GENERIC EXISTENCE OF INTERVAL P-POINTS

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Abstract A P-point ultrafilter over ω is called an interval P-point if for every function from ω to ω there exists a set A in this ultrafilter such that the restriction of the function to A is either a constant function or an interval-to-one function. In this paper we prove the following results. (1) Interval P-points are not isomorphism invariant under CH or MA. (2) We identify a cardinal invariant $\mathbf{non}^{**}(\mathcal{I}_{\text{int}})$ such that every filter base of size less than continuum can be extended to an interval P-point if and only if $\mathbf{non}^{**}(\mathcal{I}_{\text{int}}) = \mathfrak{c}$. (3) We prove the generic existence of slow/rapid non-interval P-points and slow/rapid interval P-points which are neither quasi-selective nor weakly Ramsey under the assumption $\mathfrak{d} = \mathfrak{c}$ or $\mathbf{cov}(\mathcal{B}) = \mathfrak{c}$.

1. Introduction

♦ All filters and ultrafilters $\mathcal{F}$ considered in this paper are non-principal over ω, i.e., $\mathcal{F}$ contains all co-finite subsets of ω. All ideals $\mathcal{I}$ considered in this paper are free ideals over ω, i.e., $\mathcal{I}$ contains all finite subsets of ω.

This paper is a sequel to [12, 15]. In [4, 9] quasi-selective ultrafilters are studied under CH and, as an intermediary between P-points and quasi-selective or weakly Ramsey ultrafilters, interval P-points are introduced. The question whether interval P-points are really different from either P-points or quasi-selective or weakly Ramsey ultrafilters is then asked in [4, 11]. In [12, 15] slow/rapid non-interval P-points and slow/rapid interval P-points which are neither quasi-selective nor weakly Ramsey are constructed under CH or MA. However, some interesting questions about interval P-points remain unanswered there. For example, it remains to be explored whether interval P-points are isomorphism invariant (notice that P-points are isomorphism invariant but quasi-selective ultrafilters are not) and how the generic existence of ultrafilters associated with interval P-points are related to some assumptions on some cardinal invariants. Notice that some assumptions beyond ZFC are necessary for the discussion because, by some celebrated theorems of Shelah,

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there could be only one P-point or there could be no P-point without extra assumption beyond ZFC (See [18, 1]).

In §2 we list some concepts needed in the later sections including the ideals $\mathcal{I}_{\rm int}$, $\mathcal{I}_h$, and $\mathcal{I}_i$ for $i = 1, 2, 3, 4$. In §3 we show that the interval P-points are not isomorphism invariant under CH or MA. Let $\mathfrak{c} := 2^{\aleph_0}$. In §4 we identify a cardinal invariant $\mathbf{non}^{**}(\mathcal{I}_{\rm int})$ for an ideal $\mathcal{I}_{\rm int}$ and prove that every filter base of size less than $\mathfrak{c}$ can be extended to an interval P-point if and only if $\mathbf{non}^{**}(\mathcal{I}_{\rm int}) = \mathfrak{c}$. In §5 we show that every filter base of size less than $\mathfrak{c}$ with $\mathcal{I}_2^+$-finite intersection property can be extended to a slow non-interval P-point if and only if $\mathfrak{d} = \mathfrak{c}$. In §6 we show that if $\mathbf{cov}(\mathcal{B}) = \mathfrak{c}$, then (a) every filter base of size less than $\mathfrak{c}$ with $\mathcal{I}_1^+$-finite intersection property can be extended to a rapid non-interval P-point, (b) every filter base of size less than $\mathfrak{c}$ with $\mathcal{I}_3^+$-finite intersection property can be extended to a rapid interval P-point which is neither quasi-selective nor weakly Ramsey, (c) every filter base of size less than $\mathfrak{c}$ with $\mathcal{I}_4^+$-finite intersection property can be extended to a slow interval P-point which is neither quasi-selective nor weakly Ramsey.

The notation and terminology in this paper are fairly standard. The reader is recommended to consult, for example [1, 3, 5], for basic knowledge on P-points, selective ultrafilters, etc. and other unexplained terms.

2. Filters and ideals

For the convenience of the reader, we include a list of concepts in this section for easy references. The set of all functions from Y to X is denoted by X^Y. The collection of all subsets of X with cardinality κ is denoted by $[X]^\kappa$. The set $[X]^{<\omega}$ represents the collection of all finite subsets of X.

Definition 2.1. A function $f \in \omega^A$ for some $A \subseteq \omega$ is *interval-to-one* if for every $n \in f[A]$ there exist integers $a \le b$ such that $f^{-1}(\{n\}) = [a, b] \cap A$.

An ultrafilter $\mathcal{F}$ is an *interval P-point* if for every function $f \in \omega^\omega$ there is an $A \in \mathcal{F}$ such that the restriction of f on A, denoted by $f \restriction A$, is either constant or interval-to-one.

Notice that an interval P-point is a P-point.

Definition 2.2. An ultrafilter $\mathcal{F}$ is *quasi-selective* if for every $f \in \omega^\omega$, if $f(x) \le x$ for every $x \in \omega$, then there exists an $A \in \mathcal{F}$ such that $f \restriction A$ is non-decreasing.

Definition 2.3. An ultrafilter $\mathcal{F}$ is *weakly Ramsey* if for every coloring $c : [\omega]^2 \to 3$, there exists an $A \in \mathcal{F}$ such that $|c\left[[A]^2\right]| \le 2$.

Clearly, a selective ultrafilter is quasi-selective and weakly Ramsey. Both quasi-selective ultrafilters and weakly Ramsey ultrafilters are interval P-points (see [4, 11]).

Definition 2.4. Let $f, g \in \omega^\omega$. We say that g dominates f, denoted by $f \le g$, if $f(n) \le g(n)$ for all $n \in \omega$ and g eventually dominates f, denoted by $f \le^* g$, if $f(n) \le g(n)$ for all but finitely many $n \in \omega$.

Let $A, B \subseteq \omega$. We say that A is an almost subset of B, denoted by $A \subseteq^* B$, if $|A \smallsetminus B| < \omega$.

If $A \in [\omega]^\omega$ and $A = \{a_0 < a_1 < \cdots\}$ is an ascending listing of all elements of A, then the function $e_A : n \mapsto a_n$ is called an enumeration of A.

Definition 2.5. An ultrafilter $\mathcal{F}$ is *rapid* if for every $f \in \omega^\omega$, there is an $A \in \mathcal{F}$ such that e_A dominates f.

Definition 2.6. If $\mathcal{I}$ is an ideal, then $\mathcal{I}^+$ is the collection of all subsets of ω which are not in $\mathcal{I}$.

Let $\mathcal{A}$ be a collection of sets. The ideal generated by $\mathcal{A}$ is denoted by $\langle \mathcal{A} \rangle$.

Definition 2.7. Let $\mathcal{I}$ be an ideal on ω. A collection of sets $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is said to have $\mathcal{I}^+$*-finite intersection property*, or $\mathcal{I}^+$-f.i.p. for an abbreviation, if

$$\bigcap \mathcal{F}_0 \in \mathcal{I}^+$$

for every non-empty $\mathcal{F}_0 \in [\mathcal{F}]^{<\omega}$. If $\mathcal{I}$ is the ideal of all finite subsets of ω, then $\mathcal{I}^+$-f.i.p. is the usual strong finite intersection property, or s.f.i.p.

A collection of sets $\mathcal{A}$ with s.f.i.p. is called a *filter subbase*. A filter subbase $\mathcal{A}$ is called a *filter base* if $A, B \in \mathcal{A}$ imply $A \cap B \in \mathcal{A}$.

Definition 2.8. Let $\mathcal{F}$ be a ultrafilter and $\mathcal{I}$ be an ideal. The ultrafilter $\mathcal{F}$ is called an $\mathcal{I}$-ultrafilter if for every $f \in \omega^\omega$ there is an $A \in \mathcal{F}$ such that $f[A] \in \mathcal{I}$.

The concept of $\mathcal{I}$-ultrafilter is introduced in [2] and studied in, for example, [6, 7, 10, 13, 14].

For an $n \geq 2$ and $1 \leq k \leq n$ let π_k be the projection of $\mathbb{R}^n$ to $\mathbb{R}$ such that for every $\bar{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$

$$\pi_k(\bar{a}) = a_k.$$

The projections π_k are used for the cases of $n = 2$ and $k = 1, 2$, or $n = 3$ and $k = 1, 2, 3$.

In §4 we will consider an ideal $\mathcal{I}_{\text{int}}$ over $\omega \times \omega$ instead of ω for convenience.

Definition 2.9. Let $\prec$ be the linear order on $\omega \times \omega$ defined by $(a, b) \prec (c, d)$ if and only if $a + b < c + d$ or $a + b = c + d$ and $b < d$. This definition of $\prec$ will be used in §4 only.

Notice that all points in a diagonal line in $\omega \times \omega$ with slope -1 is a $\prec$-interval and $(\omega \times \omega; \prec) \cong (\omega; <)$. For any two points $p, q \in \omega \times \omega$, let $[\![p, q]\!]$ denote the set $\{r \in \omega \times \omega \mid p \preceq r \preceq q\}$. Let $\mathcal{H}$ be the collection of all horizontal lines $\omega \times \{l\}$ for $l \in \omega$. For each $A \subseteq \omega \times \omega$ and $n \in \omega$ let $A^n := A \cap (\omega \times \{n\})$, i.e., A^n is a horizontal cross section of A. Let $l(A^n) := (\min \pi_1 [A^n], n)$ be the leftmost point of A^n and $r(A^n) := (\max \pi_1 [A^n], n)$ be the rightmost point of A^n if $|A^n| < \omega$.

Definition 2.10. Let

$$\begin{aligned} \mathcal{C}_{\text{int}} := \{&A \subseteq \omega \times \omega \mid \\ &\forall n\, |A^n| < \omega \ \wedge \forall m, n\, (m \neq n \to [\![l(A^m), r(A^m)]\!] \cap [\![l(A^n), r(A^n)]\!] = \emptyset)\} \end{aligned}$$

and

$$\mathcal{I}_{\text{int}} := \langle \mathcal{C}_{\text{int}} \cup \mathcal{H} \rangle.$$

The subscript int stands for "interval P-points". Notice that in general we have $A^n \subsetneq [\![l(A^n), r(A^n)]\!] \cap A$. But if $A \in \mathcal{C}_{\text{int}}$, then $A^n = [\![l(A^n), r(A^n)]\!] \cap A$.

Definition 2.11. A set $\mathcal{D} \subseteq \omega^\omega$ is called a dominating family if every function $f \in \omega^\omega$ is dominated by a function $g \in \mathcal{D}$. Let

$$\mathfrak{d} := \min\{|\mathcal{D}| \mid \mathcal{D} \subseteq \omega^\omega \text{ is a dominating family}\}.$$

Definition 2.12. A set $A \subseteq \omega^\omega$ is meager if A is a countable union of nowhere dense sets. Let $\mathcal{B}$ be the collection of all meager sets. Let

$$\mathbf{cov}(\mathcal{B}) := \min\left\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{B} \text{ and } \bigcup \mathcal{A} = \omega^\omega\right\}.$$

Definition 2.13. Given an ideal $\mathcal{I}$, let

$$\begin{aligned} \mathbf{non}^{**}(\mathcal{I}) := \min\{|\mathcal{A}| \mid \\ \mathcal{A} \subseteq \mathcal{I}^+ \text{ is a filter base} \wedge \forall B \in \mathcal{I}\, \exists A \in \mathcal{A}\, (|A \cap B| < \omega)\}. \end{aligned} \tag{2.1}$$

Notice that the cardinal $\mathbf{non}^{**}(\mathcal{I})$ is the smallest size of a filter base in $\mathcal{I}^+$ which can only be extended to filters in $\mathcal{I}^+$. The cardinal invariant $\mathbf{non}^{**}(\mathcal{I})$ is introduced in [13]. It is also defined in [7] and denoted by $\mathfrak{ge}$.

Definition 2.14. Let $h : \omega \to (0, 1]$ be a non-increasing real-valued function such that $\sum_{n=1}^{\infty} h(n) = \infty$. The following set $\mathcal{I}_h$ is called a *summable ideal*[1] determined by h where

$$\mathcal{I}_h := \left\{ A \subseteq \omega \;\middle|\; \sum_{n \in A} h(n) < \infty \right\}.$$

The summable ideal $\mathcal{I}_h$ is called *tall* if $\lim_{n\to\infty} h(n) = 0$.

Notice that all non-tall summable ideals are the same which is the ideal of all finite subsets of ω. For convenience we will use the following notation.

$$\sum(h, A) := \sum_{n \in A} h(n). \tag{2.2}$$

Definition 2.15. Let $\mathcal{I}_h$ be a tall summable ideal determined by h. A set $A \subseteq \omega$ is said to be *h-slow* if $A \in \mathcal{I}_h^+$. A collection $\mathcal{A}$ is called *h-slow* if every $A \in \mathcal{A}$ is h-slow. A collection $\mathcal{A}$ is called *slow* if it is h-slow for some non-increasing h with $\lim_{n\to\infty} h(n) = 0$.

A rapid ultrafilter must contain sets in every tall summable ideal (see [19]).

In order to illustrate geometric ideas it is sometimes convenient to consider ultrafilters and ideals over the set Δ defined below.

[1] In some literature, a summable ideal $\mathcal{I}_h$ is defined without requiring that h be monotonic. We choose the current form for simplicity.

Definition 2.16. Let $\Delta \subseteq \omega^3$ be the set defined by

$$\Delta := \{(n,i,j) \mid n \in \omega \wedge 0 \leq i,j < 2^{(n+1)!}\}.$$

Let $a_0 = 0$ and $a_{n+1} = a_n + 4^{(n+1)!}$. For each $a \in \omega$, there is a unique triple (n,i,j) with $0 \leq i,j < 2^{(n+1)!}$ such that $a = a_n + j + i2^{(n+1)!}$. Let $\xi : \Delta \to \omega$ be the function defined by

$$\xi(n,i,j) := a_n + j + i2^{(n+1)!}.$$

Then ξ is a bijection from Δ to ω. If one views the set

$$\Delta_n = \{(n,i,j) \mid 0 \leq i,j < 2^{(n+1)!}\}$$

as a square with vertical lines $V_i = \{(n,i,j) \mid 0 \leq j < 2^{(n+1)!}\}$ and horizontal lines $H_j = \{(n,i,j) \mid 0 \leq i < 2^{(n+1)!}\}$, then

$$\xi[H_j] = \{a_n + j + i2^{(n+1)!} \mid i = 0,1,\ldots 2^{(n+1)!} - 1\} \text{ and}$$

$$\xi[V_i] = a_n + i2^{(n+1)!} + [0, 2^{(n+1)!} - 1].$$

Notice that the lexicographical ordering of Δ is isomorphic to the natural ordering of ω. Notice also that $\xi(V_i)$ is an interval and $\xi(H_j)$ is an arithmetic progression with difference $2^{(n+1)!}$ in ω.

Let $P : \omega \to \omega$ be the fixed function with $P\,[[a_n, a_{n+1})] = \{n\}$.

Definition 2.17. A set $X \subseteq P^{-1}(\{n\})$ is called an *l-square* if there are $S, T \subseteq 2^{(n+1)!}$ with $|S| = |T| = l$ such that $\xi[\{n\} \times S \times T] = X$. We sometimes also call the set $S \times T$ an l-square.

Definition 2.18. A set $A \subseteq P^{-1}(\{n\})$ is *l-pretty* if $\xi^{-1}(A)$ is in the union of l vertical lines such that each vertical line contains l elements, i.e., $\xi^{-1}(A) \subseteq \bigcup_{i \in I} V_i$ for some $I \subseteq 2^{(n+1)!}$ with $|I| = l$ and $|\xi^{-1}(A) \cap V_i| = l$ for each $i \in I$.

A set $A \subseteq \omega$ is *pretty* if A contains l-pretty sets for arbitrarily large l.

Definition 2.19. The following four ideals are considered in §5 and §6.

$$\mathcal{I}_1 := \{X \subseteq \omega \mid \exists k \in \omega\, (A \subseteq X \text{ is an } l\text{-square} \Rightarrow l \leq k)\},$$

$$\mathcal{I}_2 := \left\{X \subseteq \omega \mid \exists k \in \omega\, \forall n \in \omega\, (A \subseteq X \cap P^{-1}(\{n\}) \text{ is an } l\text{-square } \Rightarrow l \leq 2^{n!k})\right\}.$$

$$\mathcal{I}_3 = \{X \subseteq \omega \mid \exists k \in \omega\, (A \subseteq X \text{ is } l\text{-pretty } \Rightarrow l \leq k\},$$

$$\mathcal{I}_4 := \left\{X \subseteq \omega \mid \exists k \in \omega\, \forall n \in \omega\, (A \subseteq X \cap P^{-1}(\{n\}) \text{ is } l\text{-pretty } \Rightarrow l \leq 2^{n!k})\right\}.$$

Notice that if $A \in \mathcal{I}_2^+$ or $A \in \mathcal{I}_4^+$, then A is g-slow where $g(x) = 1/2^{n!}$ whenever $x \in P^{-1}(\{n\})$.

Definition 2.20. $\mathsf{MA}(\mathit{ctble})$ is the following statement: if $\mathbb{P}$ is a countable forcing notion and $\mathcal{D}$ is a collection of fewer than $\mathfrak{c}$ dense subsets of $\mathbb{P}$, then there is a filter $G \subseteq \mathbb{P}$ generic over $\mathcal{D}$, i.e., $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$.

Notice that $\mathsf{MA}(\mathit{ctble})$ is equivalent to $\mathbf{cov}(\mathcal{B}) = \mathfrak{c}$ (see, for example, [3]).

3. Interval P-point is not isomorphism invariant

We find a permutation p of ω and construct an interval P-point $\mathcal{F}$, assuming CH, such that $p(\mathcal{F}) := \{p[F] \mid F \in \mathcal{F}\}$ is not an interval P-point.

Recall that $\pi_1((a,b)) = a$ and $\pi_2((a,b)) = b$.

Definition 3.1. Let $R = \{a_{i,j} \in \omega \times \omega \mid (i,j) \in m \times n\}$.

- We say that R is an $m \times n$ rectangle if and only if the following are true:
  (1) there are $k_0 < k_1 < \cdots < k_{m-1}$ such that $\pi_1(a_{i,j}) = k_i$ for every $(i,j) \in m \times n$,
  (2) there are $l_0 < l_1 < \cdots < l_{n-1}$ such that $\pi_2(a_{i,j}) = l_j$ for every $(i,j) \in m \times n$.

  So, if R is a rectangle, then $a_{i,j} = (k_i, l_j)$.
- We say that R is an $m \times n$ *quasi-rectangle* if
  (1) there are $k_0 < k_1 < \cdots < k_{m-1}$ such that $\pi_1(a_{i,j}) = k_i$ for every $(i,j) \in m \times n$,
  (2) $\max\{\pi_2(a_{i,j}) \mid i \in m\} < \min\{\pi_2(a_{i,j+1}) \mid i \in m\}$ for every $j \in n-1$.
- For a rectangle or quasi-rectangle $R = \{a_{i,j} \mid (i,j) \in m \times n\}$, we call $R_i = \{a_{i,j} \mid j \in n\}$ the i-th column of R and $R^j = \{a_{i,j} \mid i \in m\}$ the j-th row of R.

Warning: The elements $a_{i,j}$ in Definition 3.1 are unrelated to a_n used in Definition 2.16. We can view a rectangle or quasi-rectangle R as well as a row or column of R as a subset of $\omega \times \omega$. Hence if f is a function with domain $\omega \times \omega$, the restriction of f to $R \subseteq \omega \times \omega$ makes sense.

Later we will consider quasi-rectangles in $\{n\} \times \omega \times \omega$ with a given $n \in \omega$. They are just the images of quasi-rectangles in $\omega \times \omega$ under the map $(a,b) \mapsto (n,a,b)$.

Notice that elements in a row in the definition of quasi-rectangle may not have the same second coordinate. Notice also that a rectangle is trivially a quasi-rectangle. If R is an $m \times n$ quasi-rectangle and $m' \le m, n' \le n$, then there exists an $R' \subseteq R$ such that R' is an $m' \times n'$ quasi-rectangle.

Lemma 3.2. *Let $X \subseteq \omega$ be such that $|X| \ge n^2$ and $f \in \omega^X$. There exists a $Y \subseteq X$ with $|Y| \ge n$ such that $f \restriction Y$ is constant or one-to-one.*

Proof. If $|f(X)| \ge n$ we can form Y by selecting one element from each $f^{-1}(\{m\})$ for every $m \in f(X)$. If $|f(X)| < n$, then there exists one $m \in f(X)$ such that $Y = f^{-1}(\{m\})$ contains more than n elements. $\square$

Lemma 3.3. *Let $k > 1$, $n \ge k^{2^m}$, and $f : m \times n \to \omega$. Then there exists an $m \times k$ rectangle $R \subseteq m \times n$ such that $f \restriction R_i$ is either constant or one-to-one for every $i \in m$.*

Proof. We prove the lemma by induction on m. The case when $m = 1$ can be proven with a simple application of Lemma 3.2. Suppose the lemma is true for the case of $m-1$. Since $n \ge \left(k^2\right)^{2^{m-1}}$, we can find an $(m-1) \times k^2$ rectangle $R' \subseteq (m-1) \times n$ such that $f \restriction R'_i$ is either constant or one-to-one for $i \in m-1$. Let $C = \{a \in m \times n \mid \pi_1(a) = m-1 \text{ and } \pi_2(a) \in \pi_2[R']\}$. Notice that C is a subset of $\{m-1\} \times n$, $|C| = k^2$, and $C \cup R'$ is an $m \times k^2$ rectangle. By Lemma 3.2 again we can find $C' \subseteq C$ with $|C'| = k$ such that $f \restriction C'$ is either constant or one-to-one. Now let

$$R = C' \cup \{a \in R' \mid \pi_2(a) \in \pi_2[C']\}.$$

Then R is an $m \times k$ rectangle and $f \restriction R_i$ is either constant or one-to-one for each $i \in m$. □

Lemma 3.4. *Let $k > 1$, $n \geq k^{2^{2m}}$, and $f : 2m \times n \to \omega$. Then there exists an $m \times k$ rectangle $R \subseteq 2m \times n$ such that either $f \restriction R_i$ is constant for every column R_i of R or $f \restriction R_i$ is one-to-one for every column R_i of R.*

Proof. In the resulting rectangle R obtained in Lemma 3.3, f is constant on at least half of the columns or one-to-one on at least half of the columns. □

Remark 3.5. The two lemmas above and their proofs are still valid if all occurrences of rectangle are replaced by quasi-rectangle.

Lemma 3.6. *If R is an $m^2 \times n$ quasi-rectangle and $f : R \to \omega$ is such that $f \restriction R_i \equiv v_i$ for each column R_i of R, then there is an $m \times n$ quasi-rectangle $R' \subseteq R$ such that $|\{v_i \mid R_i \subseteq R'\}| = 1$ or $|\{v_i \mid R_i \subseteq R'\}| = m$.*

Proof. The proof is a simple application of Lemma 3.2. □

Lemma 3.7. *If R is an $m \times \frac{1}{2}n(n+1)m$ quasi-rectangle and $f \in \omega^R$ such that $f \restriction R_i$ is one-to-one for each column $R_i \subseteq R$, then there is an $m \times n$ quasi-rectangle $R' \subseteq R$ such that $f \restriction R'$ is one-to-one.*

Proof. We prove the lemma by induction on n.

If $n = 1$, then R has m rows and m columns. Since f is one-to-one on each column, one can pick an element $a_{i,1}$ from each column of R to make the values of $f(a_{i,1})$ for $i \in m$ distinct. So $R' := \{a_{i,1} \mid i \in m\}$ is an $m \times 1$ quasi-rectangle and $f \restriction R'$ is one-to-one.

Assume that the lemma is true for $n-1$. Let

$$\underline{R} = \bigcup \left\{ R^j \mid j < \frac{1}{2}(n-1)nm \right\} \text{ and}$$

$$\overline{R} = \bigcup \left\{ R^j \mid \frac{1}{2}(n-1)nm \leq j < \frac{1}{2}n(n+1)m \right\}.$$

By the induction hypothesis we can find an $m \times (n-1)$ quasi-rectangle $\underline{R}' \subseteq \underline{R}$ such that $f \restriction \underline{R}'$ is one-to-one. Since each column of $\overline{R}$ contains mn elements and $|f[\underline{R}']| = |\underline{R}'| = m(n-1)$, one can find $\overline{R}_i' \subseteq \overline{R}_i$ with $|\overline{R}_i'| = m$ such that $f[\underline{R}'] \cap f\left[\overline{R}_i'\right] = \emptyset$ for each $i \in m$. One can now select one element $a_{i,n-1}$ from $\overline{R}_i'$ for each $i \in m$ such that $f(a_{i,n-1})$ for all $i \in m$ are distinct. Let

$$R' = \underline{R}' \cup \{a_{i,n-1} \mid i \in m\}.$$

Then R' is an $m \times n$ quasi-rectangle and $f \restriction R'$ is one-to-one. □

Remark 3.8. The assumption on the dimension of R in the lemma above can be reduced. However, the current form of the lemma is enough for our purpose.

We can now summarize the lemmas above to the following.

Lemma 3.9. *Let $m \geq 2k^2$ and $n \geq \left(\frac{1}{2}k^2(k+1)\right)^{2^{2k^2}}$. Assume that R is an $m \times n$ quasi-rectangle and $f \in \omega^R$. Then R contains a $k \times k$ quasi-rectangle R' such that one of the following is true:*

1. *$f \restriction R'$ is one-to-one,*
2. *$f \restriction R'$ is constant,*
3. *$f \restriction R'_i \equiv v_i$ for each column R'_i of R' and v_i's are distinct.*

Proof. By Lemma 3.4 we have an $k^2 \times \frac{1}{2}k^2(k+1)$ quasi-rectangle $S \subseteq R$ such that either (i) $f \restriction S_i$ is one-to-one for every column S_i of S or (ii) $f \restriction S_i$ is constant for every column S_i of S.

If (i) is true, then we can choose any k columns from S to form a $k \times \frac{1}{2}k^2(k+1)$ quasi-rectangle S'. By Lemma 3.7 we can find a $k \times k$ quasi-rectangle $R' \subseteq S'$ such that $f \restriction R'$ is one-to-one.

If (ii) is true, we can first choose an $k^2 \times k$ quasi-rectangle $S' \subseteq S$ and then find a $k \times k$ quasi-rectangle $R' \subseteq S'$ such that either $f \restriction R'$ is constant or $f \restriction R'_i \equiv v_i$ for every column R'_i of R' and all v_i's are distinct by Lemma 3.6. $\square$

Definition 3.10. Let $<$ be the lexicographical order on Δ. Hopefully, no confusion will arise when $<$ is also used for the comparison of two real numbers. We define another order $\lhd$ on Δ which is the lexicographical order with the second and third coordinates interchanged, i.e., for any $(n, i, j), (n', i', j') \in \Delta$ define

$$(n, i, j) \lhd (n', i', j') \text{ if and only if}$$

$$n < n', \text{ or } n = n' \wedge j < j', \text{ or } n = n' \wedge j = j' \wedge i < i'.$$

Notice that $(\Delta; <) \cong (\omega; <) \cong (\Delta; \lhd)$. Clearly, the identity map from Δ to Δ is a bijection but not an isomorphism between $(\Delta; <)$ and $(\Delta; \lhd)$.

We now consider quasi-rectangles in the n-th plane Δ_n. Notice that a point in n-th plane is determined by the second and third coordinates (not by the first and second coordinates as described in Definition 3.1 because the first coordinate is occupied by n).

Definition 3.11. A set $A \subseteq \Delta$ is called *good* if for every $k \in \omega$, there exists an $n \in \omega$ such that A contains a $k \times k$ quasi-rectangle in $\Delta_n = \{n\} \times 2^{(n+1)!} \times 2^{(n+1)!}$.

Remark 3.12. Notice that $\pi_1[\Delta_n] = \{n\}$. If we view Δ_n as a $2^{(n+1)!} \times 2^{(n+1)!}$ square at stage n, then π_2 projects Δ_n to the set $2^{(n+1)!}$ in the horizontal axis and π_3 projects Δ_n to the set $2^{(n+1)!}$ in the vertical axis. If A is a good subset of Δ, then $\pi_2 \restriction A$ can never be an interval-to-one map with respect to the order $(\Delta, \lhd)$. Indeed this is true because if $R = \{a_{i,j} \mid (i, j) \in 2 \times 2\} \subseteq \Delta_n$ is a 2×2 quasi-rectangle with $a_{i,j} = (n, x_{ij}, y_{ij})$, then $x_{00} = x_{01}$ and $x_{10} = x_{11}$. Hence we have that $\max_\lhd\{a_{00}, a_{10}\} \lhd \min_\lhd\{a_{01}, a_{11}\}$ while $\{a_{00}, a_{01}\} \subseteq \pi_2^{-1}(\{\pi_2(a_{00})\})$ and $\{a_{10}, a_{11}\} \subseteq \pi_2^{-1}(\{\pi_2(a_{10})\})$.

Our goal is to construct an ultrafilter $\mathcal{F}$ generated by good sets such that $\mathcal{F}$ is an interval P-point with respect to $(\Delta, <)$. Hence the identity map $ID : (\Delta, <) \to (\Delta, \lhd)$ maps an interval P-point in $(\Delta, <)$ to a non-interval P-point in $(\Delta, \lhd)$.

Lemma 3.13. *If $A \subseteq \Delta$ is good and $f \in \omega^A$, then there is a $B \subseteq A$ such that B is still good and $f \restriction B$ is either constant or interval-to-one in $(\Delta, <)$.*

Proof. Since A is good, by Lemma 3.9 we can find an increasing sequence $\{n_l \mid l \in \omega\}$ and $l \times l$ quasi-rectangles $R(l) \subseteq A \cap \Delta_{n_l}$ such that $f \restriction R(l)$ is in one of the three types described in Lemma 3.9. Let $R = \bigcup\{R(l) \mid l \in \omega\}$ and

- $J_1 = \{n_l \mid f \restriction R(l) \text{ is constant}\}$,
- $J_2 = \{n_l \mid f \restriction R(l)_i \equiv v_i \text{ for } R(l)_i \subseteq R(l) \text{ and } v_i \neq v_{i'} \text{ if } i \neq i'\}$,
- $J_3 = \{n_l \mid f \restriction R(l) \text{ is one-to-one}\}$.

Case 1: J_1 is infinite.

Let $V = \{v_l \mid n_l \in J_1 \text{ and } f \restriction R(l) \equiv v_l\}$. If V is bounded, we can find one $v \in V$ and infinite $J_1' \subseteq J_1$ such that $v_l = v$ for every $n_l \in J_1'$. Let $B = \bigcup\{R(l) \mid n_l \in J_1'\}$. Then B is good and $f \restriction B$ is constant. If V is unbounded, we can find an infinite $J_1' \subseteq J_1$ such that $\{v_l \mid n_l \in J_1' \text{ is increasing}\}$. Let $B = \bigcup\{R(l) \mid n_l \in J_1'\}$. Then B is good subset of A and $f \restriction B$ is interval-to-one because each Δ_n is an interval in $(\Delta, <)$.

Case 2: J_2 is infinite.

We construct an increasing sequence $\{m_k \in J_2 \mid k \in \omega\}$ and a sequence $\{S(k) \subseteq R \cap \Delta_{m_k} \mid k \in \omega\}$ such that

(1) $S(k)$ is a $k \times k$ quasi-rectangle and
(2) $f\left[\bigcup_{i<k} S(i)\right] \cap f\left[S_k\right] = \emptyset$.

Let $m_0 = 0$ and $S_0 = \emptyset$. Suppose we have found $\{m_l \mid l < k\}$ and $\{S(l) \mid l < k\}$ satisfying (1) and (2) up to $k-1$. Let $N_{k-1} = \sum_{l<k} |f\left[S(l)\right]|$. Choose an $m_k \in J_2$ sufficiently large such that $m_k > m_{k-1}$ and $R \cap \Delta_{m_k}$ contains an $(N_{k-1}+k) \times (N_{k-1}+k)$ quasi-rectangle R'. Since $f \restriction R_i' \equiv v_i$ with different v_i for each column R_i' of R', one can find k many v_i's not in $f\left[\bigcup_{i<k} S(i)\right]$. Let R'' be the union of these columns. Notice that R'' is a $k \times (N_{k-1}+k)$ quasi-rectangle and $f\left[\bigcup_{i<k} S(i)\right] \cap f\left[R''\right] = \emptyset$. Let $S(k) \subseteq R''$ be a $k \times k$ quasi-rectangle obtained by removing N_{k-1} many rows from R''. Hence $S(k)$ is a $k \times k$ quasi-rectangle and $f\left[\bigcup_{i<k} S(i)\right] \cap f\left[S(k)\right] = \emptyset$.

Let $B = \bigcup\{S(k) \mid k \in \omega\}$. Clearly, B is a good subset of A and $f \restriction B$ is interval-to-one because each column of Δ_n is an interval in $(\Delta, <)$.

Case 3: J_3 is infinite.

The argument in this case is almost the same as in Case 2. We again construct an increasing sequence $\{m_k \in J_3 \mid k \in \omega\}$ and a sequence $\{S(k) \subseteq R \cap \Delta_{m_k} \mid k \in \omega\}$ such that

(1) $S(k)$ is a $k \times k$ quasi-rectangle and
(2) $f\left[\bigcup_{i<k} S(i)\right] \cap f\left[S_k\right] = \emptyset$.

Let $m_0 = 0$ and $S_0 = \emptyset$. Suppose we have found $\{n_l \mid l < k\}$ and $\{S(l) \mid l < k\}$ satisfying (1) and (2) up to $k-1$. Let $N_{k-1} = \sum_{l<k} |f\left[S(l)\right]|$. Choose an $m_k \in J_3$ sufficiently large such that $m_k > m_{k-1}$ and $R \cap \Delta_{m_k}$ contains a $(N_{k-1}+k) \times (N_{k-1}+k)$ quasi-rectangle R'. Since $f \restriction R'$ is one-to-one, there can be at most N_{k-1} elements $a \in R'$ with $f(a) \in f\left[\bigcup_{l<k} S(l)\right]$. After removing rows containing these a's, one obtains a $(N_{k-1}+k) \times k$ quasi-rectangle R''. Let $S(k) \subseteq R''$ be a $k \times k$ quasi-rectangle obtained by removing N_{k-1} columns from R''. Hence $S(k)$ is a $k \times k$ quasi-rectangle and $f\left[\bigcup_{i<k} S(i)\right] \cap f\left[S(k)\right] = \emptyset$.

Let $B = \bigcup\{S(k) \mid k \in \omega\}$. Clearly, B is a good subset of A and $f \restriction B$ is one-to-one which certainly implies that $f \restriction B$ is interval-to-one. □

Lemma 3.14. *If $\{A_n \subseteq \Delta \mid n \in \omega\}$ is a sequence of good subsets of Δ such that $A_{n+1} \subseteq^* A_n$ for every $n \in \omega$, i.e., $A_{n+1} \setminus A_n$ is a finite set, then there is a good subset $B \subseteq \Delta$ such that $B \subseteq^* A_n$ for all $n \in \omega$.*

Proof. This can be proven by a standard diagonal argument. □

Theorem 3.15 (CH)**.** *There is an interval P-point $\mathcal{F}$ and a bijection $p : \omega \to \omega$ such that $p(\mathcal{F})$ is not an interval P-point.*

Proof. We construct an interval P-point $\mathcal{F}$ in $(\Delta, <)$ and identity map p from Δ to Δ such that $p(\mathcal{F})$ is not an interval P-point in $(\Delta, \lhd)$. The ultrafilter $\mathcal{F}$ should be generated by good subsets in Δ.

Let $\omega^\Delta = \{f_\alpha : \alpha < \omega_1\}$.

We construct $\{A_\alpha \subseteq \Delta \mid \alpha < \mathfrak{c}\}$ inductively on α satisfying the following conditions:

(1) $\forall \alpha < \mathfrak{c}\,(A_\alpha$ is good$)$,
(2) $\forall \alpha < \beta < \mathfrak{c}\,(A_\beta \subseteq^* A_\alpha)$,
(3) $\forall \alpha < \mathfrak{c}$ $(f_\alpha \restriction A_{\alpha+1}$ is either constant or interval-to-one with respect to the order $(\Delta, <)$.

Let $A_0 = \Delta$. If α is a limit ordinal, let A_α be obtained by a diagonal argument in Lemma 3.14. Suppose that $\{A_\beta \mid \beta \leq \alpha\}$ has been obtained. By Lemma 3.13 we find $A_{\alpha+1} \subseteq^* A_\alpha$ such that $f_\alpha \restriction A_{\alpha+1}$ is either constant or interval-to-one. This completes the construction. Let $\mathcal{F}$ be a filter generated by $\{A_\alpha \mid \alpha < \mathfrak{c}\}$. Since a characteristic function χ_A for $A \subseteq \Delta$ is a constant function on some $A' \in \mathcal{F}$, which implies $A' \subseteq A$ or $A' \subseteq \omega \setminus A$, we have that $A \in \mathcal{F}$ or $\omega \setminus A \in \mathcal{F}$. So, we conclude that $\mathcal{F}$ is an interval P-point over $(\Delta; <)$. Since every $A \in \mathcal{F}$ is good, $\mathcal{F} = p(\mathcal{F})$ is not an interval P-point over $(\Delta; \lhd)$. □

Remark 3.16. The ultrafilter $\mathcal{F}$ in Theorem 3.15 can also be constructed under MA. One needs only to generate the set A_α for each limit ordinal $\alpha < \mathfrak{c}$ by a c.c.c. forcing notion. In fact, the assumption $\mathbf{cov}(\mathcal{B}) = \mathfrak{c}$ suffices (see §6). Furthermore, the ultrafilter $\mathcal{F}$ above can be made rapid or slow with a little more work.

4. Generic existence of interval P-points and $\mathbf{non}^{**}(\mathcal{I}_{int}) = \mathfrak{c}$

In this section we show that every filter base of size less than $\mathfrak{c}$ can be extended to an interval P-point if and only if $\mathbf{non}^{**}(\mathcal{I}_{\rm int}) = \mathfrak{c}$ where the ideal $\mathcal{I}_{\rm int}$ is defined in Definition 2.10 and $\mathbf{non}^{**}(\mathcal{I}_{\rm int})$ is defined in Definition 2.13.

It is proven in [16] that every filter base of size less than $\mathfrak{c}$ can be extended to a P-point if and only if $\mathfrak{d} = \mathfrak{c}$. It is also proven in [8] that every filter base of size less than $\mathfrak{c}$ can be extended to a selective ultrafilter if and only if $\mathbf{cov}(\mathcal{B}) = \mathfrak{c}$.

In [13] it is proven that every filter base of size less that $\mathfrak{c}$ can be extended to an $\mathcal{I}$-ultrafilter if and only if $\mathbf{non}^{**}(\mathcal{I}) = \mathfrak{c}$. Both P-points and selective ultrafilters are $\mathcal{I}$-ultrafilters for some ideals $\mathcal{I}$ (see [10]). If an ideal $\mathcal{I}$ could be found so that an ultrafilter $\mathcal{F}$ is an interval P-point if and only if $\mathcal{F}$ is an $\mathcal{I}$-ultrafilter, then we would have achieved our goal by applying directly the result in [13]. Unfortunately, this is impossible because $\mathcal{I}$-ultrafilters are isomorphism invariant while interval P-points are not by Theorem 3.15 at least under CH. However, the idea in [13] can still be used for our purpose.

Lemma 4.1. *If $\mathcal{F}$ is an $\mathcal{I}_{int}$-ultrafilter, then $\mathcal{F}$ is an interval P-point.*

Proof. Let $\mathcal{F}$ be an $\mathcal{I}_{\mathrm{int}}$-ultrafilter and $f \in \omega^\omega$. It suffices to find $A \in \mathcal{F}$ such that $f \restriction A$ is either constant or interval-to-one.

We define an increasing function $g : (\omega; <) \to (\omega \times \omega; \prec)$ by induction where $(\omega \times \omega; \prec)$ is defined in the paragraph before Definition 2.10. Let $g(0) = (0, f(0))$. Suppose that $g(i)$ has been defined for every $i < m$. Set

$$g(m) := (\min\{a \mid \forall i < m\,(g(i) \prec (a, f(m)))\}, f(m)),$$

i.e., $g(m)$ is the leftmost point in the horizontal line $\omega \times \{f(m)\}$ which is $\prec$-greater than $g(i)$ for every $i < m$. Notice that $\pi_2(g(m)) = f(m)$ for all $m \in \omega$.

Since $\mathcal{F}$ is an $\mathcal{I}_{\mathrm{int}}$-ultrafilter, there is an $A_0 \in \mathcal{F}$ such that $g[A_0] \in \mathcal{I}_{\mathrm{int}}$. Notice that every member of $\mathcal{I}_{\mathrm{int}}$ is a subset of the union of finitely many lines in $\mathcal{H}$ and finitely many sets in $\mathcal{C}_{\mathrm{int}}$. Hence there is an $A_1 \in \mathcal{F}$ with $A_1 \subseteq A_0$ such that $g[A_1]$ is a subset of one line in $\mathcal{H}$ or one set in $\mathcal{C}_{\mathrm{int}}$. If $g[A_1] \subseteq \omega \times \{n\}$, then $f(a) = n$ for every $a \in A_1$, i.e., f is constant on A_1. So we can assume that $g[A_1]$ is infinite and is a subset of a member in $\mathcal{C}_{\mathrm{int}}$. Clearly, $g[A_1]$ is itself in $\mathcal{C}_{\mathrm{int}}$.

For each $n \in \omega$ with $g[A_1]^n \neq \emptyset$ let $a_n, b_n \in A_1$ be such that $g(a_n) = l\,(g[A_1]^n)$ and $g(b_n) = r\,(g[A_1]^n)$. Notice that A_1 a subset of the union of all these $[a_n, b_n]$'s. If $n_1 \neq n_2$, then $[\![g(a_{n_1}), g(b_{n_1})]\!] \cap [\![g(a_{n_2}), g(b_{n_2})]\!] = \emptyset$ by the definition of $\mathcal{C}_{\mathrm{int}}$. Hence $[a_{n_1}, b_{n_1}] \cap [a_{n_2}, b_{n_2}] = \emptyset$ because g is increasing. If $a \in [a_n, b_n] \cap A_1$, then $g(a_n) \preceq g(a) \preceq g(b_n)$. Hence $g(a) \in g[A_1]^n$ because again $g[A_1] \in \mathcal{C}_{\mathrm{int}}$. Therefore, we have $f(a) = n$. This shows that f is an interval-to-one function on $A_1 \in \mathcal{F}$. $\square$

Theorem 4.2. *Every filter base of size less than $\mathfrak{c}$ can be extended to an interval P-point if and only if $\mathbf{non}^{**}(\mathcal{I}_{int}) = \mathfrak{c}$.*

Proof. "$\Rightarrow$": Assume that $\mathbf{non}^{**}(\mathcal{I}_{\mathrm{int}}) < \mathfrak{c}$. Let $\mathcal{A}$ be the set appeared in the right side of (2.1) with $|\mathcal{A}| = \mathbf{non}^{**}(\mathcal{I}_{\mathrm{int}})$. We show that $\mathcal{A}$ cannot be extended to an interval P-point over $(\omega \times \omega; \prec)$.

Suppose that $\mathcal{A} \subseteq \mathcal{F}$ and $\mathcal{F}$ is an interval P-point over $(\omega \times \omega; \prec)$. By the definition of $\mathcal{A}$ we have that $\mathcal{F} \subseteq \mathcal{I}^+_{\mathrm{int}}$. We show that there does not exist a set $A \in \mathcal{F}$ such that π_2 is either constant or interval-to-one on A.

Suppose that $\pi_2 \restriction A$ is either constant or interval-to-one for some $A \in \mathcal{F}$. If $\pi_2[A] = \{n\}$, then $A \subseteq \omega \times \{n\} \in \mathcal{I}_{\mathrm{int}}$, which contradicts $\mathcal{F} \subseteq \mathcal{I}^+_{\mathrm{int}}$. So we can assume that π_2 is interval-to-one on A. Hence we can find disjoint intervals $[\![p_n, q_n]\!]$ for each $n \in \pi_2[A]$ such that $A \cap [\![p_n, q_n]\!] = A \cap \pi_2^{-1}(n) = A^n$. But this implies that $A \in \mathcal{C}_{\mathrm{int}}$, contradicting $A \in \mathcal{I}^+_{\mathrm{int}}$. Therefore, π_2 witnesses that $\mathcal{F}$ is not interval-to-one.

"$\Leftarrow$": This part is just half of the proof in [13]. To be self-contained, we sketch a proof here.

Assume that $\mathbf{non}^{**}(\mathcal{I}_{\mathrm{int}}) = \mathfrak{c}$. Let $\mathcal{A}$ be a filter base of size less than $\mathfrak{c}$ over ω. Let $(\omega \times \omega)^\omega = \{f_\alpha \mid \alpha < \mathfrak{c}\}$ and $\mathcal{P}(\omega) = \{A_\alpha \mid \alpha < \mathfrak{c}\}$. We construct $\mathcal{A} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots$ inductively such that $\mathcal{F}_\alpha$ is a filter base of size less than $\mathfrak{c}$ and $f_\alpha[A] \in \mathcal{I}_{\mathrm{int}}$ for some $A \in \mathcal{F}_{\alpha+1}$ for every $\alpha < \mathfrak{c}$. If α is a limit ordinal, let $\mathcal{F}_\alpha$ be the union of all $\mathcal{F}_\beta$ for $\beta < \alpha$. To construct $\mathcal{F}_{\alpha+1}$ first add A_α or $\omega \smallsetminus A_\alpha$ to $\mathcal{F}_\alpha$ to form $\mathcal{F}'_\alpha$ with s.f.i.p. If $f_\alpha[A] \in \mathcal{I}_{\text{int}}$ for some $A \in \mathcal{F}'_\alpha$, let $\mathcal{F}_{\alpha+1} = \mathcal{F}'_\alpha$. So we can assume that $f_\alpha[A] \in \mathcal{I}^+_{\text{int}}$ for every $A \in \mathcal{F}'_\alpha$. Since $|f_\alpha[\mathcal{F}'_\alpha]| < \mathfrak{c} = \mathbf{non}^{**}(\mathcal{I}_{\text{int}})$, there is a $B \in \mathcal{I}_{\text{int}}$ such that $|f_\alpha[A] \cap B| = \omega$ for every $A \in \mathcal{F}'_\alpha$. Hence $\mathcal{F}'_\alpha \cup \{f_\alpha^{-1}[B]\}$ has s.f.i.p. Let $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \cup \{f_\alpha^{-1}(B)\}$. This completes the construction. Clearly, $\mathcal{F} = \bigcup_{\alpha<\mathfrak{c}} \mathcal{F}_\alpha$ is an $\mathcal{I}_{\text{int}}$-ultrafilter extending $\mathcal{A}$. Now $\mathcal{F}$ is an interval P-point extending $\mathcal{A}$ by Lemma 4.1. □

Remark 4.3. In the proof above, the order $\prec$ can be replaced by any order $\prec'$ as long as $(\omega \times \omega; \prec') \cong (\omega; <)$.

Remark 4.4. Let

$$\mathcal{C}_{\text{select}} = \{A \subseteq \omega \times \omega \mid \forall n\, |A^n| \leq 1\},$$
$$\mathcal{C}_{\text{P}} = \{A \subseteq \omega \times \omega \mid \forall n\, |A^n| < \omega\},$$

$\mathcal{I}_{\text{select}} = \langle \mathcal{H} \cup \mathcal{C}_{\text{select}} \rangle$, and $\mathcal{I}_{\text{P}} = \langle \mathcal{H} \cup \mathcal{C}_{\text{P}} \rangle$. Then $\mathcal{I}_{\text{select}} \subseteq \mathcal{I}_{\text{int}} \subseteq \mathcal{I}_{\text{P}}$. Hence $\mathbf{non}^{**}(\mathcal{I}_{\text{select}}) \leq \mathbf{non}^{**}(\mathcal{I}_{\text{int}}) \leq \mathbf{non}^{**}(\mathcal{I}_{\text{P}})$. Notice that $\mathcal{I}_{\text{select}}$ and $\mathcal{I}_{\text{P}}$ are the reflections of $\mathcal{ED}$ and Fin $\times$ Fin, respectively, in [10] along the diagonal line[2]. It is proven in [10] that $\mathcal{F}$ is a P-point if and only if $\mathcal{F}$ is an $\mathcal{I}_{\text{P}}$-ultrafilter and $\mathcal{F}$ is a selective ultrafilter if and only if $\mathcal{F}$ is an $\mathcal{I}_{\text{select}}$-ultrafilter. It is proven in [13] that $\mathbf{non}^{**}(\mathcal{I}_{\text{select}}) = \mathbf{cov}(\mathcal{B})$ and $\mathbf{non}^{**}(\mathcal{I}_{\text{P}}) = \mathfrak{d}$. We don't know which of the cases $\mathbf{cov}(\mathcal{B}) = \mathbf{non}^{**}(\mathcal{I}_{\text{int}}) < \mathfrak{d}$, $\mathbf{cov}(\mathcal{B}) < \mathbf{non}^{**}(\mathcal{I}_{\text{int}}) = \mathfrak{d}$, or $\mathbf{cov}(\mathcal{B}) < \mathbf{non}^{**}(\mathcal{I}_{\text{int}}) < \mathfrak{d}$ is consistently possible.

5. Generic existence of slow non-interval P-point and $\mathfrak{d} = \mathfrak{c}$

Ketonen's argument in [16] can be slightly modified to prove the following.

Theorem 5.1. *Every h-slow filter base of size less than $\mathfrak{c}$ can be extended to an h-slow P-point if and only if $\mathfrak{d} = \mathfrak{c}$.*

Proof. Assume $\mathfrak{d} = \mathfrak{c}$. Let $\mathcal{P}(\omega) = \{A_\alpha \mid \alpha < \mathfrak{c}\}$ and $\mathcal{F}_0$ be an h-slow filter base of size less than $\mathfrak{c}$. We construct an increasing sequence of h-slow filter bases $\{\mathcal{F}_\alpha \mid \alpha < \mathfrak{c}\}$ of sizes less than $\mathfrak{c}$ by induction such that $\mathcal{F} = \bigcup_{\alpha<\mathfrak{c}} \mathcal{F}_\alpha$ is the desired P-point.

If $\alpha < \mathfrak{c}$ is a limit ordinal, let $\mathcal{F}_\alpha$ be the union of $\mathcal{F}_\beta$ for all $\beta < \alpha$. Suppose that all $\mathcal{F}_\beta$ for $\beta \leq \alpha$ has been obtained. First add A_α or $\omega \smallsetminus A_\alpha$ to $\mathcal{F}_\alpha$ to form $\mathcal{F}'_\alpha$ so that $\mathcal{F}'_\alpha$ has $\mathcal{I}^+_h$-f.i.p. By adding fewer than $\mathfrak{c}$ sets we can assume that $\mathcal{F}'_\alpha$ is closed under finite intersection and deletion of finite sets. Let $\mathcal{F}'_\alpha = \{A_\gamma \mid \gamma < \lambda\}$ for some $\lambda < \mathfrak{c}$.

Claim 5.2. Given a countable collection $\mathcal{C} = \{F_m \in \mathcal{F}'_\alpha \mid m \in \omega\}$, there exists an h-slow set F such that $F \subseteq^* F_m$ for each $m \in \omega$ and $\mathcal{F}'_\alpha \cup \{F\}$ has $\mathcal{I}^+_h$-f.i.p.

[2]We define $\mathcal{C}_{\text{int}}$ by requiring that all sets A in it have finite y-segments A^n, rather than finite x-segments, due to the fact that in the proof of Lemma 4.1. the function f, as a convention, maps a number from x-axis to a number in y-axis. For keeping consistency, we then introduce the ideals $\mathcal{I}_{\text{selec}}$ and $\mathcal{I}_{\text{P}}$ as the reflections of $\mathcal{ED}$ and Fin $\times$ Fin, respectively, along the diagonal line $y = x$.

Proof of Claim 5.2 : By reducing to the intersection, we can assume that $F_m \supseteq F_{m+1}$ for $m \in \omega$. For each $\gamma < \lambda$ define a function $g_\gamma \in \omega^\omega$ such that for every $m \in \omega$

$$g_\gamma(m) := \min\left\{k \in \omega \;\middle|\; \sum(h, F_m \cap A_\gamma \cap k) \geq 1\right\}$$

where $\sum(h, A)$ is defined by (2.2). The function g_γ is well defined because $F_m \cap A_\gamma$ is an h-slow set. Since $\lambda < \mathfrak{d}$, there is an increasing function $g \in \omega^\omega$ which is not bounded by any g_γ for $\gamma < \lambda$. Let

$$F := \bigcup_{m \in \omega} (F_m \cap g(m)).$$

Clearly, $F \subseteq^* F_m$ because $F \setminus F_m \subseteq g(m)$. Given each $A_\gamma \in \mathcal{F}'_\alpha$, we show that $F \cap A_\gamma$ is an h-slow set. Given an $m \in \omega$, it suffices to show that there is a $k > m$ such that

$$\sum(h, F \cap A_\gamma \cap [m, k)) \geq 1.$$

Notice that $A_\gamma \setminus m = A_{\gamma'} \in \mathcal{F}'_\alpha$. Since g is not bounded above by $g_{\gamma'}$, there is an $n \geq m$ such that $g(n) > g_{\gamma'}(n)$. Thus $F \cap A_\gamma \cap [m, g(n)) \supseteq F_n \cap A_{\gamma'} \cap g_{\gamma'}(n)$ which implies that

$$\sum(h, F \cap A_\gamma \cap [m, g(n))) \geq \sum(h, F_n \cap A_{\gamma'} \cap g_{\gamma'}(n)) \geq 1$$

by the definition of $g_{\alpha'}$. This completes the proof of Claim 5.2.

Now let $\mathcal{F}_{\alpha+1} = \mathcal{F}'_\alpha \cup \{F\}$ and the inductive construction is completed.

Let $\mathcal{F} = \bigcup_{\alpha<\mathfrak{c}} \mathcal{F}_\alpha$. By a book-keeping trick one can make sure that for all countable collections $\mathcal{C} \subseteq \mathcal{F}$ there is an $F \in \mathcal{F}$ such that $F \subseteq^* C$ for every $C \in \mathcal{C}$. Hence $\mathcal{F}$ is an h-slow P-point extending $\mathcal{F}_0$.

Assume $\mathfrak{d} < \mathfrak{c}$. We want to find an h-slow filter base of size $\mathfrak{d}$ which cannot be extended to any P-point. The example is exactly the same as the example in [16]. We just add an extra requirement that every set in the filter is h-slow. Let $\{f_\alpha \mid \alpha < \mathfrak{d}\}$ be a dominating family of functions from ω to ω. Consider the filter base $\mathcal{F}_0$ on $\omega \times \omega$ containing all sets of the form $A_n = [n, \infty) \times \omega$ for all $n \in \omega$ and all sets of the form $B_\alpha = \{(a, b) \mid b > f_\alpha(a)\}$ for $\alpha < \mathfrak{d}$. Then $|\mathcal{F}_0| = \mathfrak{d} < \mathfrak{c}$ and $\mathcal{F}_0$ cannot be extended to a P-point.

Notice that every infinitely long arithmetic progression $\{a + kn \mid n \in \omega\}$ is h-slow. Define a bijection $\theta : \omega \times \omega \to \omega$ by

$$\theta(a, b) := 2^a - 1 + 2^{a+1} b.$$

If $I \in [\omega]^{<\omega}$ and $J \in [\mathfrak{d}]^{<\omega}$, then $\theta\left[\left(\bigcap_{n\in I} A_n\right) \cap \left(\bigcap_{\alpha\in J} B_\alpha\right)\right]$ is h-slow because θ maps $\{k\} \times (\omega \setminus \max\{f_\alpha(k) \mid \alpha \in J\})$ for some $k > \max I$, which is a subset of $\left(\bigcap_{n\in I} A_n\right) \cap \left(\bigcap_{\alpha\in J} B_\alpha\right)$, to an infinite arithmetic progression. Let $\mathcal{F}'_0 = \{\theta[F] \mid F \in \mathcal{F}_0\}$. Then $\mathcal{F}'_0$ is an h-slow filter base, $|\mathcal{F}'_0| < \mathfrak{c}$, and $\mathcal{F}'_0$ cannot be extended to any P-point over ω. □

Since it is a theorem of ZFC that there does not exist $1/n$-slow interval P-point by [15, Theorem 3.2], we have the following corollary.

Corollary 5.3. Every $1/n$-slow filter base of size less than $\mathfrak{c}$ can be extended to a $1/n$-slow non-interval P-point if and only if $\mathfrak{d} = \mathfrak{c}$.

Notice that the non-interval P-point in the corollary above requires that the P-point be $1/n$-slow. In the following theorem we construct a slow but not too slow non-interval P-point directly assuming $\mathfrak{d} = \mathfrak{c}$. In the proof the set Δ defined in Definition 2.16 is used.

Lemma 5.4. *If a set $A \subseteq \omega$ contains an l-square for $l \geq 2$, then $\pi_3 \circ \xi^{-1}$ is not interval-to-one on A, where l-square is defined in Definition 2.17 and ξ is defined in Definition 2.16.*

The proof of Lemma 5.4 can be found in [12]. The idea is similar to the argument in Remark 3.12.

Lemma 5.5. *Suppose $S, T \subseteq 2^{(n+1)!}$ with $|S| = |T| = 2^{3m}$. Given each coloring function with two colors $c : S \times T \to 2$, there exist $S' \subseteq S$ and $T' \subseteq T$ with $|S'| = |T'| \geq m$ such that all points in $S' \times T'$ have the same color.*

Proof. Without loss of generality let $S = T = 2^{3m}$. Recall that $\pi_2 : (i, j) \mapsto j$. Let $V_i = \{(i, j) \mid 0 \leq j < 2^{3m}\}$ be the i-th vertical line of the square $2^{3m} \times 2^{3m}$. Choose sets $T_i \subseteq V_i$ for $i = 0, 1, \ldots, 2m-1$ inductively such that $\pi_2 [T_{i+1}] \subseteq \pi_2 [T_i]$, $2|T_0| = |V_0|$, $2|T_{i+1}| = |T_i|$, and all points in T_i have the same color. Then $|T_{2m-1}| = 2^{3m-2m} = 2^m \geq m$. Notice that since all points in the set T_i has the same one of the two colors, there is a set $S' \subseteq 2m$ with $|S'| = m$ such that all points in $\bigcup_{i \in S'} T_i$ have the same color. Choose a $T' \subseteq \pi_2 [T_{2m-1}]$ with $|T'| = m$. Then all points in the square $S' \times T'$ with side length m have the same color. $\square$

By Lemma 5.5, $\mathcal{I}_1$ and $\mathcal{I}_2$ defined in Definition 2.19 are indeed ideals. Notice that all sets in $\mathcal{I}_2^+$ are g-slow where $g(x) = 1/2^{n!}$ for all $x \in [a_n, a_{n+1})$ for all $n \in \omega$.

Theorem 5.6. *The following are equivalent:*

(1) *$\mathfrak{d} = \mathfrak{c}$;*
(2) *Every filter base of size less than $\mathfrak{c}$ satisfying $\mathcal{I}_1^+$-f.i.p. can be extended to a non-interval P-point;*
(3) *Every filter base of size less than $\mathfrak{c}$ satisfying $\mathcal{I}_2^+$-f.i.p. can be extended to a g-slow non-interval P-point.*

Proof. We prove (1) $\iff$ (2) and (1) $\iff$ (3) simultaneously. The steps for the second part which are different from the first part will be inside parentheses.

Assume $\mathfrak{d} = \mathfrak{c}$. Let $\mathcal{P}(\omega) = \{A_\alpha \mid \alpha < \mathfrak{c}\}$ and $\mathcal{F}_0$ be a filter base with $\mathcal{I}_1^+$-f.i.p. ($\mathcal{I}_2^+$-f.i.p.) and $|\mathcal{F}_0| < \mathfrak{c}$. We construct an increasing sequence of filter bases $\{\mathcal{F}_\alpha \mid \alpha < \mathfrak{c}\}$ with $\mathcal{I}_1^+$-f.i.p. ($\mathcal{I}_2^+$-f.i.p.) by induction so that $|\mathcal{F}_\alpha| < \mathfrak{c}$, A_α or $\omega \setminus A_\alpha$ is in $\mathcal{F}_{\alpha+1}$ for each $\alpha < \mathfrak{c}$, and for each countable collection $\mathcal{C} \in [\mathcal{F}_\beta]^\omega$, there is a $\gamma < \mathfrak{c}$ and an $F \in \mathcal{F}_\gamma$ such that $F \subseteq^* C$ for each $C \in \mathcal{C}$. Clearly, $\mathcal{F} = \bigcup_{\alpha<\mathfrak{c}} \mathcal{F}_\alpha$ is the desired P-point.

If α is a limit ordinal, let $\mathcal{F}_\alpha$ be the union of $\mathcal{F}_\beta$ for all $\beta < \alpha$. Suppose all filter bases $\mathcal{F}_\beta$ for $\beta \leq \alpha$ has been obtained. By Lemma 5.5 we can assume that $A_\alpha \in \mathcal{F}_\alpha$ or $\omega \setminus A_\alpha \in \mathcal{F}_\alpha$. We can also assume that $\mathcal{F}_\alpha$ is closed under finite intersection and deletion of finite sets.

Claim 5.7. Given a countable collection $\mathcal{C} = \{F_m \in \mathcal{F}_\alpha \mid m \in \omega\}$, there exists a set F such that $F \subseteq^* F_m$ for each $m \in \omega$ and $\mathcal{F}_\alpha \cup \{F\}$ has $\mathcal{I}_1^+$-f.i.p. ($\mathcal{I}_2^+$-f.i.p.)

Proof of Claim 5.7 : The idea of finding F is the same as in the proof of Claim 5.2 except that we want $\mathcal{F}'_\alpha \cup \{F\}$ to have $\mathcal{I}_1^+$-f.i.p. ($\mathcal{I}_2^+$-f.i.p.) instead of $\mathcal{I}_h^+$-f.i.p.

Use the same notation as in Claim 5.2 we define a function $f_\gamma \in \omega^\omega$ for each $\gamma < \lambda$ such that for every $m \in \omega$

$$f_\gamma(m) := \min\{a_{n+1} \mid F_m \cap A_\gamma \cap a_{n+1} \text{ contains an } l\text{-square for } l \geq m\}.$$

$$(f_\gamma(m) := \min\{a_{n+1} \mid F_m \cap A_\gamma \cap a_{n+1} \text{ contains an } l\text{-square for } l \geq 2^{mn!}\}.)$$

Let $f \in \omega^\omega$ an increasing function which is not dominated by f_γ for any $\gamma < \lambda$ and let

$$F := \bigcup_{m\in\omega} (F_m \cap f(m)).$$

Clearly, $F \subseteq^* F_m$. We now show that $\{F\} \cap \mathcal{F}_\alpha$ has $\mathcal{I}_1^+$-f.i.p. ($\mathcal{I}_2^+$-f.i.p.)

Let $\gamma < \lambda$. Given each $m \in \omega$, it suffices to show that $F \cap A_\gamma$ contains an l-square for $l \geq m$ ($F \cap A_\gamma \cap \Delta_n$ contains an l-square for $l \geq 2^{mn!}$ for some n).

Since f is not dominated by f_γ, there is a $k \geq m$ such that $f(k) > f_\gamma(k)$. Notice that $F_k \cap A_\gamma \cap f_\gamma(k)$ contains an l-square for $l \geq k \geq m$ ($2^{kn!} \geq 2^{mn!}$) and

$$F \cap A_\gamma \cap f(k) \supseteq F \cap A_\gamma \cap f_\gamma(k) \supseteq F_k \cap f_\gamma(k) \cap A_\gamma.$$

We conclude that $F \cap A_\gamma$ contains an l-square for $l \geq k \geq m$ ($l \geq 2^{kn!} \geq 2^{mn!}$). This completes the proof of Claim 5.7.

Let $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \cup \{F\}$ and $\mathcal{F} = \bigcup_{\alpha<\mathfrak{c}} \mathcal{F}_\alpha$. By a book-keeping trick one can make sure that for every $\mathcal{C} \in [\mathcal{F}]^\omega$ there is an $F \in \mathcal{F}$ such that $F \subseteq^* C$ for every $C \in \mathcal{C}$. Hence $\mathcal{F}$ is a P-point extending $\mathcal{F}_0$. Since each $A \in \mathcal{F}$ contains squares of unbounded side lengths l, the map $\pi_3 \circ \xi^{-1}$ can never be an interval-to-one map on A by Lemma 5.4.

Assume $\mathfrak{d} < \mathfrak{c}$. It suffices to show that the filter base $\mathcal{F}'_0$ in the "if" part of Theorem 5.1 has $\mathcal{I}_1^+$-f.i.p. ($\mathcal{I}_2^+$-f.i.p.). Notice that $\mathcal{I}_1^+ \supseteq \mathcal{I}_2^+$.

Since $\mathcal{F}'_0$ is closed under finite intersection, it suffices to show that every $A \in \mathcal{F}'_0$ is in $\mathcal{I}_1^+$ (in $\mathcal{I}_2^+$). Notice that if $A \in \mathcal{F}'_0$, then we can find an M satisfying that if $m \geq M$, then there is an N_m such that A contains a set of the form $R = \{2^{m-1}-1+2^m k \mid k = N_m, N_m+1, \ldots\}$. Fix an $m \geq M$ and $k \in \omega$. If we choose n sufficiently large so that $a_n \geq N_m$ and $n > m$, then $\xi^{-1}(R \cap [a_n, a_{n+1})) = \{n\} \times 2^{(n+1)!} \times T$ is a $2^{(n+1)!}$ by $2^{(n+1)!-m}$ rectangle where T is an arithmetic progression of difference 2^m, and $|T| = 2^{(n+1)!-m}$. Hence $R \cap [a_n, a_{n+1})$ contains an l-square for $l = 2^{(n+1)!-m}$. Since n can be arbitrarily large, we have that $2^{(n+1)!-m} \geq 2^{kn!}$ for all sufficiently large n. □

Remark 5.8. We choose $a_n = \sum_{i=0}^{n-1} 4^{(n+1)!}$ for convenience only. Other sequences work too provided that they grow sufficiently fast. The ultrafilter which has $\mathcal{I}_2^+$-f.i.p. is a slow non-interval P-point although some sets in the ultrafilter have their enumeration functions grow quite fast. It is not determined whether the non-interval P-point with $\mathcal{I}_1^+$-f.i.p. is rapid or slow.

6. Generic existence of interval P-points and $\mathbf{cov}(\mathcal{B}) = \mathfrak{c}$

Notice that assuming $\mathfrak{d} = \mathfrak{c}$ won't guarantee the existence of a rapid ultrafilter. So a stronger condition should be assumed for the rapid version counterparts of the theorems in §5. The natural candidates of a stronger condition, according to the diagram in [3, page 424], is $\mathfrak{b} = \mathfrak{c}$ or $\mathbf{cov}(\mathcal{B}) = \mathfrak{c}$. Since $\mathfrak{b} = \mathfrak{c}$ won't guarantee the existence of rapid ultrafilters either ($\mathfrak{b} = \mathfrak{c}$ is true in Laver model for Borel conjecture in which no rapid ultrafilter exists. See [17]), it is reasonable to assume $\mathbf{cov}(\mathcal{B}) = \mathfrak{c}$. Notice that assuming $\mathbf{cov}(\mathcal{B}) = \mathfrak{c}$ is really assuming $\mathsf{MA}(ctble)$, i.e., Martin's Axiom for all countable forcing notions. Thus in this section we construct rapid non-interval P-points and rapid/slow interval P-points which are neither quasi-selective nor weakly Ramsey using countable forcing constructions.

The following proposition is [3, Theorem 7.13].

Proposition 6.1. $\mathbf{cov}(\mathcal{B}) = \mathfrak{c}$ if and only if $\mathsf{MA}(ctble)$ holds.

The following lemma is essentially in [12]. We give only a sketch of the proof. The set Δ is defined in Definition 2.16 and the ideals $\mathcal{I}_3, \mathcal{I}_4$ are defined in Definition 2.19.

Lemma 6.2. *If an ultrafilter $\mathcal{F}$ has $\mathcal{I}_i^+$-f.i.p. for $i = 3, 4$, then $\mathcal{F}$ is not weakly Ramsey.*

Proof. Without loss of generality we consider that $\mathcal{F}$ is over Δ instead of ω. For any $\{(x_1, y_1, z_1), (x_2, y_2, z_2)\} \in [\Delta]^2$ let

$$c((x_1, y_1, z_1), (x_2, y_2, z_2)) = \begin{cases} 0, & \text{if } x_1 = x_2 \text{ and } y_1 = y_2 \\ 1, & \text{if } x_1 = x_2 \text{ and } y_1 \neq y_2 \\ 2, & \text{if } x_1 \neq x_2 \end{cases}$$

A set $A \subseteq \Delta$ is called $\{i, j\}$-homogeneous for c if $c[[A]^2] = \{i, j\}$. It is now easy to check that if A contains l-pretty sets for arbitrarily large l, then A cannot be an $\{i, j\}$-homogeneous set of c for any $i, j \in 3$. □

Theorem 6.3. *Assume $\mathbf{cov}(\mathcal{B}) = \mathfrak{c}$. Then*

(1) *every filter base $\mathcal{F}_0$ of size less than $\mathfrak{c}$ satisfying $\mathcal{I}_1^+$-f.i.p. can be extended to a rapid non-interval P-point;*
(2) *every filter base $\mathcal{F}_0$ of size less than $\mathfrak{c}$ satisfying $\mathcal{I}_3^+$-f.i.p. can be extended to a rapid interval P-point which is neither quasi-selective nor weakly Ramsey;*
(3) *every filter base $\mathcal{F}_0$ of size less than $\mathfrak{c}$ satisfying $\mathcal{I}_4^+$-f.i.p. can be extended to a g-slow interval P-point for some g, which is neither quasi-selective nor weakly Ramsey.*

Proof. Let $\omega^\omega = \{g_\alpha \mid \alpha < \mathfrak{c}\}$ and $\mathcal{P}(\omega) = \{A_\alpha \mid \alpha < \mathfrak{c}\}$.

(1) We construct $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ inductively such that

a. $\mathcal{F}_\alpha$ is a filter base and $|\mathcal{F}_\alpha| < \mathfrak{c}$ for all $\alpha < \mathfrak{c}$,
b. $\mathcal{F}_\alpha$ has $\mathcal{I}_1^+$-f.i.p. for all $\alpha < \mathfrak{c}$,
c. $A_\alpha \in \mathcal{F}_{\alpha+1}$ or $\omega \smallsetminus A_\alpha \in \mathcal{F}_{\alpha+1}$ for all $\alpha < \mathfrak{c}$,
d. there is an $F \in \mathcal{F}_{\alpha+1}$ with $g_\alpha \leq^* e_F$ for all $\alpha < \mathfrak{c}$, where e_F is the enumeration of F,

e. for any $\alpha' < \mathfrak{c}$ and $\{E_n\}_{n=0}^{\infty} \subseteq \mathcal{F}_{\alpha'}$, there is an $\alpha < \mathfrak{c}$ and $F \in \mathcal{F}_\alpha$ with $F \subseteq^* E_n$ for every $n \in \omega$.

Fix a function $\eta : \mathfrak{c} \to \mathfrak{c} \times \mathfrak{c}$ for a book-keeping purpose such that for every $(\beta, \gamma) \in \mathfrak{c} \times \mathfrak{c}$ there is an $\alpha \in \mathfrak{c}$ with $\alpha \geq \beta$ and $\eta(\alpha) = (\beta, \gamma)$.

If $\alpha \leq \mathfrak{c}$ is a limit ordinal, let $\mathcal{F}_\alpha$ be the union of $\mathcal{F}_\beta$ for all $\beta < \alpha$. Suppose that $\mathcal{F}_\beta$ for all $\beta \leq \alpha$ are constructed. We now construct $\mathcal{F}_{\alpha+1}$. Fix an enumeration of $[\mathcal{F}_\alpha]^\omega = \{\mathcal{E}_{\alpha,\gamma} \mid \gamma < \mathfrak{c}\}$. Notice that the enumerations of $[\mathcal{F}_{\alpha'}]^\omega = \{\mathcal{E}_{\alpha',\gamma} \mid \gamma < \mathfrak{c}\}$ for all $\alpha' < \alpha$ have already been fixed. Let $\eta(\alpha) = (\beta_\alpha, \gamma_\alpha)$. We want to make sure that A_α or $\omega \setminus A_\alpha$ is in $\mathcal{F}_{\alpha+1}$, there is a set $F \in \mathcal{F}_{\alpha+1}$ with $e_F \geq^* g_\alpha$, and there is a set $F \in \mathcal{F}_{\alpha+1}$ with $F \subseteq^* E_n$ for every $E_n \in \mathcal{E}_{\beta_\alpha,\gamma_\alpha}$.

If $\{A_{\beta_\alpha}\} \cup \mathcal{F}_\alpha$ has $\mathcal{I}_1^+$-f.i.p., let $\mathcal{F}'_\alpha = \{A_{\beta_\alpha}\} \cup \mathcal{F}_\alpha$. Otherwise let $\mathcal{F}'_\alpha = \{\omega \setminus A_{\beta_\alpha}\} \cup \mathcal{F}_\alpha$. So $\mathcal{F}'_\alpha$ has $\mathcal{I}_1^+$-f.i.p. by Lemma 5.5.

Without loss of generality let $\mathcal{E}_{\beta_\alpha,\gamma_\alpha} = \{E_n \mid n \in \omega\}$ with $E_0 \supseteq E_1 \supseteq E_2 \supseteq \cdots$ and g_α be increasing. Fix a function $h \in \omega^\omega$ such that $h(0) = 1$ and $a_{h(n)} > g_\alpha(a_{h(n-1)})$ for all $n > 0$. Let

$$\begin{aligned}\mathbb{P}_\alpha := \{s \in [\omega]^{<\omega} \mid \forall n \in \omega\, |P[s] \cap [h(n), h(n+1))| \leq 1 \wedge \\ \forall i\, (i \in P[s] \cap [h(n), h(n+1)) \to s \cap P^{-1}(\{i\}) \text{ is an } l\text{-square for } l \leq n)\}.\end{aligned}$$

For each $s \in [\omega]^{<\omega}$ let $top(s) := \max P[s]$. For any $s, t \in \mathbb{P}_\alpha$ define

$$t \leq s \text{ if and only if } s \subseteq t,\ t \cap a_{top(s)+1} = s, \text{ and } t \setminus s \subseteq E_{top(s)}.$$

Clearly, $\mathbb{P}_\alpha$ is a countable partial order. For each $A \in \mathcal{F}'_\alpha$ and $m \in \omega$ let

$$\begin{aligned}D_{A,m} := \{s \in \mathbb{P}_\alpha \mid \\ \exists n \in P[s \cap A]\, (s \cap A \cap P^{-1}(\{n\}) \text{ is an } l\text{-square for } l \geq m)\}.\end{aligned}$$

Then $D_{A,m}$ is dense in $\mathbb{P}_\alpha$ because for every $s \in \mathbb{P}_\alpha$ the set $A \cap E_{top(s)}$ contains squares of arbitrarily long side lengths. Let $\mathcal{D} := \{D_{A,m} \mid A \in \mathcal{F}'_\alpha \wedge m \in \omega\}$. Then $|\mathcal{D}| = |\mathcal{F}'_\alpha| < \mathfrak{c}$. Hence there is a filter $G \subseteq \mathbb{P}_\alpha$ generic over $\mathcal{D}$. Let $F = \bigcup G$. Then $\{F\} \cup \mathcal{F}'_\alpha$ has $\mathcal{I}_1^+$-f.i.p. and $F \subseteq^* E_n$ for every $n \in \omega$. Let $\mathcal{F}_{\alpha+1} = \mathcal{F}'_\alpha \cup \{F\}$ and the induction is completed.

Notice that $e_F \geq^* h$ because the following: Given a sufficiently large $a \in F$. Suppose that a is the k-th element of F and $a \in P^{-1}(\{j\})$ for some $j \in [h(n), h(n+1))$. Notice that

$$|F \cap a_{h(n)}| \leq \sum_{i=1}^{n-1} i^2 \text{ and } |F \cap [a_{h(n)}, a_{h(n+1)})| \leq n^2.$$

Thus we have that $k \leq \sum_{i=1}^{n} i^2$. Since

$$a_{h(n-1)} = 1 + \sum_{i=0}^{h(n-1)-1} 4^{(i+1)!} \geq \sum_{i=1}^{n} i^2 \geq k$$

for sufficiently large n, we have $g_\alpha(k) \leq g_\alpha(a_{h(n-1)}) < a_{h(n)} \leq a_j \leq a$, i.e., the k-th element of F is greater than $g_\alpha(k)$. So $e_F \geq^* g_\alpha$.

Let $\mathcal{F} = \mathcal{F}_\mathfrak{c}$. Since every $\mathcal{E} \in [\mathcal{F}]^{<\omega}$ is in $[\mathcal{F}_{\alpha'}]^{<\omega}$ for some $\alpha' < \mathfrak{c}$ and there is an $F \in \mathcal{F}_\alpha$ for some $\alpha > \alpha'$ such that $F \subseteq^* E_n$ for every $E_n \in \mathcal{E}$, we have that $\mathcal{F}$ is a P-point. Since $\mathcal{F}$ has $\mathcal{I}_1^+$-f.i.p., we have that $\mathcal{F}$ is not an interval P-point by

Lemma 5.4. Since every $g_\alpha \in \omega^\omega$ is eventually dominated by an enumeration e_F for some $F \in \mathcal{F}$ we conclude that $\mathcal{F}$ is a rapid ultrafilter.

(2) We construct $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ inductively such that

a. $\mathcal{F}_\alpha$ is a filter base and $|\mathcal{F}_\alpha| < \mathfrak{c}$ for all $\alpha < \mathfrak{c}$,
b. $\mathcal{F}_\alpha$ has $\mathcal{I}_3^+$-f.i.p. for all $\alpha < \mathfrak{c}$,
c. there is an $F \in \mathcal{F}_{\alpha+1}$ such that g_α is constant or interval-to-one on F.
d. there is an $F \in \mathcal{F}_{\alpha+1}$ such that $g_\alpha \leq^* e_F$.

Again $\mathcal{F}_\alpha = \bigcup_{\beta<\alpha} \mathcal{F}_\beta$ if $\alpha \leq \mathfrak{c}$ is a limit ordinal. Suppose that $\alpha < \mathfrak{c}$ and $\mathcal{F}_\beta$ for all $\beta \leq \alpha$ have been obtained. We construct $\mathcal{F}_{\alpha+1}$. Without loss of generality we assume that $\mathcal{F}_{\alpha+1}$ is closed under finite intersection.

If there is an $m \in \omega$ such that $\{g_\alpha^{-1}(\{m\})\} \cup \mathcal{F}_\alpha$ has $\mathcal{I}_3^+$-f.i.p., then let $\mathcal{F}'_\alpha := \{g_\alpha^{-1}(\{m\})\} \cup \mathcal{F}_\alpha$ and generate a set F the same way as in the proof of (1) with "l-squares" replaced by "l-pretty sets." Also ignore E_n. So $e_F \geq^* g_\alpha$ is true. Hence we can assume that $\{g_\alpha^{-1}(\{m\})\} \cup \mathcal{F}_\alpha$ does not have $\mathcal{I}_3^+$-f.i.p. for any $m \in \omega$.

Again assume that g_α is increasing and fix a function $h \in \omega^\omega$ such that $h(0) = 1$ and $a_{h(n)} > g_\alpha(a_{h(n-1)})$ for all $n > 0$. Let

$$\begin{aligned}\mathbb{P}_\alpha := \{s \in [\omega]^{<\omega} \mid \forall n \in \omega\, |P[s] \cap [h(n), h(n+1))| \leq 1 \wedge \\ \forall i\, (i \in P[s] \cap [h(n), h(n+1)) \to s \cap P^{-1}(\{i\}) \text{ is an } l\text{-pretty set for } l \leq n) \wedge \\ g_\alpha \restriction s \text{ is interval-to-one}\}.\end{aligned}$$

For any $s, t \in \mathbb{P}_\alpha$ define

$$t \leq s \text{ if and only if } s \subseteq t, \text{ and } t \cap a_{top(s)+1} = s.$$

For each $A \in \mathcal{F}_\alpha$ let

$$D_{A,m} := \{s \in \mathbb{P}_\alpha \mid s \cap A \text{ contains an } l\text{-pretty set for } l \geq m\}$$

and $\mathcal{D} := \{D_{A,m} \mid A \in \mathcal{F}_\alpha \wedge m \in \omega\}$. Then $|\mathcal{D}| = |\mathcal{F}_\alpha| < \mathfrak{c}$.

Claim 6.4. $D_{A,m}$ is dense in $\mathbb{P}_\alpha$.

Proof of Claim 6.4 Given each $s \in \mathbb{P}_\alpha$, let $R = g_\alpha[s]$. Since R is finite, there is an $A' \subseteq A$, $A' \in \mathcal{F}_\alpha$ such that $A' \cap (\bigcup\{g_\alpha^{-1}(k) \mid k \in R\}) \in \mathcal{I}_3$. Let b be an upper bound of all l's for some l-pretty set in $A' \cap (\bigcup\{g_\alpha^{-1}(k) \mid k \in R\})$. Since $A' \in \mathcal{I}_3^+$, there is an $n > h(j) > top(s)$ for some j such that $A' \cap P^{-1}(\{n\})$ contains an l'-pretty set B with $l' > b|s| + m^4$. Suppose that $\xi^{-1}[B] = V = \bigcup_{i<l'} V_i$ where V_i is a subset of a vertical line such that $|V_i| = l'$. By deleting $b|s|$ lines from V, we can assume that every remaining V_i contains at most $b|s|$ elements v such that $g_\alpha \circ \xi(v) \in g_\alpha[s]$. After deleting these elements v we obtain an $(l' - b|s|)$-pretty set $V' \subseteq V$ such that $g_\alpha \circ \xi[V'] \cap g_\alpha[s] = \emptyset$. Let $l'' = l' - b|s|$ and $V' = \bigcup_{i<l''} V'_i$. Notice that we have that $l'' > m^4$ and $|V'_i| > m^4$.

Notice that for each $i < l''$, there exists $V''_i \subseteq V'_i$ such that $|V''_i| > m^2$ and $g_\alpha \circ \xi$ is one-to-one or constant on V''_i by Lemma 3.2. Let $I_0 = \{i < l'' \mid g_\alpha \circ \xi \restriction V''_i \text{ is one-to-one}\}$ and $I_1 = l'' \setminus I_0$. Then $\max\{|I_0|, |I_1|\} \geq m^2$.

If $|I_0| \geq m^2$, one can select $V'''_i \subseteq V''_i$ inductively for all $i \in I_0$ such that $|V'''_i| = m$ and $g_\alpha \circ \xi$ is one-to-one on $\bigcup_{i \in I_0} V'''_i$. Choose an $I'_0 \subseteq I_0$ with $|I'_0| = m$. Now $B' = \xi(\bigcup_{i \in I'_0} V'''_i)$ is an m-pretty set. Let $t = s \cup B'$. Then $g_\alpha \restriction t$ is interval-to-one, $t \leq s$, and $t \in D_{A,m}$.

Suppose that $|I_1| \geq m^2$ and let c_i be the constant value of $g_\alpha \circ \xi$ on V''_i for $i \in I_1$. Then there exists $I'_1 \subseteq I_1$ with $|I'| = m$ such that c_i are the same for all

$i \in I_1'$ or c_i are pairwise distinct for all $i \in I_1'$. In each of these two cases, $g_\alpha \circ \xi$ is interval-to-one on $\bigcup_{i\in I_1'} V_i''$ because, due to the lexicographical ordering of Δ, each vertical line in Δ_n corresponds to an interval in ω. Choose a $V_i''' \subseteq V_i''$ with $|V_i'''| = m$ for $i \in I_1'$. Let $B' = \xi\left(\bigcup_{i\in I_0'} V_i'''\right)$ and $t = s \cup B'$. Then $t \leq s$ and $t \in D_{A,m}$. This completes the proof of Claim 6.4.

By $\mathsf{MA}(\mathit{ctble})$ there is a filter $G \subseteq \mathbb{P}_\alpha$ generic over $\mathcal{D}$. Let $F = \bigcup G$. By the same argument as in (1) we have that $e_F \geq^* g_\alpha$ and $g_\alpha \restriction F$ is an interval-to-one function. Clearly, $\mathcal{F}_\alpha \cup \{F\}$ has $\mathcal{I}_3^+$-f.i.p. Let $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha \cup \{F\}$. This completes the inductive construction.

Now $\mathcal{F} = \mathcal{F}_\mathfrak{c}$ is a rapid interval P-point which is not weakly Ramsey. Since a rapid quasi-selective ultrafilter must be selective (see [4]), we conclude that $\mathcal{F}$ is not quasi-selective.

(3) The proof is quite similar to the proof of (2). We construct an increasing sequence of filter bases $\mathcal{F}_\alpha$ which satisfy $|\mathcal{F}_\alpha| < \mathfrak{c}$ and have $\mathcal{I}_4^+$-f.i.p. Again $\mathcal{F}_\alpha$ is the union of $\mathcal{F}_\beta$ for all $\beta < \alpha$ if α is a limit ordinal. To construct $\mathcal{F}_{\alpha+1}$ let

$$\begin{aligned}\mathbb{P}_\alpha = \{s \in [\omega]^{<\omega} \mid\ & g_\alpha \restriction s \text{ is interval-to-one } \wedge \\ & \forall n \in P[s]\,(s \cap P^{-1}(\{n\}) \text{ is an } l\text{-pretty set for some } l \geq 2^{n!}).\}\end{aligned}$$

For any $s, t \in \mathbb{P}_\alpha$ define

$$t \leq s \text{ if and only if } s \subseteq t, \text{ and } t \cap a_{top(s)+1} = s.$$

For each $A \in \mathcal{F}_\alpha$ let

$$D_{A,m} := \{s \in \mathbb{P}_\alpha \mid s \cap A \text{ contains an } l\text{-pretty set for } l \geq 2^{n!m}\}$$

and $\mathcal{D} := \{D_{A,m} \mid A \in \mathcal{F}_\alpha \wedge m \in \omega\}$. Then $|\mathcal{D}| = |\mathcal{F}_\alpha| < \mathfrak{c}$. The proof of $D_{A,m}$ being dense is similar to the proof in (2). Given $s \in \mathbb{P}_\alpha$, to extend s to $t \in D_{A,m}$ we start with an l-pretty set with $l > 2^{n!b}|s| + \left(2^{n!m}\right)^4$ here instead of $l > b|s| + m^4$ in (2).

The ultrafilter $\mathcal{F} = \mathcal{F}_\mathfrak{c}$ is an interval P-point because every function $g \in \omega^\omega$ is either constant or interval-to-one on some set $F \in \mathcal{F}$. $\mathcal{F}$ is g-slow where $g(x) = 1/2^{n!}$ if $x \in [a_n, a_{n+1})$ because for each $A \in \mathcal{F}$ there are infinitely many n such that $A \cap P^{-1}(\{n\})$ contains an l-pretty set for some $l > 2^{n!}$. So there are infinitely many n with $\sum(g, A \cap P^{-1}(\{n\})) \geq 1$.

We show that $\mathcal{F}$ is not quasi-selective by the following: Suppose $\mathcal{F}$ is quasi-selective. By [4, Proposition 1.7] for every function $f \in \omega^\omega$ with $f(n) \leq 4^n$ there is an $A \in \mathcal{F}$ such that f is non-decreasing on A. Now define a function f by letting $f(x) = a_{n+1} - x$ for all $n \in \omega$ and all $x \in [a_n, a_{n+1})$. So for each $x \in [a_n, a_{n+1})$ we have $f(x) \leq a_{n+1} = a_n + 4^{(n+1)!} \leq 2^{a_n} + 2^{4^{n!}} \leq 2^{2a_n} = 4^{a_n} \leq 4^x$. Hence f is non-decreasing on some $A \in \mathcal{F}$. Thus by the definition of A, we have that $|A \cap [a_n.a_{n+1})| \leq 1$. But this contradicts that $A \in \mathcal{I}_4^+$. So $\mathcal{F}$ is not quasi-selective. $\square$

Remark 6.5. By the same idea of the proof for Theorem 6.3, one can improve Theorem 3.15 by assuming $\mathbf{cov}(\mathcal{B}) = \mathfrak{c}$ instead of CH. One can also make the interval P-point $\mathcal{F}$ over Δ to be rapid or slow.

Let $\mathcal{I}$ be the ideal of all $A \subseteq \Delta$ such that A does not contain good sets. Then construct a sequence of increasing filter bases $\mathcal{F}_\alpha$ inductively for all $\alpha < \mathfrak{c}$ such that $|\mathcal{F}_\alpha| < \mathfrak{c}$, $\mathcal{F}_{\alpha+1}$ contains either the α-th set A_α or its complement, g_α is either constant or interval-to-one on some $F \in \mathcal{F}_{\alpha+1}$, and $\mathcal{F}_\alpha$ has $\mathcal{I}^+$-f.i.p.

At a limit stage, take the union of previously obtained filter bases. At the stage $\alpha + 1$, do exactly the same as in part (2) or (3) of Theorem 6.3 except the partial order $\mathbb{P}_\alpha$ is modified with "l-pretty set" replaced by "$l \times l$ quasi-rectangle".

7. Questions

Question 7.1. Can each of the following be a consequence of ZFC plus $\mathbf{cov}(\mathcal{B}) < \mathfrak{d}$:

- $\mathbf{non}^{**}(\mathcal{I}_{\text{int}}) < \mathfrak{d}$;
- $\mathbf{non}^{**}(\mathcal{I}_{\text{int}}) = \mathfrak{d}$;
- $\mathbf{cov}(\mathcal{B}) < \mathbf{non}^{**}(\mathcal{I}_{\text{int}})$;
- $\mathbf{cov}(\mathcal{B}) = \mathbf{non}^{**}(\mathcal{I}_{\text{int}})$?

Question 7.2. Can each of the following be consistent with ZFC plus $\mathbf{cov}(\mathcal{B}) < \mathfrak{d}$:

- $\mathbf{cov}(\mathcal{B}) = \mathbf{non}^{**}(\mathcal{I}_{\text{int}}) < \mathfrak{d}$;
- $\mathbf{cov}(\mathcal{B}) < \mathbf{non}^{**}(\mathcal{I}_{\text{int}}) = \mathfrak{d}$;
- $\mathbf{cov}(\mathcal{B}) < \mathbf{non}^{**}(\mathcal{I}_{\text{int}}) < \mathfrak{d}$?

Question 7.3. Do all three conclusions of Theorem 6.3 imply $\mathbf{cov}(\mathcal{B}) = \mathfrak{c}$?

References

[1] T. Bartoszyński and H. Judah, *Set Theory, On the Structure of The Real Line*, A. K. Peters, 1995.
[2] J. E. Baumgartner, *Ultrafilters on ω*. Journal of Symbolic Logic, 1995, 2, 624 – 639
[3] A. Blass, *Combinatorial cardinal characteristics of the continuum*, in: M. Foreman, A. Kanamori (Eds.), Handbook of Set Theory, Springer-Verlag, 2010, 395 – 489.
[4] A. Blass, M. Di Nasso, and M. Forti, *Quasi-selective ultrafilters and asymptotic numerosities*, Advances in Mathematics, Vol. 231 (2012), 1462 – 1486.
[5] D. Booth, *Ultrafilters on a countable set*, Annals of Mathematical Logic, 2 (1970), 1 – 24.
[6] J. Brendle, *Between p-points and nowhere dense ultrafilters*, Israel Journal of Mathematics, vol. 113 (1999), 205 – 230
[7] J. Brendle and J. Flašková, *Generic existence of ultrafilters on the natural numbers*, Fundamenta Mathematicae 236 (2017), 201 – 245
[8] R. M. Canjar, *On the generic existence of special ultrafilters*, Proceedings of the American Mathematical Society, 110 (1990), 233 – 241.
[9] M. Di Nasso, *Fine asymptotic densities for sets of natural numbers*, Proceedings of the American Mathematical Society, 138 (2010) 2657 – 2665.
[10] J. Flašková, *Description of some ultrafilters via $\mathcal{I}$-ultrafilter*, Proceedings of RIMS, vol. 1619 (2008)
[11] M. Forti, *Quasi-selective and weakly Ramsey ultrafilters*, https://arxiv.org/pdf/1012.4338.pdf
[12] J. He, R. Jin, S. Zhang, *Rapid Interval P-point*, Topology and Its Applications, 283, 1 September 2020, 107333
[13] J. Y. Hong and S. Zhang, *Cardinal invariants related to the $\mathcal{I}$-ultrafilters*, Scientia Sinica Mathematica, Vol. 43, Issue 1 (2013), 1 – 6
[14] J. Y. Hong and S. Zhang, *Relations between the $\mathcal{I}$-ultrafilters*, Archive for Mathematical Logic, 56 (2017), Issue 1-2, 161 – 173
[15] R. Jin, *Slow P-point Ultrafilters*, Journal of Symbolic Logic, 85, 1, (2020), 26 – 36
[16] J. Ketonen, *On the existence of P-points in the Stone-Čech compactification of integers*, Fundamenta Mathematicae (1976), Vol. 92, Issue: 2, 91 – 94

[17] A. Miller, *There are no Q-Points in Laver's Model for the Borel Conjecture*, Proceedings of the American Mathematical Society, Vol. 78, No. 1 (Jan., 1980), 103 – 106.
[18] S. Shelah, *Proper and Improper Forcing*, second ed., Springer-Verlag, 1998.
[19] P. Vojtáš, *On* ω^* *and absolutely divergent series*, Topology Proceedings 19 (1994), 335 – 348.

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