)$$

Then $z \geq 3$ and $z - 1, z \notin A, A(0, z) = \frac{1}{2}(z + 1)$, and $A \cap [0, z] \in FT_{[0, z]}$ by the minimality of $z$. Notice that $z$ is a balanced hole.

If $z = 2k - 3 = n - 1$, then $2A \supseteq [0, n - 2] \cup (n + A)$. Hence $|2A| = 3k - 3$ implies that $2A = [0, z - 1] \cup (n + A)$. So $A \in TP_{I_{0,n}}$. Therefore, we can now assume that $z < 2k - 3 = n - 1$.

We now want to show that either $|2A| > 3k - 3$ or $A$ is a bi-arithmetic progression of difference 1 or 4.

Let $a = \max A \cap [0, z]$ and $b = \min A \cap [z, n]$. By part 3 of Proposition 1.10, we can assume that

$$|2(A \cap [0, b])| \geq 3A(0, b) - 3. \quad (32)$$

Since $A(z, n) = A(b, n) = \frac{1}{2}(n - z + 1) \geq \frac{1}{2}(z + 2 - z + 1) > 1$, the set $A \cap [a, n]$ contains at least three elements.

Suppose that $\gcd(A \cap [b, n] - b) > 1$. Then $A(z, n) = \frac{1}{2}(n - z + 1)$ implies that $b = z + 1$ and $A \cap [b, n]$ is an arithmetic progression of difference 2. Since $A \cap [b, n]$ is an arithmetic progression of difference 2, we have that

$$2A \supseteq [0, z - 1] \cup (A \cap [b, n + 2] + \{0, 1\}) \cup (n + A). \quad (33)$$
So $|2A| = 3k - 3$ implies that two sides in (33) are the same set. Let $E_0$ be the set of all even numbers and $O_0$ be the set of all odd numbers in $A \cap [0,a]$. If there is an $x > 0$ such that $x \notin E_0$ and $x + 2 \in E_0$, then $n + x = (n - 2) + (x + 2)$ is in $2A$ but not in the right side of (33). So $E_0$ is a set of consecutive even numbers. By the same reason we can assume that $O_0$ is a set of consecutive odd numbers.

If $a = z - 1 = b - 2$, then $a$ is even and $A(0,z) = \frac{1}{2}(z + 1) = |E_0|$. So $O_0 = \emptyset$, which contradicts $1 \in A$. Hence we can assume that $a < b - 2$ and $b - 2 \notin A$. Now $b + (n - 2)$ is in $2A$ but not in the right side (33), a contradiction to the fact that two sides of (33) are the same set.

**Remark 2.7** Notice that in the proof of above we have that $|2A| \geq 3k - 2$ by identifying an element in $((2A \setminus (n + A)) \cap [n,2n]$. If in some case we can also show that $z \in 2A$, then $|2A| \geq 3k - 1$. This fact will be mentioned later.

We can now assume that $\gcd(A \cap [b,n] - b) = 1$.

By part 3 of Proposition 1.10, we have that $|2(A \cap [a,n])| \geq 3A(a,n) - 3$. Together with (32), we conclude that (3), (4), and (5) are true.

**Case 1.1** $H \cap [0,a] = \emptyset$.

This case implies that $z = 2a + 1$, $2a < b$, and $A \cap [0,b] = [0,a] \cup \{b\}$.

By applying Theorem 1.2 we have that either $A \cap [a,n]$ is a bi-arithmetic progression, or $n - a + 1 \leq 2A(a,n) - 1$, or $A \cap [a,n]$ is Freiman isomorphic to $K_6$ in (1).

Notice that $n - a + 1 \leq 2A(a,n) - 1 = (n - z + 1) + 1$ implies that $2a + 1 = z \leq a + 1$, which is absurd. So we can assume that $A \cap [a,n]$ is either a bi-arithmetic progression or Freiman isomorphic to $K_6$.

**Case 1.1.1** $A \cap [a,n]$ is Freiman isomorphic to $K_6$ in (1).

Let $\varphi : K_6 \to A \cap [a,n]$ be the Freiman isomorphism. Notice that $A(b + 1, n - 1) = 3$.

Suppose that $b + 1 \notin A$. Let $b' = \min A \cap [b + 1,n]$.

If $a > 1$, then there is an $x \in \{a-1,a-2\}$ such that $x + b' \notin \{2a,a+b,2b\}$. Hence $x + b'$ is in the set in (5), which contradicts that the set is empty.
If \(a = 1\), then \(z = 3\), \(b \geq 4\), and \(n = 12\) because \(k = 7\). If \(2b \neq b'\), then \(b'\) is in the set in (5). So we can assume that \(b' = 2b \geq 8\). Notice that \(a = 1\) is a vertex of \(\varphi(K_6)\) and \(b\) is not a vertex by part 2 of Proposition 1.6. So \(2b - a = 2b - 1\) is another vertex of \(\varphi(K_6)\). This contradicts the minimality of \(b'\).

We can now assume that \(b + 1 \in A\). If \(b + 1\) is a vertex of \(\varphi(K_6)\), then \(c = \frac{1}{2}(a + b + 1) \in A \cap [a + 1, b - 1]\), which contradicts \(A \cap [a + 1, b - 1] = \emptyset\). So \(b + 1\) is not a vertex in \(\varphi(K_6)\). Hence \(2b + 2 - a\) is in \(A\) and is a vertex. So \(A = [0, a] \cup \{b, b + 1, 2b - a, 2b - a + 1, 2b - a + 2\}\), which implies that \((a - 1) + (2b - a) = 2b - 1\) is in the empty set in (5).

**Case 1.1.2** \(A \cap [a, n]\) is a bi-arithmetic progression of difference \(d\).

Let \(A \cap [a, n] = I_0 \cup I_1\) be the bi-arithmetic progression decomposition and \(a \in I_0\).

If \(d = 1\), then \(A \cap [a, n] = \{a\} \cup [b, n]\) such that \(n - b > b - a\). Hence \(A = [0, a] \cup [b, n]\) is a bi-arithmetic progression of difference 1.

If \(d = 2\), then \(\gcd((A \cap [b, n]) - b) = 2\) because \(b - a \geq 3\). But this contradicts the assumption that \(\gcd((A \cap [b, n]) - b) = 1\).

If \(d = 3\), then \(b \in I_0\) and \(b - a = 3\). Hence \(z = a + 2\), which implies that \(a = 1\) because \(z = 2a + 1\). Let \(c = \min I_1\). If \(c = b + 2\), then \(a - 1 + c\) is in the set in (5). If \(c = b + 1\) or \(c > b + 3\), then \(a - 1 + b + 3\) is in the set in (5). But both contradict that the set in (5) is empty.

If \(d = 4\), then \(A(b + 1, b + 3) \leq 1\). If \(a \not\equiv b \pmod{4}\), then \(b = a + 3\) because \(b - a \geq 3\) and \(a + 4 \in I_0\). But \(b = a + 3\) implies \(a = 1\). So \(A\) is a bi-arithmetic progression of difference 4. Hence we can assume that \(a \equiv b \pmod{4}\). If \(A(b + 1, b + 3) = 0\) or \(b + 1 \in A\), let \(x = b + 4\). If \(b + 3 \in A\), let \(x = b + 3\). Then \(a - 1 + x\) is in the empty set in (5). Notice that \(b + 2 \not\in A\) because otherwise \(\gcd(A \cap [b, n] - b) = 2\).

If \(d \geq 5\), then \(\frac{1}{2}(n - z + 1) = A(z, n) \leq \frac{7}{2}(n - z + 3)\), which implies that \(n - z \leq 7\). Now \(A(z, n) = \frac{1}{2}(n - z + 1), z < b, \) and \(d = 5\) imply that \(n - z = 7, d = 5, b = z + 1,\) and \(A \cap [b, n] = \{b, b + 1, b + 5, b + 6\}\). If \(a \equiv b \pmod{5}\), then \(a - 1 + b + 5\) is in the empty set in (5). If \(a \not\equiv b \pmod{5}\), then \(b - a = 4\), which implies that \(a = 2\) because \(a + 3 = z = 2a + 1\). Hence \(b + 5 = (a - 2) + (b + 5) = 2b - 1\) is in the empty set in (5).
Case 1.2 \( H \cap [0, a] \neq \emptyset \).

Notice that \( z \leq 2a \). If \( b > z + 1 \), then \( |2(A \cap [0, b])| > 3|A \cap [0, b]| - 3 \) by part 5 of Proposition 1.8. Hence we can assume that \( b = z + 1 \). By (4), \( A \cap [0, z] \) is an additively minimal backward triangle. Hence \( A \cap [0, z] \) is anti-symmetric.

Since \( A \cap [0, z] \) is anti-symmetric and \( 1 \in A \), we have that \( z - 1 \notin A \). So \( b - a \geq 3 \). If \( n - a + 1 \leq 2A(a, n) - 1 = (n - z + 1) + 1 \), then \( z - 1 \leq a \).

Hence we can assume, by Theorem 1.2, that \( A \cap [a, n] \) is either a bi-arithmetic progression or a Freiman isomorphism image of \( K_6 \) in (1).

**Case 1.2.1** \( A \cap [a, n] \) is Freiman isomorphic to \( K_6 \) in (1).

Let \( \varphi : K_6 \rightarrow A \cap [a, n] \) be the Freiman isomorphism.

Since \( A(z, n) = 5 = \frac{1}{2}(n - b + 2) \), we have that \( n - b = 8 \). Notice that \( a \) is a vertex, \( b \) is not a vertex, and \( 2b - a \) is a vertex of \( \varphi(K_6) \). Let \( c \) be the third vertex in \( \varphi(K_6) \). If \( 2b - a = n \), then \( \frac{1}{2}(a + c) \in A \cap [a + 1, b - 1] = \emptyset \), which is absurd. So we can assume that \( 2b - a < c = n \).

Notice that \( \frac{1}{2}(n + 2b - a) \) is in \( A \cap [b, n] \). Clearly, \( n - (2b - a) \) is even and \( \leq 5 \) because \( n - b = 8 \) and \( b - a \geq 3 \). If \( n - (2b - a) = 4 \), then \( A \cap [b, n] = \{b, b+2, b+4, b+6, b+8\} \), which contradicts \( \gcd(A \cap [b, n] - b) = 1 \).

If \( n - (2b - a) = 2 \), then \( A \cap [b, n] = \{b, b+1, b+6, b+7, b+8\} \) and \( a = b - 6 \). If \( a - 1 \in A \), then \( (a - 1) + (b + 6) \) is in the empty set in (5). So we can assume that \( a - 1 \notin A \). Let \( a' = \max A \cap [0, a - 1] \). If \( a' + b + 1 \neq 2a \), then \( a' + b + 1 \) is in the empty set in (5). If \( a' + b + 1 = 2a \), then \( a' + b + 6 \) is in the empty set in (5). Both are absurd. This completes the proof of Case 1.2.1.

**Case 1.2.2** \( A \cap [a, n] \) is a bi-arithmetic progression of difference \( d \).

Let \( A \cap [a, n] = I_0 \cup I_1 \) be the bi-arithmetic progression decomposition and \( a \in I_0 \).

If \( d = 1 \), then \( A \cap [a, n] = \{a\} \cup [b, n] \) with \( n - b < b - a \). Let \( A' = n - A \), \( z' = n - z \), \( b' = n - a \), and \( a' = n - b \). Then \( A' \cap [0, z'] = [0, a'] \), \( A' \cap [a', b'] = [0, a'] \cup \{b'\} \), and \( z' \) is a balanced hole of \( A' \). The same proof for Case 1.1 works for \( A' \).

If \( d = 2 \), then \( \gcd((A \cap [b, n]) - b) = 2 \) because \( b - a \geq 3 \), which contradicts the assumption that \( \gcd((A \cap [b, n]) - b) = 1 \).
If \( d = 3 \), then \( a, b \in I_0 \), \( b - a = 3 \), and \( z = a + 2 \). Let \( c = \min I_1 \) and \( a' = \max A \cap [0, a - 1] \). If \( a' = a - 1 \), then \( A(0, a' - 1) = \frac{1}{2}a' \), which contradicts the minimality of \( z \). Thus we can assume that \( a' < a - 1 \). If \( a' \equiv a \pmod{3} \), then \( c + a' \) is in the empty set in (5). So we can assume that \( a' \not\equiv a \pmod{3} \). If \( c = b + 1 \), then \( c + a' \) is in the empty set in (5). So we can assume that \( c > b + 1 \). If \( b + 3 \not\in A \), then \( c = b + 2 \) and \( A \cap [z, n] = \{b, b + 2\} \) by the fact that \( A(z, n) = \frac{1}{2}(n - z + 1) \), which contradicts the assumption that \( \gcd(A \cap [b, n] - b) = 1 \). So we can assume that \( b + 3 \in A \). But now \( a' + b + 3 \) is in the empty set in (5).

If \( d = 4 \), then \( b - a = 4 \) or \( b - a = 3 \) because \( \gcd(A \cap [b, n] - b) = 1 \). Let \( a' = \max(A \cap [0, a - 1]) \) and \( c = \min\{x \in A \cap [b + 1, n] : x \not\equiv b \pmod{4}\} \).

Suppose that \( a' = a - 1 \). If \( b - a = 4 \), then \( c \not\equiv b + 2 \) and \( b + 4 \in A \) by the fact that \( A(b - 1, n) = \frac{1}{2}(n - b) \). If \( c \not\equiv b + 3 \), then \( a' + b + 4 \) is in the empty set in (5). If \( c = b + 3 \), then \( a' + c \) is in the empty set in (5). If \( b - a = 3 \), then \( z - a = 2 \) and \( A(0, a' - 1) = \frac{1}{2}a' \), which contradicts the minimality of \( z \).

So we can now assume that \( a' < a - 1 \). If \( b + 1 \in A \), then \( a' + b + 1 \) is in the empty set in (5) unless \( a' + b + 1 = 2a \). If \( a' + b + 1 = 2a \), then \( 2a \not\in (b + A \cap [0, b]) \), which leads to a contradiction to (4). So we can assume that \( b + 1 \not\in A \), which implies that \( b \not\equiv a + 3 \). So we have that \( b = a + 4 \). Since \( A(z, n) = \frac{1}{2}(n - z + 1) \), we have that \( b + 3 \in A \) and \( n > b + 4 \). If \( a' \equiv a \pmod{4} \), then \( a' + b + 3 \) is in the empty set in (5). If \( a' \not\equiv a \pmod{4} \), then \( a' + b + 4 \) is in the empty set in (5).

If \( d > 5 \), then \( \frac{1}{2}(n - z + 1) = A(z, n) \leq \frac{2}{5}(n - z + 3) \), which implies that \( n - z = 7 \) and \( A \cap [b, n] = \{b, b + 1, b + 5, b + 6\} \) Notice that \( b - a = 5 \) or 4. Let \( a'' = \max A \cap [0, a - 1] \).

If \( a' < a - 1 \), then \( a' + b + 1 \) is in the empty set in (5) unless \( a' + b + 1 = 2a \). If \( a' + b + 1 = 2a \), then \( 2a \not\in (b + A \cap [0, b]) \), which leads to a contradiction to (4). So we can assume that \( a' = a - 1 \). If \( a - b = 5 \), then \( a' + b + 5 \) is the empty set in (5). If \( a - b = 4 \), then let \( a'' = \max\{x \in A \cap [0, a - 1] : x \not\equiv b, b + 1\} \). Notice that \( a'' \) exists because otherwise \( A \) is a subset of a bi-arithmetic progression of difference 5, which contradicts the assumptions that \( H \cap [0, a] \neq \emptyset \) and \( A \cap [0, z] \) is backward triangle in \([0, z] \). If \( a'' \equiv b + 2 \) or \( b + 3 \pmod{5} \), then \( a'' + b + 1 \) is in the empty set in (5). If \( a'' \equiv b + 4 \pmod{5} \), then \( a'' + b + 5 \) is the empty set in (5).
This completes the proof of Case 1.2.2 as well as Case 1.

**Case 2** \( 1 \not\in A \).

If \( n - 1 \in A \), then by the proof of Case 1 for \( n - A \), we can conclude that \( n - A \in \mathcal{TPI}_{0,n} \) or \( A \) is an arithmetic progression of difference 1 or 4. Thus we can assume that \( n - 1 \not\in A \).

We want to show that \( A \in \mathcal{TPI}_{m,n} \) or \( n - A \in \mathcal{TPI}_{m,n} \) for some \( m \in [1, \frac{1}{2}n - 2] \) or \( A \in \mathcal{TPI}_{u,n} \) for some \( u \in [4, n - 6] \). Let \( E \) be the set of all even numbers and

\[
a = \max \{ x \in [0, n] : A \cap [0, x] = E \cap [0, x] \}.
\]

Notice that \( 2 \leq a \leq n - 2 \). Notice also that \( a \in A \) implies \( a + 1 \in A \) and if \( a \not\in A \) implies \( a + 1 \not\in A \) by the maximality of \( a \).

**Case 2.1** \( a \in A \).

In this case we show that \( A \in \mathcal{TPI}_{m,n} \) for \( m = a/2 \).

Let \( A' = A \cap [a, n] \). Then \( a, a + 1 \in A \). Notice that

\[
3k - 3 = |2A|
\]

\[
\geq |2(A \cap [0, a])| + |a + 1 + A \cap [0, a - 2]| + |2(A \cap [a, n])| - 1
\]

\[
= 3A(0, a - 2) + |2(A \cap [a, n])|.
\]

Hence \( |2(A \cap [a, n])| \leq 3A(a, n) - 3 \). Notice also that \( A(a, n) = \frac{1}{2}(n - a + 2) \).

So \( |2(A \cap [a, n])| = 3A(a, n) - 3 \) by part 3 of Proposition 1.10. Let \( n' = n - a \). Now \( A' = A \cap [a, n] - a \) in \([0, n']\) satisfies all conditions for Case 1. So either \( A' \in \mathcal{TPI}_{0,n'} \) or \( A' \) is a bi-arithmetic progression of difference 1 or 4.

Suppose that \( A' \) is a bi-arithmetic progression of difference 1. Since \( n - 1 \not\in A \), we have that \( A \cap [a, n] = [a, x] \cup \{n\} \). Since \( |A \cap [a, n]| = \frac{1}{2}(n - a + 1) \), we have that \( A' \in \mathcal{TPI}_{0,n'} \). Hence \( A \in \mathcal{TPI}_{m,n} \) for \( m = a/2 \).

Suppose that \( A' \) is a bi-arithmetic progression of difference 4. If \( a + 5 \not\in A \), then \( A' = \{0, 1, 4\} \) by the fact that \( |A'| = \frac{1}{2}(n' + 2) \). Since \( \{0, 1, 4\} \in \mathcal{TPI}_{0,4} \), we have again that \( A \in \mathcal{TPI}_{m,n} \) for \( m = a/2 \). If \( a + 5 \in A \), then

\[
3k - 3 = |2A| \geq |2(A \cap [0, a])| + |a + 1 + A \cap [0, a - 2]|
\]

\[
+ |\{(a + 5) + (a - 2)\}| + |2(A \cap [a, n])| - 1
\]

\[
\geq 3A(a, n) - 3 = 3k - 2,
\]
which is absurd.

**Case 2.2** \( a \notin A \).

In this case we show that \( a > 1 \) implies \(|2A| > 3k - 3\) and \( a = 1 \) implies that \( A \) is either \( n - A \in \mathcal{TPI}_{m,n} \) or \( A \in \mathcal{TPI}_{u,n} \).

Recall that \( a + 1 \notin A \) and \( a - 1 \in A \). Notice that \( A(0, a) = \frac{1}{2}(a + 1) \). Thus \( a \) is a balanced hole. Notice also that \( A(n - 2) = \frac{1}{2}(n - a - 1) \) because \( n \in A \) and \( n - 1 \notin A \). Let

\[
u = \min \left\{ x \in [a, n - 2] : A(a, x) \geq \frac{1}{2}(x - a + 1) \right\}.
\]

Notice that \( u > a + 2 \) because \( a, a + 1 \notin A \). Notice also that \( u, u - 1 \in A \), \( A(u, u) = \frac{1}{2}(u - a + 1) \), and \( A(x, u) > \frac{1}{2}(u - x + 1) \) for every \( x \in [a + 1, u] \) by the minimality of \( u \). So \( A \cap [a, u] \) is a backward triangle in \([a, u]\). Notice that \( A(u, n) = \frac{1}{2}(n - u + 2) \).

**Case 2.2.1** \( a > 1 \).

In this case we derive a contradiction by showing \(|2A| > 3k - 3\).

Since \( 0, 2 \in A \) and \( 1 \notin A \), \( A \) as well as \( A \cap [0, u] \) can be neither a bi-arithmetic progression of difference 1 nor a bi-arithmetic progression of difference 4. Notice that since \( 0, 2 \in A \) and \( 1 \notin A \), \( u - (A \cap [0, u]) \) cannot be in \( \mathcal{TPI}_{0,u} \). Hence by applying the proof of Case 1 to \( u - (A \cap [0, u]) \), we have that \(|2(A \cap [0, u])| > 3A(0, u) - 3\).

If \( \gcd(A \cap [u, n] - u) > 1 \), then

\[
|2A| \geq |2(A \cap [0, u])| + |2(A \cap [u, n])| - 1 + |u - 1 + A \cap [u + 2, n]| > 3k - 3.
\]

So we can assume that \( \gcd(A \cap [u, n] - u) = 1 \). Hence \(|2(A \cap [u, n])| \geq 3A(u, n) - 3 \). Let \( v = \min A \cap [u + 1, n] \).

If \( v > u + 1 \), then \( u - 1 + v \) is in

\[
(2A) \setminus ((2(A \cap [0, u])) \cup (2(A \cap [u, n]))).
\]

Hence

\[
|2A| \geq |2(A \cap [0, u])| + |2(A \cap [u, n])| > 3k - 3.
\]
Thus we can assume that $v = u + 1$.

Recall that in the argument in the proof of Case 1 before case 1.1, we showed that if $b < n$, $\gcd(A \cap [b, n] - b) > 1$, and $z = b - 1 \in 2A$ then $|2A| \geq 3k - 1$ (see Remark 2.7). So we can apply the same argument to $A' = u - (A \cap [0, u])$ with $z' = u - a$ and $b' = u - a + 1$ to show that

$$|2(A \cap [0, u]) \cup \{u + a\}| = |2A'| + |z' - 1| \geq 3|A'| - 1 = 3A(0, u) - 1.$$ 

Hence

$$|2A| \geq |2(A \cap [0, u]) \cup \{a - 1 + v\}| + |2(A \cap [u, n])| - 1 \geq 3A(0, u) - 1 + 3A(u, n) - 4 = 3k - 2.$$ 

Case 2.2.2 $a = 1$.

In this case we show that either $n - A \in TPI_{m,n}$ for some $m > 0$ or $A \in TPI_{u,n}$ for some $u \in [4, n - 6]$.

Notice that $A \cap [1, u]$ is a backward triangle in $[1, u]$ and $|2(A \cap [0, u])| \geq 3A(0, u) - 3$ by part 3 of Proposition 1.10.

Case 2.2.2.1 $u + 1 \in A$.

If $|2(A \cap [0, u])| > 3A(0, u) - 3$, then

$$3k - 3 = |2A| \geq |2(A \cap [0, u])| + |2(A \cap [u, n])| - 1 \geq 3A(0, u) - 2 + 3A(u, n) - 3 - 1 = 3k - 3.$$ 

Hence $|2(A \cap [u, n])| = 3A(u, n) - 3$. By applying the proof of Case 1 to the set $A' = A \cap [u, n] - u$ and $n' = n - u$, we can conclude that either $A' \in TPI_{0,n'}$ or $A'$ is a bi-arithmetic progression of difference 1 or 4. We now want to show that $|2A| > 3k - 3$ by identifying one element in the set in (35), which implies that $|2A| > 3A(0, u) - 3 + 3A(u, n) - 3 - 1 + 1 = 3k - 3$.

If $A' \in TPI_{0,n'}$, then $u - 1 + n$ is in the set in (35). If $A'$ is a bi-arithmetic progression of 1, then $A \cap [u, n] = [u, x] \cup \{n\}$. So again $A' \in TPI_{0,n'}$. If $A'$ is a bi-arithmetic progression of difference 4, then $u - 1 + u + 4$ is in the set in (35).

Thus we can assume that $|2(A \cap [0, u])| = 3A(0, u) - 3$. So $A' = u - (A \cap [0, u]) \in TPI_{0,n'}$ for $n' = u$ or $A'$ is a bi-arithmetic progression of difference
1 or 4. Notice that if $A'$ is a bi-arithmetic progression of difference 1, then $A' \in TP_{0,n}$ because $1 \not\in A$. And if $A'$ is a bi-arithmetic progression of difference 4, then $A' = \{0, 1, 4\} \in TP_{0,4}$ because $A'$ is an forward triangle in $[0, u - 1]$ and $A'(0, 3) = \frac{1}{2}(n' + 1)$. As a consequence we have that $u + 1$ is in the set in (35).

If $|2(A \cap [u, n])| > 3A(u, n) - 3$, then $|2A| > 3A(0, u) - 3 + 3A(u, n) - 3 = 3k - 3$. Hence we can now assume that $|2(A \cap [u, n])| = 3A(u, n) - 3$.

Recall that we have assumed that $|2(A \cap [0, u])| = 3A(0, u) - 3$, $|2(A \cap [u, n])| = 3A(u, n) - 3$, and $u - (A \cap [0, u]) \subseteq A$, and $u - (A \cap [0, u]) \in TP_{0,u}$. By applying the proof of Case 1 to $A' = (A \cap [u, n]) - u$, we have that either $(A \cap [u, n]) - u \in TP_{0,n-u}$ or $(A \cap [u, n]) - u$ a bi-arithmetic progression of difference 4. Notice that if $A \cap [u, n]$ is a bi-arithmetic progression of difference 1, then $(A \cap [u, n]) - u \in TP_{0,n-u}$. We now want to show that $|2A| > 3k - 3$ by identifying two elements in the set in (35), which implies that $|2A| \geq 3k - 2$.

If $(A \cap [u, n]) - u \in TP_{0,n-u}$, then $u + 1, u - 1 + n$ are in the set in (35). If $A \cap [u, n]$ is a bi-arithmetic progression of difference 4, then $0 + u + 1, u - 1 + u + 4$ are in the set in (35).

**Case 2.2.2.2** $u + 1 \not\in A$.

Let $v = \min A \cap [u + 1, n]$. Notice that $v + u - 1$ is in the set in (35). If $|2(A \cap [0, u])| > 3A(0, u) - 3$, then $|2A| > 3k - 3$. Hence we can assume that $|2(A \cap [0, u])| = 3A(0, u) - 3$. By applying the proof of Case 1, we have that $u - (A \cap [0, u]) \in TP_{0,u}$. If $\gcd((A \cap [u, n]) - u) = d > 1$, then $d = 2$ and $A \cap [u, n]$ is an arithmetic progression of difference 2. So $n - A \in TP_{m,n}$ for $m = (n - u)/2$. Hence we can assume that $\gcd(A \cap [u, n] - u) = 1$. If $|2(A \cap [u, n])| > 3A(u, n) - 3$, then $|2A| > 3k - 3$. Hence we can assume that $|2(A \cap [u, n])| = 3A(u, n) - 3$.

Suppose that $v = u + 2$. We want to show that $A \in TP_{m,n}$.

Let $a' = \max \{x \in [u, n] : A \cap [u, x] = (u + E) \cap [u, x]\}$. $a'$ is well defined because $\gcd((A \cap [u, n]) - u) = 1$. Notice that $a' \geq u + 2$.

If $a' \not\in A$, then $a' > u + 2$. By the proof of Case 2.2.1 we have that $|2(A \cap [u, n])| > 3A(u, n) - 3$. So we can assume that $a' \in A$, which implies that $a' + 1 \in A$ by the maximality of $a'$. By applying the proof of Case 2.1
to the set \( A' = (A \cap [u, n]) - u \), we have that \( A' \in TP_{m,n-u} \). Hence

\[
A = \{0\} \cup C \cup \{u, u + 2, \ldots, u + 2m\} \cup D \cup \{n\}
\]

where \( C \in BT_{am}[1, u] \), and \( D \in FT_{am}[u + 2m, n - 1] \). Thus

\[
|2A| = |2(A \cap [0, u])| + |[2u + 1, 2u + 4m - 1]| + |2(A \cap [u + 2m, n])|\
= 3A(0, u) - 3 + 4m - 1 + 3A(u + 2m, n) - 3\
= 3A(0, u) - 3 + 4A(u + 2, u + 2m - 2) + 3 + 3A(u + 2m, n) - 3\
= 3k - 3 + A(u + 2, u + 2m - 2) > 3k - 3
\]

unless \( m = 1 \). If \( m = 1 \), then \( A \in TP_{II_{u,n}} \).

Now we assume that \( v > u + 2 \). We show that \(|2A| > 3k - 3\) by identifying two elements in the set in (35).

If \( u - 2, u - 3 \not\in A \), then \( u = 4 \) and \( A \cap [0, u] = \{0, 3, 4\} \) by the minimality of \( u \). If \( A \) is not a bi-arithmetic progression of difference 4, we can define

\[
c = \min\{x \in [u, n] : A \cap [0, x] \\
is not a subset of a bi-arithmetic progression of difference 4\}.
\]

Since \( 3, 4 \in A \), we have that \( c \) is either congruent to 5 or congruent to 6 modulo 4.

Suppose that \( c = v \). Recall that \( v > 6 \). So we can assume that \( v \geq 9 \). Then \( v + 3, v + 0 \) are in the set in (35).

Suppose that \( c > v \). Then \( v \) is congruent to 3 or 4 modulo 4. If \( c \equiv 5 \pmod{4} \), then \( c + 0, v + 3 \) are in the set in (35). If \( c \equiv 6 \pmod{4} \), then \( c + 3, v + 3 \) are in the set in (35).

So we can assume that \( A(u - 3, u) \geq 3 \). If \( u - 2 \in A \), then \( v + u - 1, v + u - 2 \) are in the set in (35). So we can assume that \( u - 2 \not\in A \) and \( u - 3 \not\in A \).

If \( v \geq u + 4 \), then \( v + u - 1, v + u - 3 \) are in the set in (35). So we can assume that \( v = u + 3 \). Let \( v' = \min A \cap [v + 1, n] \). If \( v' = v + 1 \), then \( v + u - 1, v' + u - 3 \) are in the set in (35). If \( v' > v + 1 \), then \( v + u - 1, v' + u - 1 \) are in the set in (35) unless \( v' + u - 1 = 2v \). But if \( v' + u - 1 = 2v \), then \( v + u - 1, v' + u - 3 \) are in the set in (35).

This completes the proof of Theorem 2.2.
References


