

Abstract densities and ideals of sets

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Abstract

Abstract upper densities are monotone and subadditive functions from the power set of positive integers to the unit real interval that generalize the upper densities used in number theory, including the upper asymptotic density, the upper Banach density, and the upper logarithmic density.

We answer a question posed by G. Grekos in 2013, and prove the existence of translation invariant abstract upper densities onto the unit interval, whose null sets are precisely the family of finite sets, or the family of sequences whose series of reciprocals converge. We also show that no such density can be atomless. (More generally, these results also hold for a large class of summable ideals.)

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Introduction

Several notions of densities for sets of natural numbers are used in number theory for different purposes, including the upper and lower asymptotic densities, the upper and lower Banach (or uniform) density, the upper and lower logarithmic density, the Schnirelmann density, *etc.*. The idea of an abstract notion of density that encompasses the basic features of the known densities have been repeatedly considered in the literature (see, *e.g.* [2, 1, 6, 4, 10, 5]).

At the open problem session of the Workshop “*Densities and their applications*”, held in St. Etienne in July 2013, the following question was asked by G. Grekos:

- *Is there an “abstract density” δ such that $\delta(A) = 0$ if and only if A is finite? Or such that $\delta(A) = 0$ if and only if $\sum_{a \in A} 1/a < \infty$? More generally, given an ideal of sets $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$, is there an “abstract density” δ such that $\delta(A) = 0$ if and only if $A \in \mathcal{I}$?*

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Recall that a nonempty family $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is an *ideal* if it closed under subsets and under taking finite unions, and $\mathbb{N} \notin \mathcal{I}$.

To avoid trivial examples, such an abstract density should behave “nicely”, in the sense that it should share as many as possible of the properties of the familiar densities as considered in number theory. In this paper we investigate around the above questions.

The paper is organized as follows. In the first section, we discuss the general properties of an abstract density; in the second section we present our main results; the third section contains the proofs.

1 Abstract Densities

Let \mathbb{N} be the set of positive integers. Intervals of natural numbers will be denoted by using the subscript \mathbb{N} ; so, *e.g.*, we will write $[1, n]_{\mathbb{N}}$ to mean $[1, n] \cap \mathbb{N}$.

Let us start by isolating the fundamental features that an abstract notion of density on \mathbb{N} must have.

Definition 1.1 An *abstract density* on \mathbb{N} is a function $\delta : \mathcal{P}(\mathbb{N}) \rightarrow [0, 1]$ defined on the family of all subsets of \mathbb{N} , taking values in the unit real interval, and that satisfies the following properties:

- (1) $\delta(\mathbb{N}) = 1$,
- (2) $\delta(F) = 0$ for every finite $F \subset \mathbb{N}$,
- (3) *Monotonicity*: If $A \subseteq B$ then $\delta(A) \leq \delta(B)$.

We remark that virtually all upper and lower densities that have been considered in number theory are examples of abstract densities in the above sense. For example, it is easily seen that the following seven densities for subsets of \mathbb{N} satisfy properties (1), (2), (3).¹

For $A \subseteq \mathbb{N}$ and $a \leq b$ in \mathbb{N} , we denote $A(a, b) := |A \cap [a, b]|$ and by $A(a) := A(1, a)$.

- The *lower asymptotic density*:

$$\underline{d}(A) := \liminf_{n \rightarrow \infty} \frac{A(n)}{n};$$

- The *upper asymptotic density*:

$$\bar{d}(A) := \limsup_{n \rightarrow \infty} \frac{A(n)}{n};$$

¹ Also other densities that have been considered in the literature satisfy properties (1), (2), (3), *e.g.* upper Buck, upper Polya, upper analytic, and exponential densities (see [7] for the definitions).

- The *lower Banach density*:

$$\underline{BD}(A) := \liminf_{n \rightarrow \infty} \inf_{k \in \mathbb{N}} \frac{A(k+1, k+n)}{n};$$

- The *upper Banach density* (or simply *Banach density* or *uniform density*):

$$\overline{BD}(A) := \limsup_{n \rightarrow \infty} \sup_{k \in \mathbb{N}} \frac{A(k+1, k+n)}{n};$$

- The *lower logarithmic density*:

$$\underline{ld}(A) := \liminf_{n \rightarrow \infty} \frac{\sum_{a \in A \cap [1, n]} a^{-1}}{\sum_{a=1}^n a^{-1}};$$

- The *upper logarithmic density*:

$$\overline{ld}(A) := \limsup_{n \rightarrow \infty} \frac{\sum_{a \in A \cap [1, n]} a^{-1}}{\sum_{a=1}^n a^{-1}};$$

- The *Schnirelmann density*:²

$$\sigma(A) = \inf_{n \geq 1} \frac{A(n)}{n}.$$

Since here we are interested in abstract densities δ whose family of null sets is closed under finite unions, also the following property should be satisfied:

$$“\delta(A) = \delta(B) = 0 \Rightarrow \delta(A \cup B) = 0.”$$

In consequence, a natural assumption is subadditivity.

Definition 1.2 An abstract density is called *abstract upper density* if it satisfies:

(4) *Subadditivity*: $\delta(A \cup B) \leq \delta(A) + \delta(B)$.

The name “upper density” is justified by the fact that the three upper densities itemized above (namely, upper asymptotic density, upper Banach density, and upper logarithmic density) are indeed subadditive. On the contrary, the three corresponding lower densities, as well as Schnirelmann density, are not.

The following is easily proved:

² Schnirelmann density is included here because it is widely used in number theory. However, it is worth remarking that it is a “one of a kind” density, that was designed to guarantee the implication: “ $\sigma(A) = 1 \Rightarrow A = \mathbb{N}$.”

- Let δ be an abstract upper density. If the symmetric difference $A \triangle B$ is finite, then $\delta(A) = \delta(B)$.

The monotonicity and subadditivity properties are independent of each other.

Proposition 1.3 *Let $\delta : \mathcal{P}(\mathbb{N}) \rightarrow [0, 1]$. Then*

- (i) *Properties (1), (2), (3) do not imply (4);*
- (ii) *Properties (1), (2), (4) do not imply (3).*

PROOF. (i). The lower asymptotic density is an example of an abstract density that is not subadditive. (E.g., let $A = \bigcup_{n \in \mathbb{N}} (a_{2n}, a_{2n+1}]_{\mathbb{N}}$ where $\{a_n\}$ is any increasing sequence with $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \infty$; then it is easily verified that $\underline{d}(A) = \underline{d}(A^c) = 0$, while $\underline{d}(A \cup A^c) = 1$.)

(ii). Let $E \subseteq \mathbb{N}$ be the set of all even numbers, let $O \subseteq \mathbb{N}$ be the set of all odd numbers, let E_1 be the set of multiples of 4, and let $E_2 = E \setminus E_1$ be the set of numbers that are congruent to 2 modulo 4. Define the function $\delta : \mathcal{P}(\mathbb{N}) \rightarrow [0, 1]$ as follows:

$$\delta(A) = \begin{cases} 0 & \text{if } A \text{ is finite,} \\ 1 & \text{if } A \cap O \text{ is infinite,} \\ 1 & \text{if } A \cap O \text{ is finite, and exactly one of } A \cap E_1 \text{ and } A \cap E_2 \text{ is infinite,} \\ \frac{1}{2} & \text{if } A \cap O \text{ is finite, and both } A \cap E_1 \text{ and } A \cap E_2 \text{ are infinite.} \end{cases}$$

It is readily seen that δ satisfies (1) and (2). Moreover, δ is subadditive. Indeed, if both A and B are infinite, then $\delta(A \cup B) \leq 1 = \frac{1}{2} + \frac{1}{2} \leq \delta(A) + \delta(B)$; and if at least one of the two sets is finite, say B , then $\delta(A \cup B) = \delta(A) \leq \delta(A) + \delta(B)$. However, δ is not monotonic; for example, $E_1 \subseteq E$ but $\delta(E_1) = 1 > \delta(E) = 1/2$. \square

For $A \subseteq \mathbb{N}$ and $m \in \mathbb{N}$, denote by $A + m := \{a + m \mid a \in A\}$ the *rightward translation* of A by m ; and by $A - m := \{k \in \mathbb{N} \mid k + m \in A\}$ the *leftward translation* of A by m . A natural property to be considered for densities is the following:

- (5) *Translation invariance: $\delta(A + m) = \delta(A - m) = \delta(A)$ for every $A \subseteq \mathbb{N}$ and for every $m \in \mathbb{N}$.*

It is easily seen that an abstract upper density that is rightward translation invariant, is also leftward translation invariant. Indeed, in this case, $\delta(A - m) = \delta((A - m) + m) = \delta(A)$, because the symmetric difference $((A - m) + m) \triangle A$ is finite.

With the only exception of Shnirelmann density, we remark that all other densities itemized at the beginning of this section are translation invariant (see, e.g., [7]).

There is a trivial example of an abstract density that fulfills all properties (1)–(5).

Example 1.4 Let $\delta : \mathcal{P}(\mathbb{N}) \rightarrow [0, 1]$ be the function defined by setting $\delta(A) = 0$ if A is finite, and $\delta(A) = 1$ if A is infinite. Then δ is a translation invariant upper abstract density.

The above example calls for additional properties that make abstract densities “non-trivial”. Two such properties that have been repeatedly considered in the literature are the following:

- (6) *Richness*: For every real number $r \in [0, 1]$ there exists $A \subseteq \mathbb{N}$ with $\delta(A) = r$;
- (7) *Atomless-ness*: For every $A \subseteq \mathbb{N}$ with $\delta(A) > 0$, there exists $B \subset A$ such that $0 < \delta(B) < \delta(A)$.

Notice that even by assuming (1) – (5), richness and atomless-ness are independent of each other. The following example is motivated by [9].

Example 1.5 For every $A \subseteq \mathbb{N}$, let

$$\delta(A) := \begin{cases} \frac{1}{2}(1 + \bar{d}(A)) & \text{if } \bar{d}(A) > 0, \\ 0 & \text{if } \bar{d}(A) = 0. \end{cases}$$

Then δ is a translation invariant abstract upper density that is atomless but not rich, because $\text{range}(\delta) = \{0\} \cup (\frac{1}{2}, 1]$.

The notion of asymptotic density also makes sense when relativized to any infinite set.

Definition 1.6 Let $X \subseteq \mathbb{N}$ be an infinite set and $f : \mathbb{N} \rightarrow [0, 1]$ be a non-increasing function such that $\sum_{n \in X} f(n) = \infty$. For every $A \subseteq \mathbb{N}$, the *upper f -weighted density of A relative to X* is defined as

$$\bar{d}_{f, X}(A) := \limsup_{n \rightarrow \infty} \frac{\sum_{k \in A \cap X, k < n} f(k)}{\sum_{k \in X, k < n} f(k)}.$$

Notice that relative upper f -weighted densities are abstract upper densities that are rich; however, in general, they are not translation invariant.

Example 1.7 Let $\langle b_n \mid n \in \mathbb{N} \rangle$ be a “rapidly growing” sequence of natural numbers, in the sense that $\lim_{n \rightarrow \infty} \frac{b_{n+1} - b_n}{b_n} = \infty$. Let $B := \bigcup_{n=1}^{\infty} [b_{2n}, b_{2n+1}]_{\mathbb{N}}$, and for $A \subseteq \mathbb{N}$ let

$$\delta_B(A) := \begin{cases} 1 & \text{if } \bar{d}_B(A) > 0, \\ 0 & \text{if } \bar{d}_B(A) = 0 \end{cases} \quad \text{and} \quad \delta(A) := \max\{\delta_B(A), \bar{d}_{\mathbb{N} \setminus B}(A)\}.$$

Then δ is a translation invariant abstract upper density that is rich but not atomless. Indeed, by the choice of B , both \bar{d}_B and $\bar{d}_{\mathbb{N} \setminus B}$ are translation invariant. Moreover, δ is rich because $\bar{d}_{\mathbb{N} \setminus B}$ is rich. Finally, B is an “atom”, that is, $\delta(B) = 1$ and for every $B' \subseteq B$ one has either $\delta(B') = 0$ or $\delta(B') = \delta(B)$.

A stronger property that directly implies both richness and atomless-ness is the following intermediate value property:

- (8) *Darboux property*:³ For every $A \subseteq \mathbb{N}$ and for every $0 \leq r \leq \delta(A)$, there exists a subset $B \subseteq A$ such that $\delta(B) = r$.

2 The results

Recall that an *ideal* (over \mathbb{N}) is a nonempty family $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ with $\mathbb{N} \notin \mathcal{I}$, which is closed under taking finite unions and subsets; that is $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$, and $B \subseteq A \in \mathcal{I} \Rightarrow B \in \mathcal{I}$.

We say that an ideal $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is *translation invariant* if $A \in \mathcal{I} \Leftrightarrow A + k \in \mathcal{I}$ for every $k \in \mathbb{Z}$.

The first easy example of a translation invariant ideal is given by the family of finite sets:

$$\mathbf{fin} := \{A \subseteq \mathbb{N} \mid A \text{ is finite}\}.$$

Another relevant example of translation invariant ideal is given by the family of those sequences whose series of reciprocals converge:

$$\mathbf{rcp} := \left\{ A \subseteq \mathbb{N} \mid \sum_{a \in A} 1/a < \infty \right\}.$$

Recall the following general notion (*e.g.*, see [3]). For every non-increasing function $f : \mathbb{N} \rightarrow [0, \infty)$ with $\sum_{n=1}^{\infty} f(n) = \infty$, the family

$$\mathcal{I}_f = \left\{ A \subseteq \mathbb{N} \mid \sum_{n \in A} f(n) < \infty \right\}$$

is the *summable ideal* determined by f . It is easily verified that such a family \mathcal{I}_f is indeed a translation invariant ideal that includes all finite sets.

Notice that both $\mathbf{fin} = \mathcal{I}_f$ and $\mathbf{rcp} = \mathcal{I}_g$ are summable ideals, where f is the constant function with value 1, and $g(n) = 1/n$ is the “reciprocal” function, respectively.

Abstract densities and ideals are closely related notions; indeed, as one can easily verify, the family of *zero sets*

$$\mathcal{Z}_\delta := \{A \subseteq \mathbb{N} \mid \delta(A) = 0\}$$

³ The name “Darboux property” has been used in the literature because it resembles the intermediate value property of derivatives, as stated in Darboux’s theorem in real analysis (see the remarks in [8, §2]).

of any (translation invariant) abstract upper density δ is a (translation invariant) ideal over \mathbb{N} .

The following question was posed by G. Grekos at the open problem session of the Workshop “*Densities and their applications*”, held in St. Etienne in July 2013:

Question. (G. Grekos) Given an ideal \mathcal{I} on \mathbb{N} , for example $\mathcal{I} = \text{fin}$ or $\mathcal{I} = \text{rcp}$, does there exist a “nice density” δ such that $\mathcal{Z}_\delta = \mathcal{I}$?

In this section we present the main results that we obtained in this paper to address the above question. Proofs of theorems will be given in the next section.

Two sets $A, B \subseteq \mathbb{N}$ are called \mathcal{I} -almost disjoint (\mathcal{I} -AD for short) if $A \cap B \in \mathcal{I}$; and are called \mathcal{I} -translation almost disjoint (\mathcal{I} -TAD for short) if for every $s, t \in \mathbb{Z}$, the translates $A + s, B + t$ are \mathcal{I} -almost disjoint. When \mathcal{I} is translation invariant, the latter condition is equivalent to having $(A + k) \cap B \in \mathcal{I}$ for every $k \in \mathbb{Z}$.

The above notions are extended to families of sets in a natural way: A family of infinite sets $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ is \mathcal{I} -AD (or \mathcal{I} -TAD) if every pair A, B of distinct elements of \mathcal{A} are \mathcal{I} -AD (or \mathcal{I} -TAD, respectively). In order to make definitions more meaningful, it is also assumed that such families \mathcal{A} do not contain any member of \mathcal{I} .

Following the usual terminology, we will simply say “almost disjoint” and “translation almost disjoint” to mean fin -AD and fin -TAD, respectively.

Theorem 2.1 *Let \mathcal{I}_f be a summable ideal determined by f and assume that there exists an infinite \mathcal{I}_f -TAD family. Then there exists an abstract upper density δ that is translation invariant and rich, and such that $\mathcal{Z}_\delta = \mathcal{I}_f$.*

Theorem 2.2 *Let \mathcal{I}_f be the summable ideal determined by f . Then there exists an \mathcal{I}_f -TAD family \mathcal{A} of the cardinality of the continuum.*

Corollary 2.3 *There exist abstract upper densities δ_1 and δ_2 such that $\mathcal{Z}_{\delta_1} = \text{fin}$ and $\mathcal{Z}_{\delta_2} = \text{rcp}$, respectively, that are translation invariant and rich.*

On the negative side, one cannot ask for the abstract densities of Corollary 2.3 to be also atomless, and even more so, to satisfy Darboux property.

We say that A is \mathcal{I} -almost included in B , and we write $A \subseteq_{\mathcal{I}} B$, if $A \setminus B \in \mathcal{I}$. An ideal \mathcal{I} has the *diagonal intersection property* (DIP for short) if for all sequences $\langle B_n \mid n \in \mathbb{N} \rangle$ where $B_n \notin \mathcal{I}$ and $B_{n+1} \subseteq_{\mathcal{I}} B_n$ for every n , there exists a set $A \notin \mathcal{I}$ such that $A \subseteq_{\mathcal{I}} B_n$ for all n .

Proposition 2.4 *If \mathcal{I}_f is a summable ideal determined by a non-increasing function f , then \mathcal{I}_f satisfies the DIP.*

PROOF. Suppose that $\langle B_n \mid n \in \mathbb{N} \rangle$ is a sequence of sets of natural numbers where $B_n \notin \mathcal{I}_f$ and $B_{n+1} \setminus B_n \in \mathcal{I}_f$ for all n . Let $a_1 = \min B_1$ and, proceeding by induction, assume that elements $a_1 < a_2 < \dots < a_{n(s)}$ have been found such that $a_1, a_2, \dots, a_{n(s)} \in \bigcap_{i=1}^t B_i$ and $\sum_{i=1}^{n(s)} f(a_i) \geq t$ for each $t \leq s$. Since $\bigcap_{i=1}^{s+1} B_i \notin \mathcal{I}_f$, there exist elements $a_{n(s)+1} < \dots < a_{n(s+1)}$ with $a_{n(s)+1} > a_{n(s)}$ such that $a_{n(s)+1}, \dots, a_{n(s+1)} \in \bigcap_{i=1}^{s+1} B_i$ and $\sum_{i=n(s)+1}^{n(s+1)} f(a_i) \geq 1$. Let $A = \{a_i \mid i \in \mathbb{N}\}$. Clearly, $A \notin \mathcal{I}_f$ because $\sum_{i=1}^{n(s)} f(a_i) \geq s$ for all s implies that $\sum_{a \in A} f(a) = \infty$; besides, for every s we have that $A \setminus B_s \subseteq \{a_i \mid i = 1, \dots, n(s)\} \in \text{fin} \subset \mathcal{I}_f$. \square

Corollary 2.5 *Both the ideal fin and the ideal rcp satisfy the DIP.*

Theorem 2.6 *If an abstract upper density δ is atomless then the ideal \mathcal{Z}_δ of its zero sets does not satisfy the DIP.*

Corollary 2.7 *If δ is an atomless abstract upper density, then $\mathcal{Z}_\delta \neq \text{fin}$ and $\mathcal{Z}_\delta \neq \text{rcp}$.*

3 The proofs

PROOF OF THEOREM 2.1. Let $\mathcal{I} = \mathcal{I}_f$ and \mathcal{A}_0 be an infinite \mathcal{I} -TAD family. By a direct application of *Zorn's Lemma*, we can pick a maximal \mathcal{I} -TAD family $\mathcal{A} \supseteq \mathcal{A}_0$. Enumerate its elements $\mathcal{A} = \{A_\alpha \mid \alpha < \mu\}$, where $\mu = |\mathcal{A}|$ is an infinite cardinal. Notice that for every $m \in \mathbb{N}$ and for every $k \in \mathbb{Z}$, the translation invariance of \mathcal{I} guarantees that $(A + k) \cap B \in \mathcal{I}$ if and only if $(A + k + m) \cap (B + m) \in \mathcal{I}$; in consequence, A and B are \mathcal{I} -TAD if and only if A and $B + m$ are \mathcal{I} -TAD. Notice also that, by maximality, for every $B \notin \mathcal{I}$ there must be $A_\alpha \in \mathcal{A}$ such that A_α and B are not \mathcal{I} -TAD.

We are now ready to construct the desired abstract upper density δ . For $n \in \mathbb{N}$ and $B \subseteq \mathbb{N}$, we set $\delta_n(B) := 0$ if A_n and B are \mathcal{I} -TAD; otherwise we set

$$\delta_n(B) := \frac{1}{n+1} + \frac{1}{n(n+1)} \cdot \sup_{k \in \mathbb{Z}} (\bar{d}_{f, A_n+k}(B)) \in \left[\frac{1}{n+1}, \frac{1}{n} \right].$$

In case $\mu > \aleph_0$, for infinite ordinals $\alpha < \mu$ we set $\delta_\alpha(B) = 0$ if A_α and B are \mathcal{I} -TAD; otherwise we set $\delta_\alpha(B) = 1$. Finally, we define $\delta : \mathcal{P}(\mathbb{N}) \rightarrow [0, 1]$ by letting:

$$\delta(B) := \sup_{\alpha < \mu} \delta_\alpha(B).$$

Let us now verify that δ satisfies the required properties. Notice first that $\delta(B) = 0$ for every $B \in \mathcal{I}$. Indeed, for every α , it directly follows from the definition of δ_α that $\delta_\alpha(B) = 0$ whenever $B \in \mathcal{I}$.

All pairs A_α and \mathbb{N} are not \mathcal{I} -TAD, since $(A_\alpha + k) \cap \mathbb{N} = A_\alpha + k \notin \mathcal{I}$ for every $k \in \mathbb{Z}$. (Here we used the facts that $\mathcal{A} \cap \mathcal{I} = \emptyset$ and that \mathcal{I} is translation invariant.) Now, trivially $\bar{d}_{f, A_1}(\mathbb{N}) = 1$, and so $\delta_1(\mathbb{N}) = 1/2 + 1/2 \cdot 1 = 1$, and hence $\delta(\mathbb{N}) = 1$.

If $B \subseteq B'$, then it is readily verified that $\delta_\alpha(B) \leq \delta_\alpha(B')$ for every α . By passing those inequalities to the limit superior as $\alpha < \mu$, we obtain the monotonicity property $\delta(B) \leq \delta(B')$.

Next, we show that for every α and for every $B, C \subseteq \mathbb{N}$, one has the inequality $\delta_\alpha(B \cup C) \leq \delta_\alpha(B) + \delta_\alpha(C)$. Clearly, this will prove the subadditivity of δ , because

$$\begin{aligned} \delta(B \cup C) &= \sup_{\alpha < \mu} \delta_\alpha(B \cup C) \leq \sup_{\alpha < \mu} (\delta_\alpha(B) + \delta_\alpha(C)) \\ &\leq \sup_{\alpha < \mu} \delta_\alpha(B) + \sup_{\alpha < \mu} \delta_\alpha(C) = \delta(B) + \delta(C). \end{aligned}$$

If $n \in \mathbb{N}$, the desired inequality for δ_n follows directly from the definition. Indeed,

$$\delta_n(B \cup C) \leq \frac{1}{n} \leq \frac{1}{n+1} + \frac{1}{n+1} \leq \delta_n(B) + \delta_n(C)$$

if $B, C \notin \mathcal{I}$. When α is infinite, δ_α only assumes the values 0 and 1. So, if by contradiction $\delta(B \cup C) > \delta_\alpha(B) + \delta_\alpha(C)$, then it must be $\delta_\alpha(B \cup C) = 1$ and $\delta_\alpha(B) = \delta_\alpha(C) = 0$. But this is impossible because if both pairs A_α, B and A_α, C are \mathcal{I} -TAD, then also the pair $A_\alpha, B \cup C$ is \mathcal{I} -TAD.

The density δ is translation invariant because for every α , for every $B \subseteq \mathbb{N}$, and for every $m \in \mathbb{N}$, we have $\delta_\alpha(B + m) = \delta_\alpha(B)$. Indeed, if $n \in \mathbb{N}$ then for every $k \in \mathbb{Z}$ one has $\bar{d}_{f, A_n+k}(B) = \bar{d}_{f, A_n+k+m}(B + m)$, so $\sup_{k \in \mathbb{Z}} (\bar{d}_{f, A_n+k}(B)) = \sup_{h \in \mathbb{Z}} (\bar{d}_{f, A_n+h}(B + m))$, and hence $\delta_n(B) = \delta_n(B + m)$. If α is infinite, $\delta_\alpha(B) = \delta_\alpha(B + m)$ because A_α and B are \mathcal{I} -TAD if and only if A_α and $B + m$ are \mathcal{I} -TAD. Indeed, by the translation invariance of \mathcal{I} , for every $k \in \mathbb{Z}$ one has that $(A+k) \cap B \in \mathcal{I}$ if and only if $((A+k) \cap B) + m = (A+k+m) \cap (B+m) \in \mathcal{I}$.

Let us now turn to the richness property. Clearly, $\delta(B) = 0$ whenever $B \in \mathcal{I}$. Given $r \in (0, 1]$, pick $n_0 \in \mathbb{N}$ and $\lambda \in [0, 1]$ such that $r = \frac{1}{n_0+1} + \frac{1}{n_0(n_0+1)} \cdot \lambda$. Since $\bar{d}_{f, A_{n_0}}$ is rich, there exists a $B \subseteq A_{n_0}$ not in \mathcal{I} such that $\bar{d}_{f, A_{n_0}}(B) = \lambda$. For every $k \in \mathbb{Z}$ one has

$$\begin{aligned} \frac{\sum_{m \in (A_{n_0}+k) \cap B, m < n} f(m)}{\sum_{m \in (A_{n_0}+k), m < n} f(m)} &\leq \frac{\sum_{m \in B, m < n} f(m)}{(\sum_{m \in A_{n_0}, m < n} f(m)) - |k|} \\ &= \frac{\sum_{m \in B, m < n} f(m)}{(\sum_{m \in A_{n_0}, m < n} f(m)) - |k|} \cdot \frac{(\sum_{m \in A_{n_0}, m < n} f(m))}{(\sum_{m \in A_{n_0}, m < n} f(m)) - |k|}. \end{aligned}$$

By passing to the limit superiors as m goes to infinity, we get $\bar{d}_{f, A_{n_0}+k}(B) \leq \bar{d}_{f, A_{n_0}}(B)$, and hence $\sup_{k \in \mathbb{Z}} \bar{d}_{f, A_{n_0}+k}(B) = \bar{d}_{f, A_{n_0}}(B) = \lambda$. For $\alpha \neq n_0$, the sets A_α and A_{n_0} are

\mathcal{I} -TAD, and hence also the sets A_α and B are \mathcal{I} -TAD. Then $\delta_\alpha(B) = 0$ for $\alpha \neq n_0$, and so $\delta(B) = \delta_{n_0}(B) = r$, as desired.

We have noticed already that if $B \in \mathcal{I}$ then $\delta(B) = 0$, so let us assume that $B \notin \mathcal{I}$. By the maximality of the \mathcal{I} -TAD family $\{A_\alpha \mid \alpha < \mu\}$, there exists α such that A_α and B are not \mathcal{I} -TAD. If such an $\alpha = n$ is finite, then $\delta(B) \geq \delta_n(B) \geq \frac{1}{n+1} > 0$; and if such an α is infinite, then $\delta(B) = \delta_\alpha(B) = 1$. This shows that $\mathcal{Z}_\delta = \mathcal{I}$. \square

PROOF OF THEOREM 2.2. We prove a lemma first.

Lemma 3.1 *Let f be a non-increasing function, and let \mathcal{I}_f be the summable ideal determined by f . Then there exist increasing functions $g, h : \mathbb{N} \rightarrow \mathbb{N}$ such that*

1. The sequences $\{g(n) - g(n-1)\}_{n \in \mathbb{N}}$ and $\{h(n) - h(n-1)\}_{n \in \mathbb{N}}$ are non-decreasing,
2. $\lim_{n \rightarrow \infty} (g(n) - g(n-1)) = \infty$,
3. $\lim_{n \rightarrow \infty} (h(n) - h(n-1)) = \infty$,
4. $\sum_{n=1}^{\infty} f(g(h(n))) = \infty$.

Proof of Lemma. We first define by induction a function $g(n)$ and a sequence $l_0 < l_1 < l_2 < \dots$ so that $g(l_m) - g(l_m - 1) \geq 2^m$. Set $l_0 = 0$ and $g(0) = 0$. Suppose we have found $g(0) < g(1) < \dots < g(l_m)$ such that

- $\{g(i) - g(i-1)\}_{i \leq l_m}$ is non-decreasing,
- $g(l_m) - g(l_m - 1) \geq 2^m$, and
- $\sum_{i=0}^{l_m} f(g(i)) \geq m$.

We now define l_{m+1} and $g(i)$ for $i = l_m + 1, l_m + 2, \dots, l_{m+1}$.

Let $d = g(l_m) - g(l_m - 1)$. Since $\sum_{i > g(l_m)} f(i) = \infty$, there exists an $i_0 < 2d$ such that $\sum_{k=2}^{\infty} f(g(l_m) + i_0 + 2dk) = \infty$ because $[g(l_m), \infty)_{\mathbb{N}}$ is the disjoint union of $2d$ arithmetic sequences of common difference $2d$. Let l_{m+1} be sufficiently large such that $\sum_{k=2}^{l_{m+1}} f(g(l_m) + i_0 + 2dk) \geq 1$. We set

- $g(l_m + 1) = g(l_m) + i_0$ and $g(l_m + 2) = g(l_m) + i_0 + 2d$ if $i_0 \geq d$ or
- $g(l_m + 1) = g(l_m) + d$ and $g(l_m + 2) = g(l_m) + i_0 + 2d$ if $i_0 < d$.

We also set $g(l_m + j + 1) = g(l_m + j) + 2d$ for $j = 2, 3, \dots, l_{m+1}$. It is easy to check that

- $\{g(i) - g(i-1)\}_{i \leq l_{m+1}}$ is non-decreasing,
- $g(l_{m+1}) - g(l_{m+1} - 1) \geq 2d \geq 2^{m+1}$, and

- $\sum_{i=0}^{l_m+1} f(g(i)) \geq m + 1$.

Notice that the purpose of choosing the value of $g(l_m + 1)$ to be $g(l_m) + i_0$ or to be $g(l_m) + d$ is to guarantee that the sequence $\{g(i) - g(i - 1)\}_{i \leq l_m+1}$ be non-decreasing.

Clearly, $\lim_{n \rightarrow \infty} (g(n) - g(n - 1)) = \infty$ because $g(l_m) - g(l_m - 1) \geq 2^m$.

Now we define the function $h(n)$ exactly as above, by replacing $f(n)$ with $f(g(n))$. This completes the proof of Lemma 3.1 \square

Let us go back to the proof of Theorem 2.2.

Assume that the functions g, h are constructed as in Lemma 3.1. By (3) we can pick a sequence $n_1 < n_2 < \dots$ such that $h(n_m) - h(n_m - 1) \geq 2^m$ for every positive integer m . Let $2^{\mathbb{N}} := \{\sigma \mid \sigma : \mathbb{N} \rightarrow \{0, 1\}\}$ and denote $2^{[n]} := \{\sigma \mid \sigma : [1, n]_{\mathbb{N}} \rightarrow \{0, 1\}\}$. Notice that $|2^{[n]}| = 2^n$. We construct a collection of sets $\mathcal{A} = \{A_\sigma \subseteq G \mid \sigma \in 2^{\mathbb{N}}\}$ where $G = \text{range}(g)$ such that for any distinct $\sigma, \tau \in 2^{\mathbb{N}}$,

- $|A_\sigma \cap [g(h(n - 1)), g(h(n)) - 1]| = 1$ for all $n \in \mathbb{N}$,
- $A_\sigma \cap A_\tau \subseteq [0, g(h(n_m - 1))]$ where $m = \min\{n \mid \sigma(n) \neq \tau(n)\}$.

The theorem follows from the construction of \mathcal{A} : Clearly, \mathcal{A} is an AD of the cardinality of the continuum. Since n -th element of A_σ is between $g(h(n - 1))$ and $g(h(n)) - 1$, we have that $\sum_{a \in A_\sigma} f(a) \geq \sum_{n=2}^{\infty} f(g(h(n))) = \infty$, which implies $A_\sigma \notin \mathcal{I}_f$. If $x \in G \cap (G + k)$, then $x, x - k \in G$, and so $G \cap (G + k)$ must be finite because $\lim_{n \rightarrow \infty} (g(n) - g(n - 1)) = \infty$. Therefore, G and $G + k$ are almost disjoint for any non-zero $k \in \mathbb{Z}$. As a consequence, $A \cap (B + k)$ is a finite set for any distinct pair $A, B \in \mathcal{A}$ and any $k \in \mathbb{Z}$. Thus \mathcal{A} is a TAD.

We now construct \mathcal{A} . For any positive integer m and $\sigma \in 2^{\mathbb{N}}$, let $\sigma \upharpoonright m$ represent the restriction of the function σ on $[1, m]_{\mathbb{N}}$, let $A_{\sigma \upharpoonright m} := A_\sigma \cap [0, g(h(n_m)) - 1]$, and let $\mathcal{A} \upharpoonright m := \{A_{\sigma \upharpoonright m} \mid \sigma \in 2^{\mathbb{N}}\}$. We construct \mathcal{A} by defining $\mathcal{A} \upharpoonright m$ inductively on m .

Let $A_{\sigma \upharpoonright 1} = \{0\}$. Now suppose we have obtained $\mathcal{A} \upharpoonright m$. Since

$$\{h(n) - h(n - 1)\}_{n \in \mathbb{N}}$$

is non-decreasing, we have that $h(n_m + i) - h(n_m - 1 + i) \geq h(n_m) - h(n_m - 1) \geq 2^m$ for $0 \leq i < n_{m+1} - n_m$. Thus there are at least 2^m -many distinct elements in each of the sets

$$\{G \cap [g(h(n_m - 1 + i)), g(h(n_m + i)) - 1] \mid 0 \leq i < n_{m+1} - n_m\}.$$

Let $\{a_{i,\gamma} \mid \gamma \in 2^{[m]}\}$ be an enumeration of a set of cardinality 2^m in

$$G \cap [g(h(n_m - 1 + i)), g(h(n_m + i)) - 1]$$

for $i = 1, 2, \dots, n_{m+1} - n_m - 1$ and $A'_{\sigma \upharpoonright m} = A_{\sigma \upharpoonright m} \cup \{a_{i,\sigma \upharpoonright m} \mid i = 1, 2, \dots, n_{m+1} - n_m - 1\}$. It is easy to see that

- $A_{\sigma \upharpoonright m} = A'_{\sigma \upharpoonright m} \cap [0, g(h(n_m)) - 1]$;
- $A'_{\sigma \upharpoonright m} \cap A'_{\tau \upharpoonright m} = A_{\sigma \upharpoonright m} \cap A_{\tau \upharpoonright m}$ for any $\sigma \upharpoonright m \neq \tau \upharpoonright m$;
- $A'_{\sigma \upharpoonright m} \cap [g(h(n_m - 1 + i)), g(h(n_m + i)) - 1] = \{a_{i, \sigma \upharpoonright m}\}$ for $0 < i < n_{m+1} - n_m$.

Since $h(n_{m+1}) - h(n_{m+1} - 1) \geq 2^{m+1}$, we can label 2^{m+1} distinct elements in $G \cap [g(h(n_{m+1} - 1)), g(h(n_{m+1})) - 1]$ by $\{b_\gamma \mid \gamma \in 2^{[m+1]}\}$. Now let $A_{\sigma \upharpoonright m+1} = A'_{\sigma \upharpoonright m} \cup \{b_\gamma\}$ where $\gamma = \sigma \upharpoonright (m+1)$. Notice that each set $A'_{\sigma \upharpoonright m}$ has two different extensions in $\mathcal{A} \upharpoonright (m+1)$ depending on the value of $\sigma(m+1)$.

This completes the construction of $\mathcal{A} \upharpoonright (m+1)$. Now let $A_\sigma = \bigcup_{m=1}^{\infty} A_{\sigma \upharpoonright m}$. The set $\mathcal{A} = \{A_\sigma \mid \sigma \in 2^{\mathbb{N}}\}$ is the desired AD family. \square

If $\mathcal{I} = \text{fin}$ or $\mathcal{I} = \text{rcp}$, the constructions of g and h in Lemma 3.1 can be simplified. In fact g and h can be the same function.

It is readily verified that the sequence $g(n) = h(n) = n^2$ satisfies the required properties for the summable ideal $\text{fin} = \mathcal{I}_f$, where f is the constant function with value 1.

For the summable ideal $\text{rcp} = \mathcal{I}_f$ where $f(n) = 1/n$ let $\varphi(n) = \lfloor n \cdot \log \log n \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part. Since $\varphi(\varphi(n)) = o(n \log n)$ and $\sum_{n=2}^{\infty} \frac{1}{n \log n} = \infty$, we have that $\sum_{n=3}^{\infty} \frac{1}{\varphi(\varphi(n))} = \infty$. Moreover, it is easily checked that $\{\varphi(n+1) - \varphi(n)\}_{n \geq 16}$ is a non-decreasing and unbounded sequence of natural numbers. So, the previous theorem applies by taking $g(n) = h(n) = \varphi(n+15)$. \square

Remark 3.2 *We do not know whether g and h in Lemma 3.1 can be the same function for a generic summable ideal \mathcal{I}_f .*

PROOF OF THEOREM 2.6. For an upper abstract density δ , let us write $A \subseteq_{\delta} B$ if A is included in B up to a set of zero density, that is, when $\delta(A \setminus B) = 0$. Clearly, $A \subseteq_{\delta} B$ implies that $\delta(A) \leq \delta(B)$. The desired result directly follows from the following

Lemma 3.3 *Let δ be an upper abstract density. If δ is atomless then there exists a decreasing sequence $\langle B_n \mid n \in \mathbb{N} \rangle$ of sets of positive density such that the following holds: For every $A \subseteq \mathbb{N}$, if $A \subseteq_{\delta} B_n$ for all n , then $\delta(A) = 0$.*

Proof of Lemma. For each $X \subseteq \mathbb{N}$ with $\delta(X) > 0$, define

$$\gamma(X) = \inf\{\delta(B) \mid B \subseteq X \text{ and } \delta(B) > 0\}.$$

Since δ is atomless, $\gamma(X) < \delta(X)$. Moreover, γ is non-increasing, that is, if $Y \subseteq X$ and $\delta(Y) > 0$, then $\gamma(X) \leq \gamma(Y)$. Now let $B_1 = \mathbb{N}$. At the inductive step, let

$\eta_n = \delta(B_n) - \gamma(B_n) > 0$, and let $\varepsilon_n = \frac{1}{2}(\gamma(B_n) + \delta(B_n)) > \gamma(B_n)$. Then we can pick a subset $B_{n+1} \subset B_n$ such that $0 < \delta(B_{n+1}) \leq \varepsilon_n$. Since $\gamma(B_{n+1}) \geq \gamma(B_n)$, we have

$$\eta_{n+1} = \delta(B_{n+1}) - \gamma(B_{n+1}) \leq \varepsilon_n - \gamma(B_n) = \frac{\eta_n}{2}.$$

In consequence, $\lim_{n \rightarrow \infty} \eta_n = 0$, and so $\lim_{n \rightarrow \infty} \gamma(B_n) = \lim_{n \rightarrow \infty} \delta(B_n)$. If $A \subseteq_{\delta} B_n$ for all n then $\gamma(B_n) \leq \delta(A) \leq \delta(B_n)$ for all n , and hence $\delta(A) = \lim_{n \rightarrow \infty} \gamma(B_n) = \lim_{n \rightarrow \infty} \delta(B_n)$. If by contradiction $\delta(A) > 0$, we could pick $A' \subset A$ such that $0 < \delta(A') < \delta(A)$, and we would have $\delta(A') < \gamma(B_n)$ for all but finitely many n . This is not possible because $0 < \delta(A' \cap B_n) = \delta(A') < \gamma(B_n)$, against the definition of $\gamma(B_n)$. \square

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