

# Freiman's Inverse Problem with Small Doubling Property

## Long Version

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### Abstract

Let  $\mathbb{N}$  be the set of all non-negative integers,  $A \subseteq \mathbb{N}$  be a finite set, and  $2A$  be the set of all numbers of the form  $a + b$  for all  $a$  and  $b$  in  $A$ . In [Fr1] the arithmetic structure of  $A$  was optimally characterized when  $|2A| \leq 3|A| - 3$ .<sup>1</sup> In [Fr1] the structure of  $A$  was also characterized without proof when  $|2A| = 3|A| - 2$ . Since then the efforts have been made to generalize these results, see [LS, Le, HP] for example. However, no optimal characterization of the structure of  $A$  has been obtained without imposing extra conditions, until now, when  $|2A| > 3|A| - 2$ . In this paper we optimally characterize, with the help of nonstandard analysis, the arithmetic structure of  $A$  when  $|2A| = 3|A| - 3 + b$ , where  $b$  is positive but not too large. Precisely, we prove that there is a real number  $\epsilon > 0$  and there is  $K \in \mathbb{N}$  such that if  $|A| > K$  and  $|2A| = 3|A| - 3 + b$  for  $0 \leq b \leq \epsilon|A|$ , then  $A$  is either a subset of an arithmetic progression of length at most  $2|A| - 1 + 2b$  or a subset of a bi-arithmetic progression<sup>2</sup> of length at most  $|A| + b$ . An application of this result to the inverse problem for upper asymptotic density is presented near the end of the paper. In the application we improve the most important part of the main theorem in [Ji2].

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<sup>1</sup>One can easily prove that  $|2A| \geq 2|A| - 1$  is always true and  $|2A| = 2|A| - 1$  implies that  $A$  is an arithmetic progression

<sup>2</sup>The definition of bi-arithmetic progression can be found in the beginning of Introduction.

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## 1 Introduction

Inverse problems study the structural properties of the sets  $A_i$  when the sum of the sets  $\sum_{i=1}^n A_i = \{\sum_{i=1}^k a_i : a_i \in A_i\}$  satisfies certain conditions. When  $A_i = A$  for every  $i$ , we write  $nA$  for  $\sum_{i=1}^n A_i$ . Note that the term  $nA$  should not be confused with the term  $n * A = \{an : a \in A\}$ , which will also be used later in this paper. For a number  $x$  we write  $x \pm A$  for the set  $\{x\} \pm A$  and write  $A \pm x$  for the set  $A \pm \{x\}$ . G. A. Freiman and many others have studied inverse problems for the addition of finite sets and have obtained many results showing that if  $A + B$  is small relative to the size of  $A$  and the size of  $B$ , then  $A$  and  $B$  must have some arithmetic structure (cf. [Na, DLY]). In this paper we consider the addition of two copies of the same finite set  $A$  of natural numbers. Let  $a, d, k \in \mathbb{N}$  with  $d, k \geq 1$ . A set of the form  $\{a, a + d, a + 2d, \dots, a + (k - 1)d\}$  is called an *arithmetic progression* of length  $k$  with difference  $d$ . A set of the form  $I \cup J$  is called a *bi-arithmetic progression* of length  $k$  with difference  $d$  if both  $I$  and  $J$  are arithmetic progressions of difference  $d$ ,  $|I| + |J| = k$ , and  $I + I$ ,  $I + J$ ,  $J + J$  are pairwise disjoint. We will write *a.p.* and *b.p.* as abbreviations for “arithmetic progression” and “bi-arithmetic progression”,

respectively. For two integers  $m, n$  the term  $[m, n]$  represents exclusively the interval of integers. For a set of integers  $A$ , we write  $A[m, n]$  for the set  $A \cap [m, n]$  and  $A(m, n)$  for the number  $|A[m, n]|$ . The reader needs to be able to distinguish  $2A(a, b)$ , which is 2 times the number  $A(a, b)$ , from  $(2A)(a, b)$ , which is the number of elements in the set  $(2A) \cap [a, b]$ .

Suppose  $|A| = k$ . It is well known that if  $|2A| = 2k - 1$ , then  $A$  must be an *a.p.* Note that it is always true that  $|2A| \geq 2k - 1$ . In [Fr1] Freiman obtained the interesting generalizations of these facts by showing that

- (i) if  $k > 3$  and  $|2A| = 2k - 1 + b < 3k - 3$ , then  $A$  is a subset of an *a.p.* of length at most  $k + b$ ;
- (ii) if  $k > 6$  and  $|2A| = 3k - 3$ , then either  $A$  is a subset of an *a.p.* of length at most  $2k - 1$  or  $A$  is a *b.p.*

Note that the characterizations of the structure of  $A$  above are optimal. In [Fr1] the structure of  $A$  was also characterized when  $|A| > 10$  and  $|2A| = 3k - 2$ . The proof of the  $3k - 3$  theorem above in [Fr1] was not short while the proof of the  $3k - 2$  theorem was omitted there because, commented by Freiman, it was too tedious<sup>3</sup>. There has been no further optimal characterization, until now, of the structure of  $A$  when, for example,  $|2A| = 3k - 1$  without imposing some extra conditions on  $A$ . On the other hand, Freiman made the following conjecture in [Fr3] as mentioned in [Le] and re-stated in [Fr2] a few years ago.

**Conjecture 1.1 (G. A. Freiman)** *There exists a natural number  $K$  such that for any finite set of natural numbers  $A$  with  $|A| = k > K$  and  $|2A| = 3k - 3 + b$  for  $0 \leq b < \frac{1}{3}k - 2$ ,  $A$  is either a subset of an *a.p.* of length at most  $2k - 1 + 2b$  or a subset of a *b.p.* of length at most  $k + b$ .*

Note that Conjecture 1.1 is clearly false if  $b = \frac{1}{3}k - 2$  as shown in the following easy example.

**Example 1.2** *Suppose  $k = 3a$  and  $c > 2k$ . Let  $A = [0, a - 1] \cup [c, c + a - 1] \cup [2c, 2c + a - 1]$ . Then  $|A| = k$  and  $|2A| = 3|A| - 3 + \frac{1}{3}k - 2$ . But  $A$  is neither a subset of an *a.p.* of length  $2k - 1 + 2(\frac{1}{3}k - 2)$  nor a subset of a *b.p.* of length  $k + (\frac{1}{3}k - 2)$ .*

<sup>3</sup>The conclusion of Freiman's  $3k - 2$  Theorem in [Fr1] seems missing at least one case. For example, if  $A = [0, k - 3] \cup \{4k, 4k + 2\}$ , then  $|2A| = 3k - 2$ . This case of  $A$  was not covered by the theorem.

It is easy to prove Freiman’s conjecture if one adds an extra condition that the set  $A$  is a subset of a *b.p.* We state this in Theorem 1.3 as a simple consequence of Theorem A.3<sup>4</sup>.

Let  $A$  and  $B$  be two subsets of two torsion–free groups, respectively. A bijection  $\phi : A \mapsto B$  is called a  *$F_2$ -isomorphism*<sup>5</sup> if for all  $a_1, a_2, a_3, a_4 \in A$ ,  $a_1 + a_2 = a_3 + a_4$  if and only if  $\phi(a_1) + \phi(a_2) = \phi(a_3) + \phi(a_4)$ . A set

$$P = P(x_0; x_1, x_2; b_1, b_2) = \{x_0 + ix_1 + jx_2 : 0 \leq i < b_1 \text{ and } 0 \leq j < b_2\}$$

with  $b_1 \geq b_2 > 0$  is called a  *$F_2$ -progression* if the map  $\phi : [0, b_1 - 1] \times [0, b_2 - 1] \mapsto P$  with  $\phi(i, j) = x_0 + ix_1 + jx_2$  is  *$F_2$ -isomorphism*.  $P$  is said to have rank 2 if  $b_2 > 1$  and rank 1 if  $b_2 = 1$ .

**Theorem 1.3** *Suppose  $A$  is a subset of a *b.p.*  $I \cup J$  such that  $|A| = k > 10$  and both  $A \cap I$  and  $A \cap J$  are non-empty. If  $|2A| = 3k - 3 + b$  for  $0 \leq b < \frac{1}{3}k - 2$ , then  $I$  and  $J$  can be chosen so that  $|I| + |J| \leq k + b$ .*

**Proof:** Without loss of generality we can assume that  $I \cup J$  has the shortest length. Clearly,  $I \cup J$  is  *$F_2$ -isomorphic* to the set

$$M = \{(0, 0), (1, 0), \dots, (l_1 - 1, 0)\} \cup \{(0, 1), (1, 1), \dots, (l_2 - 1, 1)\} \quad (\text{I})$$

in  $\mathbb{Z}^2$  where  $l_1$  is the length of  $I$  and  $l_2$  is the length of  $J$ . Let  $\phi$  be the  *$F_2$ -isomorphism* from  $I \cup J$  to  $M$ . Then  $|\phi(A)| = k$  and  $|2\phi(A)| = |2A| = 3|A| = 3k - 3 + b < \frac{10}{3}k - 5$ . By Theorem A.3 we have that  $l_1 + l_2 \leq k + b$ . Hence  $A$  is a subset of a *b.p.* of length at most  $k + b$ .  $\square$ (Theorem 1.3)

It is worth mentioning that another interesting generalization of Freiman’s  $3k - 3$  Theorem in [HP], where the condition  $|2A| = 3k - 3$  is replaced by  $|A + t * A| = 3k - 3$  for an integer  $t$ . The most interesting case of this generalization is when  $t = -1$ . However, this generalization does not concern the case when  $|2A| = 3k - 3 + b$  with  $b > 0$ .

The author recently received an unpublished manuscript by V. Lev in which he proved Conjecture 1.1 for a set  $A$  with extra conditions that  $0 = \min A$ ,  $\gcd(A) = 1$ ,

<sup>4</sup>In the Appendix some existing theorems are included for quick references.

<sup>5</sup> $F$  is the initial of Freiman.

$p = \max A$  is a prime, and  $p \geq 35|A|$ . The idea of the proof is similar to the idea used in the proof of [Na, Theorem 2.11].

In the last couple of years the author developed some ideas with the help of nonstandard analysis in the research of the inverse problem for upper asymptotic density [Ji2] and found that these ideas can be applied to the case when  $|2A| = 3k - 3 + b$  with some relatively small  $b > 0$ . Furthermore, the main result obtained in this paper can actually improve the most important part of the main theorem back in [Ji2]. The following is the main result of this paper.

**Theorem 1.4** *There exists a positive real number  $\epsilon$  and a natural number  $K$  such that for every finite set of natural numbers  $A$  with  $|A| = k$ , if  $k > K$  and  $|2A| = 3k - 3 + b$  where  $0 \leq b \leq \epsilon k$ , then  $A$  is either a subset of an a.p. of length at most  $2k - 1 + 2b$  or a subset of a b.p. of length at most  $k + b$ .*

Theorem 1.4 gives an affirmative answer to Conjecture 1.1 when  $0 \leq b \leq \epsilon|A|$ . Note that Theorem 1.4 gives a new result even when  $b = 2$ . Note also that the upper bound  $2k - 1 + 2b$  of the length of the a.p. and the upper bound  $k + b$  of the length of the b.p. in Theorem 1.4 are optimal as shown in the following two easy examples.

**Example 1.5** *For  $k > 15$  let  $A = [0, k - 3] \cup \{k + 10, 2k + 20\}$ . Then  $|A| = k$  and  $|2A| = 3k - 3 + 11$ . The shortest a.p. containing  $A$  has length  $2k - 1 + 2 \times 11$  and  $A$  is not a subset of a b.p. of length  $k + 11$ .*

**Example 1.6** *For  $k > 14$  let  $A = [0, k - 3] \cup \{3k, 3k + 12\}$ . Then  $|A| = k$  and  $|2A| = 3k - 3 + 11$ . The shortest b.p. containing  $A$  has length  $k + 11$  and  $A$  is not a subset of an a.p. of length  $2k - 1 + 2 \times 11$ .*

The proofs in this paper use methods from nonstandard analysis. The reader is assumed to have some basic knowledge of nonstandard analysis such as the existence of infinitesimals, differences among standard sets, internal sets, and external sets, the transfer principle, countable saturation, etc. For details we recommend the reader to consult [Li], [He], [Ji1], or other introductory nonstandard analysis textbooks.

Notations involved in nonstandard methods need to be introduced. Some of these notations may not be common in other literature. We work within a fixed countably saturated nonstandard universe  ${}^*V$ . For each standard set  $A \subseteq \mathbb{N}$ , we write  ${}^*A$  for the nonstandard version of  $A$  in  ${}^*V$ . For example,  ${}^*\mathbb{N}$  is the set of all natural numbers in

${}^*\mathbb{V}$ , and if  $A$  is the set of all even numbers in  $\mathbb{N}$ , then  ${}^*A$  is the set of all even numbers in  ${}^*\mathbb{N}$ . If we do not specify that  $A, B, C, D$  are sets of standard natural numbers, then they are always assumed to be *internal* subsets of  ${}^*\mathbb{N}$ . The lower case letters are used for integers. The integers in  ${}^*\mathbb{N} \setminus \mathbb{N}$  are called hyperfinite integers. From now on, the letters  $H$  and  $N$  are exclusively reserved for hyperfinite integers. The Greek letters  $\alpha, \beta, \gamma, \delta$ , and  $\epsilon$  are reserved exclusively for standard real numbers.

For the convenience of handling nonstandard arguments, we introduce some notations of comparison (quasi-order). For real numbers  $r, s$  in  ${}^*\mathbb{R}$ , by  $r \approx s$  we mean that  $r - s$  is an infinitesimal, i.e. the absolute value of  $r - s$  is less than any standard positive real numbers; by  $r \ll s$  ( $r \gg s$ ) we mean  $r < s$  ( $r > s$ ) and  $r \not\approx s$ ; by  $r \lesssim s$  ( $r \gtrsim s$ ) we mean  $r < s$  ( $r > s$ ) or  $r \approx s$ . Given a hyperfinite integer  $H$  and two real numbers  $r, s$ , by  $r \sim_H s$  we mean  $\frac{s-r}{H} \approx 0$ ; by  $r \prec_H s$  ( $r \succ_H s$ ) we mean  $r < s$  ( $r > s$ ) and  $r \not\sim_H s$ ; by  $r \preceq_H s$  ( $r \succeq_H s$ ) we mean  $r \prec_H s$  ( $r \succ_H s$ ) or  $r \sim_H s$ . It is often said that a quantity  $a$  is insignificant with respect to  $H$  if  $a \sim_H 0$ . When using  $\sim, \prec, \preceq$ , etc. insignificant quantities can often be neglected. For example, instead of writing  $A(0, H) \sim_H \alpha(H + 1)$ , we can write its equivalent form  $A(0, H) \sim_H \alpha H$ . For another example, when  $a \leq b \leq c$ , we often write  $A(a, c) \sim_H A(a, b) + A(b, c)$  instead of  $A(a, c) = A(a, b) + A(b + 1, c)$ . We will omit the subscript  $H$  when it is clearly given. For a real number  $r \in {}^*\mathbb{R}$  bounded by a standard real number, let  $st(r)$ , the *standard part of  $r$* , be the unique standard real number  $\alpha$  such that  $r \approx \alpha$ . Note that  $\approx, \ll, \lesssim, \sim_H, \prec_H$ , and  $\preceq_H$  are external relations. If  $A \subseteq [0, H]$  is a hyperfinite set with  $a = \min A$  and  $b = \max A$ , then  $A$  is said to be *full* (in  $I$ ) if  $A$  is a subset of an *a.p.  $I$*  such that  $|A| \sim I(a, b)$ . We say that  $A$  is *full in a b.p.  $I_0 \cup I_1$*  if  $A \subseteq I_0 \cup I_1$  and  $|A| \sim I_0(l_0, u_0) + I_1(l_1, u_1)$  where  $u_i = \max(A \cap I_i)$  and  $l_i = \min(A \cap I_i)$  for  $i = 0, 1$ . Note that if  $A \subseteq [0, H]$  be a subset of an *a.p.  $I$*  and  $|I| \sim 0$ , then  $A$  is trivially full according to the definition. We always assume that  $A \cap I_0$  and  $A \cap I_1$  are non-empty when we say that  $A$  is a subset of the *b.p.  $I_0 \cup I_1$* .

In order to apply nonstandard methods, we need to translate Theorem 1.4 into the following nonstandard version. Then we prove the nonstandard version in the rest of the paper.

**Theorem 1.7** *Let  $H$  be a hyperfinite integer and  $A \subseteq [0, H]$  be an internal set. Suppose  $0 = \min A$ ,  $H = \max A$ ,  $|A| \succ 0$ ,  $\gcd(A) = 1$ , and  $|2A| = 3|A| - 3 + b$  for  $0 \leq b \sim 0$ . Then either  $H + 1 \leq 2|A| - 1 + 2b$  or  $A$  is a subset of a b.p. of length at*

most  $|A| + b$ .

Note that in the theorem above we express the statement ” $A$  is a subset of an *a.p.* of length at most  $2|A| - 1 + 2b$ ” by ” $H + 1 \leq 2|A| - 1 + 2b$ ” because we have assumed that  $0 = \min A$  and  $\gcd(A) = 1$ . The new expression will give us some notational convenience.

**Proof of Theorem 1.4 from Theorem 1.7:** Suppose Theorem 1.4 is not true. By negating the statement of Theorem 1.4 we have that with  $\epsilon_k = \frac{1}{k}$  for each  $k \in \mathbb{N}$ , there is a finite set  $A_k \subseteq [0, h_k]$  satisfying the following:  $0 = \min A_k$ ,  $h_k = \max A_k$ ,  $|A_k| > k$ ,  $\gcd(A_k) = 1$ ,  $|2A_k| = 3|A_k| - 3 + b_k$  where  $0 \leq b_k \leq \frac{|A_k|}{k}$ ,  $h_k + 1 > 2|A_k| - 1 + 2b_k$ , and  $A_k$  is not a subset of a *b.p.* of length at most  $|A_k| + b_k$ .

Let  $K$  be a hyperfinite integer and let  $A = A_K$  be the term in the internal sequence  $\langle A_k : k \in {}^*\mathbb{N} \rangle$ . Then we have the following:  $0 = \min A$ ,  $H = h_K = \max A$ ,  $|A| > K$ ,  $\gcd(A) = 1$ ,  $|2A| = 3|A| - 3 + b$  for some  $b \geq 0$  with  $\frac{b}{|A|} \leq \frac{1}{K} \approx 0$ ,  $H + 1 > 2|A| - 1 + 2b$ , and  $A$  is not a subset of a *b.p.* of length at most  $|A| + b$ . If, in addition, we have  $\frac{|A|}{H} \gg 0$ , then the set  $A$  contradicts Theorem 1.7. Hence it suffices to prove  $\frac{|A|}{H} \gg 0$  or equivalently  $|A| \succ 0$ .

Suppose  $|A| \sim 0$ . By Theorem A.5 the set  $A$  is a subset of a  $F_2$ -progression  $P = P(x_0; x_1, x_2; b_1, b_2)$  such that  $\frac{|A|}{|P|} \gg 0$ . If  $P$  has rank 1, then  $P$  is an *a.p.* Since  $\gcd(A) = 1$ , then  $[0, H] \subseteq P$ . This contradicts  $|A| \sim 0$ . Hence we can assume that  $P$  has rank 2. Let  $\phi : P \mapsto [0, b_1 - 1] \times [0, b_2 - 1]$  be a  $F_2$ -isomorphism and  $B = \phi(A)$ . Then  $B$  is not a subset of a straight line. Since  $B$  is a  $F_2$ -isomorphic image of  $A$ , we have  $|2B| = |2A|$ . Hence by Theorem A.3,  $B$  is  $F_2$ -isomorphic to a subset of  $M$  in (I) such that  $l_1 + l_2 \leq |B| + b$ . This shows that  $A$  is a subset of a *b.p.* of length at most  $|A| + b$ , which contradicts the assumption that  $A$  is not a subset of a *b.p.* of length at most  $|A| + b$ .  $\square$ (Theorem 1.4)

The approach of eliminating the possibility of  $|A| \sim 0$  in the proof above is from [Bo]. In fact the same approach can be used to prove that there exists a small positive number  $\delta$  such that Conjecture 1.1 is true when an extra condition  $|A| < \delta(\max A - \min A)$  is added. It is possible but much more tedious to prove  $|A| \succ 0$  in the proof above directly without citing Theorem A.5.

We prove Theorem 1.7 in the next several sections. The proof is done in two steps. In the first step we deal with the case when  $A \subseteq [0, H]$  contains significantly less than half of the elements in  $[0, H]$ . In the second step we deal with the case

when  $A \subseteq [0, H]$  contains roughly half of the elements in  $[0, H]$ . The main theorem in each step is preceded by a list of lemmas, which prove the theorem under various circumstances. Before these two steps we present a list of general lemmas. In this paper, theorems, lemmas, cases, and claims are numbered in such a way that the reader should be able to see how they are nested.

## 2 General Lemmas

In this section we state some lemmas from the author's previous papers without proof and state some other new lemmas with proof. The first lemma in this section will play an important role in the proof of Theorem 1.7. It uses a concept called cut from nonstandard analysis.

An infinite initial segment  $U$  of  ${}^*\mathbb{N}$  is called a *cut* if  $U + U \subseteq U$ . Clearly  $U = \mathbb{N}$  and  $U = {}^*\mathbb{N}$  are cuts. A cut  $U \neq {}^*\mathbb{N}$  is an additive semigroup and is external because it has no maximum element. For example,  $\mathbb{N}$  is external. For a hyperfinite integer  $H$ , the set

$$U_H = \bigcap_{n \in \mathbb{N}} [0, [H/n]] \tag{II}$$

is an external cut. We often write  $x > U$  for  $x \in {}^*\mathbb{N} \setminus U$  and write  $x < U$  for  $x \in U$ . Note that if  $x < U$  and  $y > U$ , then  $\frac{x}{y} \approx 0$ .

Let  $U$  be a cut. A *b.p.*  $B = I \cup J$  is called a  *$U$ -unbounded b.p.* if both  $I \cap U$  and  $J \cap U$  are upper unbounded in  $U$ . Note that a  *$U$ -unbounded b.p.* always has the difference greater than 2.

Suppose  $U$  is a cut. Given a function  $f : U \mapsto {}^*\mathbb{R}$  (not necessarily internal) bounded by a standard real number, the *lower  $U$ -density* of  $f$  is defined by

$$\underline{d}_U(f) = \sup\{\inf\{st(f(n)) : n \in U \setminus [0, m]\} : m \in U\}.$$

Given a set  $A \subseteq [0, H]$ , let  $f_A(x) = \frac{A(0;x)}{x+1}$  for each  $x \in [0, H]$ . The *lower  $U$ -density* of  $A$  is defined by

$$\underline{d}_U(A) = \underline{d}_U(f_A).$$

For any  $x \in {}^*\mathbb{N}$ , we define the *lower  $(x + U)$ -density* and *lower  $(x - U)$ -density* of  $A$  by

$$\underline{d}_{x+U}(A) = \underline{d}_U((A - x) \cap {}^*\mathbb{N})$$



and

$$\underline{d}_{x-U}(A) = \underline{d}_U((x - A) \cap {}^*\mathbb{N}).$$

**Remark 2.1** (1) For any  $A \subseteq \mathbb{N}$ ,  $\underline{d}(A) = \underline{d}_{\mathbb{N}}({}^*A)$ , where  $\underline{d}(A) = \liminf_{n \rightarrow \infty} \frac{A(0, n-1)}{n}$  is the standard definition of the lower asymptotic density of  $A$ .

(2) It is easy to check that for each  $a \in U$ ,

$$\underline{d}_U(A + a) = \underline{d}_U(A)$$

and

$$\underline{d}_U(A \setminus [0, a]) = \underline{d}_U(A).$$

(3) Let  $H$  be hyperfinite and  $A \subseteq [0, H]$ . If  $\underline{d}_U(A) > \gamma$ , then there exist  $x \in U$  and  $y \in [0, H] \setminus U$  such that for any  $x \leq z \leq y$ ,  $\frac{A(0, z)}{z+1} > \gamma$ . Clearly one can find  $x \in U$  such that for every  $z \geq x$  in  $U$ ,  $\frac{A(0, z)}{z+1} > \gamma$ . Now the set of all  $z \in [x, H]$  such that  $\frac{A(0, z)}{z+1} > \gamma$  is internal and contains all elements in  $U \cap [x, H]$ , hence contains all elements in  $[x, y]$  for some  $y > U$ .

(4) If  $\underline{d}_U(A) = \alpha$ , then there is  $x \in U$  such that for every  $y \in U$  with  $y > x$ , one has  $\frac{A(0, y)}{y+1} \gtrsim \alpha$ . This can be proven by first choosing  $x_n \in U$  for each  $n \in \mathbb{N}$  such that for all  $z \geq x_n$  in  $U$ ,  $\frac{A(0, z)}{z+1} > \alpha - \frac{1}{n}$  and then choosing  $x \in U$  such that  $x \geq x_n$  for every  $n \in \mathbb{N}$ . The element  $x$  exists because, by countable saturation, the cofinality of  $U$  is uncountable, i.e. any countable increasing sequence in  $U$  is upper bounded in  $U$ .

(5) If  $\underline{d}_U(A) > \frac{1}{2}$ , then there exists  $a \in U$  such that  $A(0, a-1) = \frac{1}{2}a$  and  $A(a, c) > \frac{1}{2}(a - c + 1)$  for every  $c \in U$  with  $c \geq a$ . As a by-product we have  $a, a+1 \in A$  and  $A(a, a+3) \geq 3$ . The existence of  $a$  is guaranteed by  $\underline{d}_U(A) > \frac{1}{2}$ .

From now on, the only cut we need is  $U_H$  defined by (II) for a given  $H$ . Hence when  $H$  is clearly given, the letter  $U$  always represents the cut  $U_H$ . Note that with  $H$  fixed we have that  $x < U$  iff  $x \sim 0$  or equivalently  $x > U$  iff  $x \succ 0$ .

The first lemma of this section below is [Ji2, Lemma 2.12].

**Lemma 2.2** Let  $H$  be hyperfinite,  $U = U_H$ , and  $A \subseteq {}^*\mathbb{N}$  be such that  $0 < \underline{d}_U(A) = \alpha < \frac{2}{3}$ . If  $A \cap U$  is neither a subset of an a.p. of difference greater than 1 nor a subset of a  $U$ -unbounded b.p., then there is  $\gamma > 0$  such that for every  $N > U$ , there is  $K \in A \setminus U$  with  $K < N$  such that

$$\frac{(2A)(0, 2K)}{2K+1} \geq \frac{3}{2} \frac{A(0, K)}{K+1} + \gamma. \quad (\text{III})$$

The following is [Ji2, Lemma 2.4].

**Lemma 2.3** *Let  $A \subseteq [0, H]$ . Suppose  $0, H \in A$ . If  $0 \leq x_1 \prec x_2 \leq H$  satisfy the following*

- (1)  $(2A)(2x_1, 2x_2) \succ 3A(x_1, x_2)$ ,
  - (2) if  $0 \prec x_1$ , then  $\gcd(A[0, x]) = 1$  and  $A(0, x) \preceq \frac{1}{2}(x+1)$  for some  $x \sim x_1$  in  $A$ ,
  - (3) if  $x_2 \prec H$ , then  $\gcd(A[x, H] - x) = 1$  and  $A(x, H) \preceq \frac{1}{2}(H-x+1)$  for some  $x \sim x_2$  in  $A$ ,
- then  $|2A| \succ 3|A|$ .

**Lemma 2.4** *Let  $A \subseteq [0, H]$ . Suppose  $0, H \in A$ ,  $|A| \preceq \frac{1}{2}H$ ,  $d_U(A) = \frac{1}{2}$ , there is  $a \succ 0$  in  $A$  such that  $\gcd(A[a, H] - a) = 1$ , and for every  $N \succ 0$  there is a  $K \in A$  with  $0 \prec K \leq N$  such that (III) is true. Then  $|2A| \succ 3|A|$ .*

**Proof:** Let  $0 < \epsilon < 1$  be such that for any  $N \succ 0$  there is  $K \in A$  with  $0 \prec K \leq N$  such that

$$\frac{(2A)(0, 2K)}{2K+1} > \frac{(3+\epsilon)A(0, K)}{2(K+1)}.$$

Let  $\delta > 0$  be such that  $\delta < \frac{\epsilon}{6}$  and let  $y \in A$  be such that  $y \succ 0$ ,  $y \leq a$ ,  $A(0, y) \geq (\frac{1}{2} - \delta)(y+1)$ , and  $(2A)(0, 2y) > (3+\epsilon)A(0, y)$ .

If  $A(y, H) \preceq \frac{1}{2}(H-y)$ , then the lemma follows from Lemma 2.3 and Theorem A.1. So we can assume  $A(y, H) \succ \frac{1}{2}(H-y)$ . By Theorem A.4 we have  $|A[y, H] + A[y, H]| \succeq H - y + A(y, H)$ . Hence

$$\begin{aligned} |2A| &\succeq (2A)(0, 2y) + |A[y, H] + A[y, H]| \\ &\succeq (3+\epsilon)A(0, y) + H - y + A(y, H) \\ &\succeq 3A(0, y) + \epsilon A(0, y) + 2|A| - y + A(y, H) \\ &\succeq 3|A| + \epsilon A(0, y) + 2A(0, y) - y \\ &\succeq 3|A| + (\epsilon + 2)(\frac{1}{2} - \delta)y - y \\ &\succeq 3|A| + (\frac{\epsilon}{2} - \epsilon\delta - 2\delta)y \\ &\succ 3|A| + (\frac{\epsilon}{2} - \frac{\epsilon}{6} - \frac{2\epsilon}{6})y \\ &= 3|A|. \end{aligned}$$

□(Lemma 2.4)

It is worth to mention here that Lemma 2.2, Lemma 2.3, and Lemma 2.4, combined together, will frequently be used to show  $|2A| \succ 3|A|$  in various situations. For example, if  $|A| \preceq \frac{1}{2}H$  and  $A \cap U$  does not have “nice arithmetic structures”, then one can find an arbitrarily small  $y \succ 0$  in  $A$  such that  $(2A)(0, 2y) \succ 3A(0, y)$ . By Lemma 2.3 or Lemma 2.4 one needs only to make sure that  $A[x, H]$  is not a subset of an *a.p.* of difference  $> 1$  for some  $x \succ 0$  in order to conclude that  $|2A| \succ 3|A|$ .

The next lemma is trivial and will be frequently referred as the pigeonhole principle.

**Lemma 2.5** *Let  $d \geq 1$ . Suppose  $a, b \in [0, d-1]$ ,  $A \subseteq a + (d * \mathbb{N})$ ,  $B \subseteq b + (d * \mathbb{N})$ ,  $x \in a + (d * \mathbb{N})$ ,  $y \in b + (d * \mathbb{N})$ , and  $t \in (d * \mathbb{N})$ . If  $A(x, x+t) + A(y-t, y) > \frac{t}{d} + 1$ , then  $x + y \in (2A)$ .*

For convenience we give a name for each of the following two sets with special structural properties. Let  $a \prec b$  in  $[0, H]$ . A set  $F \subseteq [a, b]$  is called a *forward triangle* from  $a$  to  $b$  if  $|F| \sim \frac{1}{2}(b-a)$  and for every  $x$  with  $a \prec x \prec b$ ,  $F(a, x) \succ \frac{1}{2}(x-a)$ . A set  $B \subseteq [a, b]$  is called a *backward triangle* from  $a$  to  $b$  if the set  $(b+a) - B$  is a forward triangle from  $a$  to  $b$ . By the symmetry of the forward triangle and the backward triangle, we often prove a result about forward (backward) triangle and assume the symmetric result about backward (forward) triangle without proof.

Note that if  $F$  is a forward triangle from  $a$  to  $b$ , then there is  $z \sim a$  such that  $z, z+1 \in A$  and  $A(z, z+3) \geq 3$ . The number  $z$  can be obtained by letting  $z-1$  be the greatest number in  $a+U$  such that  $A(a, z-1) \leq \frac{1}{2}(z-a)$ .

**Lemma 2.6** *Let  $A \subseteq [0, H]$  be such that  $0, H \in A$  and  $0 \prec |A| \preceq \frac{1}{2}H$ . Let  $0 < \alpha \leq \frac{1}{2}$  and  $0 \prec x \prec H$ .*

(1) *If  $A(0, x) \preceq \alpha x$  and  $|A| \succeq \alpha H$ , then there exists  $y \succeq x$  such that  $A(0, y) \sim \alpha y$  and either  $y \sim H$  or for any  $z \succ y$  in  $[0, H]$ ,  $A(0, z) \succ \alpha z$ .*

(2) *If  $A(0, x) \succ \frac{1}{2}x$ , then there are  $0 \leq y \prec x \prec y' \leq H$  such that  $A(0, y) \sim \frac{1}{2}y$  and  $A[y, y']$  is a forward triangle.*

(3) *If  $\underline{d}_U(A) > \frac{1}{2}$ , then there is  $y \succ 0$  such that  $A[0, y]$  is a forward triangle.*

(4) *If  $\underline{d}_U(A) < \frac{1}{2}$  and  $|A| \sim \frac{1}{2}H$ , then there are  $0 \leq y \prec y' \leq H$  such that  $A(y', H) \sim \frac{1}{2}(H-y')$  and  $A[y, y']$  is a backward triangle.*

**Proof:** (1) Let

$$\beta = \sup\left\{st\left(\frac{z}{H+1}\right) : z \in [0, H] \text{ and } A(0, z) \preceq \alpha z\right\},$$

where  $st$  is the standard part map. By the completeness of the standard real line,  $\beta$  is well defined. Let  $y \in [0, H]$  be such that  $\frac{y}{H+1} \approx \beta$ . Clearly  $y \succeq x$ .

It is easy to see that if  $y \prec H$ , then  $A(0, z) \succ \alpha z$  for any  $y \prec z \leq H$  by the supremality of  $\beta$ . It is also easy to see that both  $A(0, y) \succ \alpha y$  and  $A(0, y) \prec \alpha y$  are impossible by the fact that  $\beta$  is the least upper bound.

(2) By the same idea as in (1) we can find  $y' \succ x$  such that  $A(0, y') \sim \frac{1}{2}y'$  and  $A(0, z) \succ \frac{1}{2}z$  for any  $x \leq z \prec y'$ . Let  $0 \leq y \prec x$  be such that  $A(0, y) \sim \frac{1}{2}y$  and  $A(0, z) \succ \frac{1}{2}z$  for any  $y \prec z \leq x$ . It is easy to see that  $A(y, y')$  is a forward triangle.

(3) By the definition of  $\underline{d}_U$  and (1) above there exists  $y \succ 0$  such that  $A(0, y) \sim \frac{1}{2}y$  and  $A(0, z) \succ \frac{1}{2}z$  for every  $0 \prec z \prec y$ . Clearly  $A[0, y]$  is a forward triangle.

(4) Choose  $x \succ 0$  such that  $A(0, x) \prec \frac{1}{2}x$ . Hence  $A(x, H) \succ \frac{1}{2}(H - x)$ . Now the conclusion follows from (2) above with the order of  $[0, H]$  reversed.  $\square$ (Lemma 2.6)

The following lemma in nonstandard analysis, which is already used in (3) of Remark 2.1, will be frequently– sometimes implicitly–used.

**Lemma 2.7** *Let  $X \subseteq {}^*\mathbb{N}$  be a proper external initial segment of non-negative integers and let  $A \subseteq {}^*\mathbb{N}$  be an internal set. (a) If  $A \cap X$  is upper unbounded in  $X$ , then  $A \setminus X \neq \emptyset$ . (b) If  $A \setminus X$  is lower unbounded in  ${}^*\mathbb{N} \setminus X$ , then  $A \cap X \neq \emptyset$ .*

**Proof:** If (a) of the lemma is not true, then

$$X = \{v \in {}^*\mathbb{N} : (\exists x \in A) (v \leq x)\},$$

which means that  $X$  is internal. The proof of (b) is similar.  $\square$ (Lemma 2.7)

**Lemma 2.8** *Suppose  $a \prec b$  in  $[0, H]$ .*

(1) *If  $T$  is a forward triangle from  $a$  to  $b$ , then  $[a', b'] \subseteq (2T)$  for some  $a' \sim 2a$  and  $b' \sim a + b$ .*

(2) *If  $B$  is a backward triangle from  $a$  to  $b$ , then  $[a', b'] \subseteq (2B)$  for some  $a' \sim a + b$  and  $b' \sim 2b$ .*

**Proof:** Given each  $x$  with  $a \prec x \prec b$ , since  $T(a, x) \succ \frac{1}{2}(x - a)$ , then by the pigeonhole principle,  $T[a, x] \cap (x + a - T[a, x]) \neq \emptyset$ . This implies  $x + a \in (2T)$ . Since  $2T$  is an internal set, then by Lemma 2.7 there are  $a' \sim 2a$  and  $b' \sim a + b$  such that  $[a', b'] \subseteq (2T)$ . The proof of the second part follows from the symmetry.  $\square$ (Lemma 2.8)

The following is [Ji2, Lemma 2.5]

**Lemma 2.9** *Let  $A \subseteq [0, H]$  for a hyperfinite integer  $H$ . If  $A[0, a]$  is a forward triangle from 0 to  $a$  and  $(2A)(a, c) \sim 0$  for some  $0 \prec a \prec c$ , then there is  $b \sim \frac{a}{2}$  such that  $A[0, a] \subseteq [0, b]$ .*

The following is a technical lemma, which will be used in the next two sections.

**Lemma 2.10** *Suppose  $0 \prec a \prec H$ ,  $A(0, a) \succ 0$ ,  $\gcd(A[0, a]) = 1$ ,  $A[0, a]$  is a subset of a b.p.  $I \cup J$  of difference  $d \geq 3$ ,  $A[0, a + 1]$  is not a subset of a b.p. of difference  $d$ ,  $|2A| \sim 3|A|$ , and  $|A[a + 1, H] + A[a + 1, H]| \sim 3A(a + 1, H)$ . Then  $A[0, a]$  is full in  $I \cup J$  and  $\max(A \cap I) \sim \max(A \cap J) \sim a$ .*

**Proof:** Let  $A_0 = A[0, a] \cap I$ ,  $A_1 = A[0, a] \cap J$ ,  $l_i = \min A_i$ , and  $u_i = \max A_i$  for  $i = 0, 1$ . Since

$$\begin{aligned} |2A| &\succeq |2A_0| + |2A_1| + |A_0 + A_1| + |A[a + 1, H] + A[a + 1, H]| \\ &\succeq 3|A_0| + 3|A_1| + 3A(a + 1, H) \sim 3|A|, \end{aligned}$$

then  $|2A| \sim 3|A|$  implies that  $|2A_i| \sim 2|A_i|$  for  $i = 0, 1$ . Hence by Theorem A.1 we have that  $A_0$  is full in  $I$  and  $A_1$  is full in  $J$ .

Without loss of generality we assume  $u_0 < u_1$ . If  $u_1 \prec a$ , then

$$|2A| \succeq 3A(0, a) + 3A(a + 1, H) + |a + 1 + A[2u_1 - a, u_1]| \succ 3|A|.$$

If  $u_1 \sim a$  and  $u_0 \prec a$ , then

$$|2A| \succeq 3A(0, a) + 3A(a + 1, H) + |a + 1 + A_1[2u_0 - a, a]| \succ 3|A|$$

because  $(a + 1 + A_1[2u_0 - a, a]) \cap (A[0, a] + A[0, a]) = \emptyset$ . Hence we have  $u_i \sim a$  for  $i = 0, 1$ .  $\square$ (Lemma 2.10)

### 3 First Step: When $\frac{|A|}{H}$ is significantly less than $\frac{1}{2}$ .

In this section we always assume that  $H$  is a hyperfinite integer,  $A \subseteq [0, H]$ ,  $0, H \in A$ , and  $\gcd(A) = 1$ . We will prove Theorem 1.7 under one extra condition

$$|A| \prec \frac{1}{2}H. \tag{IV}$$

We will prove that if  $|2A| \sim 3|A|$  and (IV) is true, then  $A$  must be a subset of a *b.p.*, which, by Theorem 1.3, implies Theorem 1.7. In this section the condition  $|2A| = 3|A| - 3 + b$  is not explicitly used. Hence the letter  $b$  is not reserved. In order to make the lemmas in this section available for the other sections, we will not automatically assume (IV). The condition (IV) will be explicitly stated when it is needed.

We will first prove various versions of the main theorem of the section as lemmas when some additional structural properties of  $A$  are assumed. After all needed versions are proven we combine them into the main theorem.

**Lemma 3.1** *If there are  $0 \prec a < b \prec H$  in  $A$  such that  $A = T \cup P$  where  $T = A[0, a]$  is a forward triangle from 0 to  $a$  and  $P = A[b, H]$  is a subset of an a.p. of difference  $d > 1$  with  $|P| \succ 0$ , then either  $A$  is a subset of a b.p. of difference 3 or  $|2A| \succ 3|A|$ .*

**Proof:** Let  $P$  be a subset of an a.p.  $I$  of difference  $d > 1$  such that  $b \in P$  is the least element of  $I$ , and  $H \in P$  is the largest element of  $I$ . Since  $T$  is a forward triangle, there exist  $z, z + 1 \in A \cap U$  such that for every  $x$  with  $z \leq x \prec a$ ,

$$\frac{T(z, x)}{x - z + 1} > \frac{1}{2}.$$

Without loss of generality (except in Case 3.1.1.2) we can assume  $z = 0$ . Under this assumption we have  $0, 1 \in A$  and  $A(0, 3) \geq 3$ .

**Claim 3.1.1** If  $P$  is not full in  $I$ , then  $(T + P)(b, H) \succ 2|P|$ .

Proof of Claim 3.1.1: Since  $P$  is not full, then  $|I \setminus P| \succ 0$ . Let  $\mathcal{I}$  be the collection of all intervals  $[x, y] \subseteq [b, H]$  such that  $y - x \geq 2d - 2$ ,  $[x, y] \cap P = \emptyset$ , and  $x - 1, y + 1 \in P$ . Then

$$I \setminus P = \bigcup_{[x, y] \in \mathcal{I}} ([x - 1 + d, y + 1 - d] \cap I).$$

Hence

$$\begin{aligned} |I \setminus P| &= \sum_{[x, y] \in \mathcal{I}} \frac{1}{d}(y - x + 2 - d) \\ &\preceq \sum_{[x, y] \in \mathcal{I}} \frac{1}{d}(y - x) \preceq \sum_{[x, y] \in \mathcal{I}} \frac{1}{2}(y - x) \preceq \sum_{[x, y] \in \mathcal{I}} (y - x - 1). \end{aligned}$$

If there is an interval  $[x, y] \in \mathcal{I}$  such that  $y - x \geq \frac{a}{2}$ , then

$$\begin{aligned}
(P+T)(b, H) &\succeq |P| + |1+P| + |(x-1+T)(x+1, y)| \\
&\succeq 2|P| + T(2, y-x+1) \\
&\succeq 2|P| + \min\{|T| - 2, \frac{1}{2}(y-x+2) - 2\} \\
&\succeq 2|P| + \frac{1}{4}a \succ 2|P|.
\end{aligned}$$

So we can assume  $y - x < \frac{a}{2}$  for every  $[x, y] \in \mathcal{I}$ . Since for each interval  $[x, y] \in \mathcal{I}$ , we have

$$\begin{aligned}
(P+T)(x+1, y) &\geq |x-1+T[2, y-x+1]| \\
&= T(2, y-x+1) > \frac{1}{2}(y-x+2) - 2 \\
&= \frac{1}{2}(y-x-1) - \frac{1}{2}.
\end{aligned}$$

Hence  $(P+T)(x+1, y) \geq \frac{1}{2}(y-x-1)$  because the left-side is an integer. So

$$\begin{aligned}
(P+T)(b, H) &\succeq |P| + |1+P| + \sum_{[x,y] \in \mathcal{I}} (P+T)(x+1, y) \\
&\succeq |P| + |1+P| + \sum_{[x,y] \in \mathcal{I}} \frac{1}{2}(y-x-1) \\
&\succeq 2|P| + \frac{1}{2}|I \setminus P| \succ 2|P|.
\end{aligned}$$

□(Claim 3.1.1)

By the claim above we can assume that  $P$  is full in  $I$  because otherwise

$$|2A| \succeq a + (P+T)(b, H) + |H+A| \succ 2|T| + 2|P| + |A| = 3|A|.$$

Next we divide the proof of the lemma into three cases with  $d = 2$ ,  $d = 3$ , and  $d \geq 4$ .

**Case 3.1.1**  $d = 2$ .

Let  $c \prec H$  in  $P$  be such that  $H - c < \frac{a}{2}$ . Then

$$\begin{aligned}
|2A| &\succeq a + |\{0, 1\} + P| + (H+T)(H, c+a) \\
&\quad + (c+P)(c+b, H+b) + (H+P)(H+b, 2H) \\
&\sim 2|T| + 2|P| + T(0, c+a-H) + \frac{1}{2}(H-c) + |P| \\
&\succ 3|T| + 3|P| = 3|A|
\end{aligned}$$

because  $T(c + a - H + 1, a) \prec \frac{1}{2}(H - c)$ .  $\square$ (Case 3.1.1)

**Case 3.1.2**  $d = 3$ . Let  $A$  be the original set with  $z, z + 1 \in A \cap U$ . If  $A$  is a subset of a b.p. of difference 3, then the lemma is trivially true. Suppose  $A$  is not a subset of a b.p. of difference 3. Let  $c = \min\{x \in A : x \equiv z + 2 \pmod{3}\}$ . Note that either  $c \in T$  or  $c = b$ .

Suppose  $c \succ 0$ . Let  $b \leq x \prec H$  be such that  $x \in A$  and  $H - x < c$ . Note that  $A[0, c - 1] \subseteq (z + (3 * \mathbb{Z})) \cup (z + 1 + (3 * \mathbb{Z}))$  and  $A[x, H] + c \subseteq b + z + 2 + (3 * \mathbb{Z})$ . Hence  $(A[x, H] + c) \cap (H + A[0, c - 1]) = \emptyset$ . So we have

$$\begin{aligned} |2A| &\succeq a + 2|P| + |H + A| + |c + A[x, H - 1]| \\ &\sim 2|T| + 2|P| + |A| + A(x, H) \succ 3|A|. \end{aligned}$$

Suppose  $c \sim 0$ . Then

$$|2A| \succeq a + |\{z, z + 1, c\} + P| + |H + A| \sim 3|A| + |P| \succ 3|A|.$$

$\square$ (Case 3.1.2)

**Case 3.1.3**  $d \geq 4$ .

Since  $T(0, 3) \geq 3$ , then

$$|2A| \succeq a + |T[0, 3] + P| + |H + A| \sim 3|A| + |P| \succ 3|A|.$$

This ends the proof of Lemma 3.1.  $\square$ (Lemma 3.1)

**Lemma 3.2** *Suppose there are  $0 \prec a < b \prec H$  such that  $A = F \cup B$ , where  $F$  is a forward triangle from 0 to  $a$  and  $B$  is a backward triangle from  $b$  to  $H$ . If  $|2A| \sim 3|A|$ , then  $\bar{a} = \max F \sim \frac{a}{2}$  and  $\bar{b} = \min B \sim \frac{b+H}{2}$ . Hence  $F$  is full in  $[0, \bar{a}]$  and  $B$  is full in  $[\bar{b}, H]$ . So  $A$  is a full subset of the b.p.  $[0, \bar{a}] \cup [\bar{b}, H]$  of difference 1.*

**Proof:** Suppose  $\bar{b} = \min B \sim b$ . Let  $0 \prec x \prec \min\{a, \frac{H-b}{2}\}$ . Then by Lemma 2.8

$$\begin{aligned} |2A| &\succeq a + |\bar{b} + F[0, x]| + B(\bar{b} + x, H) \\ &\quad + |H + F| + |[H + b, 2H]| \\ &\sim 2|F| + F(0, x) + B(\bar{b} + x, H) + |F| + 2|B| \\ &\succ 3|F| + \frac{1}{2}(x + 1) + B(\bar{b} + x, H) + 2|B| \\ &\succ 3|F| + 2|B| + B(\bar{b}, H) \sim 3|A|. \end{aligned}$$



Hence we can assume that  $\bar{b} \succ b$ . But this implies

$$\begin{aligned} |2A| &\succeq a + (2A)(a, \bar{b}) + A(\bar{b}, H) + |H + F| + |[H + b, 2H]| \\ &\sim 2|F| + (2A)(a, \bar{b}) + |B| + |F| + 2|B| \sim 3|A| + (2A)(a, \bar{b}). \end{aligned}$$

Hence  $|2A| \sim 3|A|$  implies  $(2A)(a, \bar{b}) \sim 0$ . By Theorem 2.9,  $\bar{a} \sim \frac{a}{2}$ . By a symmetric argument, we can also show that  $\bar{b} \sim \frac{b+H}{2}$ .  $\square$ (Lemma 3.2)

**Lemma 3.3** *Suppose there are  $0 \prec a < b \prec H$  such that  $A = F \cup C$ , where  $F \subseteq [0, a]$  is a forward triangle from 0 to  $a$  and  $C \subseteq [b, H]$  with  $b \in C$ ,  $|C| \preceq \frac{1}{2}(H - b + 1)$ , and  $\gcd(C - b) = 1$ . Then  $|2A| \succ 3|A|$ .*

**Proof:** First we assume that there is an  $x \in C$  such that  $0 \prec x - b < \frac{a}{2}$ . Then

$$\begin{aligned} |2A| &\succeq a + |b + F[0, x - b]| \\ &\quad + |x + F[0, a + b - x]| + |C[b, H] + C[b, H]| \\ &\succeq 2|F| + F(0, x - b) + F(0, a + b - x) + 3|C| \\ &\succ 2|F| + 3|C| + F(a + b - x, a) + F(0, a + b - x) \\ &\sim 3|F| + 3|C| = 3|A|. \end{aligned}$$

If the assumption above is not true, let  $x = \min\{z \in C : z \geq b + \frac{a}{2}\}$  and  $y = \max\{z \in C : z < b + \frac{a}{2}\}$ . Then  $y \sim b$ ,  $(2C)(2b, b + x) \sim 0$ , and  $(2C)(2b, 2H) \sim (2C)(b + x, 2H)$ . Hence

$$\begin{aligned} |2A| &\succeq a + |b + F[0, x - b]| + |x + F| + (2C)(b + x, 2H) \\ &\succeq 3|F| + F(0, x - b) + 3|C| \sim 3|A| + F(0, x - b) \succ 3|A|. \end{aligned}$$

This ends the proof.  $\square$ (Lemma 3.3)

**Lemma 3.4** *Suppose there are  $0 \prec a < b \prec c \prec H$  such that  $A = F \cup B \cup C$ , where  $F$  is a forward triangle from 0 to  $a$ ,  $B$  is a backward triangle from  $b$  to  $c$ , and  $C \subseteq [c + 1, H]$  with  $|C| \preceq \frac{1}{2}(H - c)$ . Then  $|2A| \succ 3|A|$ .*

**Proof:** Without loss of generality we can assume  $c, c - 1 \in B$ . By Lemma 2.3 we have that  $|A[0, c] + A[0, c]| \succ 3A(0, c)$  implies  $|2A| \succ 3|A|$ . So we can now assume  $|A[0, c] + A[0, c]| \sim 3A(0, c)$ . Let  $\bar{a} = \max F$  and  $\bar{b} = \min B$ . By Lemma 3.2, we have  $\bar{a} \sim \frac{a}{2}$ ,  $\bar{b} \sim \frac{b+c}{2}$ ,  $F$  is full in  $[0, \bar{a}]$ , and  $B$  is full in  $[\bar{b}, c]$ .

**Case 3.4.1** There is  $x \in C$  with  $x \succ c$  such that  $C(c, x) \sim 0$ .

We have

$$\begin{aligned}
|2A| &\succeq 3|F| + 3|B| + |x + B[2c - x, c]| \\
&\quad + |(\{c - 1, c\} \cup C[x, H]) + (\{c - 1, c\} \cup C[x, H])| \\
&\succeq 3|F| + 3|B| + B(2c - x, c) + 3|\{c - 1, c\} \cup C[x, H]| \\
&\sim 3|F| + 3|B| + 3|C| + B(2c - x, c) \succ 3|A|.
\end{aligned}$$

□(Case 3.4.1)

**Case 3.4.2** For every  $x \in C$ , if  $x \succ c$ , then  $C(c, x) \succ 0$ .

The assumption implies that for every  $y \succ c$ , there is  $x \in C$  with  $c \prec x \prec y$ . Let  $x \in C$  be such that  $0 \prec x - c < \bar{a}$ . Then

$$\begin{aligned}
|2A| &\succeq a + B(b, c) + |c + F[0, x - c]| + |x + F[0, \bar{a}]| \\
&\quad + |[c + b, 2c]| + |(\{c - 1, c\} \cup C) + (\{c - 1, c\} \cup C)| \\
&\succeq 2|F| + |B| + F(0, x - c) + |F| + 2|B| + 3|C| \succ 3|A|.
\end{aligned}$$

This ends the proof. □(Lemma 3.4)

**Lemma 3.5** Suppose there is  $a$  with  $0 \prec a \prec H$  such that  $F = A[0, a]$  is a forward triangle from 0 to  $a$  and  $A(a, H) \preceq \frac{1}{2}(H - a)$ . If  $|2A| \sim 3|A|$ , then  $A$  is a full subset of a b.p. of difference 3 or a full subset of a b.p. of difference 1.

**Proof:** Note that if  $A$  is a subset of a b.p. then  $A$  must be a full subset of that b.p. when  $|2A| \sim 3|A|$ . Let  $b = \min A[a + 1, H]$ . If  $b \sim H$ , then

$$|2A| \succeq a + (2A)(a + 1, H) + |H + F| \sim 3|A| + (2A)(a + 1, H).$$

Hence  $(2A)(a + 1, H) \sim 0$ . By Lemma 2.9,  $\bar{a} = \max F \sim \frac{1}{2}(a + 1)$ . This shows  $2\bar{a} \prec b$  and  $\bar{a} + H \prec 2b$ . Hence  $[0, \bar{a}] \cup [b, H]$  is the desired b.p. of difference 1. So we can assume  $b \prec H$ . If  $A(b, H) \sim 0$ , then

$$|2A| \succeq a + |b + A[0, H - b]| + |H + A[0, a]| \sim 3|A| + A(0, H - b) \succ 3|A|.$$

Hence we can assume  $A(b, H) \succ 0$ .

Suppose  $A$  is not a full subset of a *b.p.* of difference 3. By Lemma 3.1 we can assume  $\gcd(A[b, H] - b) = 1$ . If  $A(b, H) \preceq \frac{1}{2}(H - b)$ , then the lemma follows from Lemma 3.3. So now we can assume that  $A(b, H) \succ \frac{1}{2}(H - b)$ .

By (2) of Lemma 2.6 there are  $a < c \prec b \prec c' \leq H$  such that  $A[c, c']$  is a backward triangle and  $A(c', H) \sim \frac{1}{2}(H - c')$ . If  $c' \prec H$ , then by Lemma 3.4 we have  $|2A| \succ 3|A|$ . Hence we can assume  $c' \sim H$ . But now  $A$  becomes the union of a forward triangle  $A[0, a]$  and a backward triangle  $A[c, H]$ . Now the lemma follows from Lemma 3.2.  $\square$ (Lemma 3.5)

**Lemma 3.6** *Suppose there are  $0 \prec a \prec a' \prec H$  such that  $A(0, a) \preceq \frac{1}{2}a$ ,  $A(a', H) \preceq \frac{1}{2}(H - a')$ ,  $A[a, a']$  is a forward triangle from  $a$  to  $a'$ , and  $A[a, H]$  is not a subset of a *b.p.* of difference 3. Then  $|2A| \succ 3|A|$ .*

**Proof:** By Lemma 2.3 we can assume  $|A[a, H] + A[a, H]| \sim 3A(a, H)$ . By Lemma 3.5 we can assume that  $A[a, H]$  is full in a *b.p.*  $[a, c] \cup [c', H]$  for some  $c < c'$  in  $[a, H]$ . If  $c' \prec H$ , then the lemma follows from Lemma 3.4. So we can assume  $c' \sim H$ . Note that  $c \sim \frac{a+a'}{2}$ . Hence we have  $2c \prec a + H$ . If there is  $x \prec a$  in  $A$  with  $x \geq 2c - H$  such that  $A(x, a) \succ 0$ , then

$$\begin{aligned} |2A| &\succeq (2A)(0, 2a) + (2A)(2a, 2c) \\ &\quad + (2A)(H + x, H + a) + |H + A[a, c]| \\ &\succeq 3A(0, a) + 3A(a, c) + A(x, a) \succ 3|A|. \end{aligned}$$

Otherwise choose  $x \prec a$  in  $A$  such that  $A(x, a) \sim 0$ . Without loss of generality let  $a, a + 1 \in A$ . Then

$$(2A)(0, x + a) \succeq |(A[0, x] \cup \{a, a + 1\}) + (A[0, x] \cup \{a, a + 1\})| \succeq 3A(0, x) \sim 3A(0, a).$$

So we have

$$\begin{aligned} |2A| &\succeq (2A)(0, x + a) + |x + A[a, 2a - x]| \\ &\quad + (2A)(2a, 2c) + |H + A[a, c]| \\ &\succeq 3A(0, a) + 3A(a, c) + A(a, 2a - x) \succ 3|A|. \end{aligned}$$

$\square$ (Lemma 3.6)

**Lemma 3.7** *Suppose there are  $0 \prec a \prec a' \prec H$  such that  $A(0, a) \preceq \frac{1}{2}a$ ,  $A(a', H) \preceq \frac{1}{2}(H - a')$ ,  $A[a, a']$  is a forward triangle from  $a$  to  $a'$ , and  $A[a', H]$  is not a subset of an a.p. of difference 3. Then  $|2A| \succ 3|A|$ .*

**proof:** By Lemma 3.6 it suffices to show that  $A[a, H]$  is not a subset of a *b.p.* of difference 3. If  $A[a, H]$  is a subset of a *b.p.* of difference 3, then  $|2A| \sim 3|A|$  implies that  $A[a, H]$  is a full subset of the *b.p.* This implies that  $A[a, a']$  is a subset of the union of an *a.p.* of difference 3 of length  $\sim \frac{1}{3}(a' - a)$  and an *a.p.* of difference 3 of length  $\sim \frac{1}{6}(a' - a)$ -both have the left-end points  $\sim a$ , and  $A[a', H]$  is a full subset of an *a.p.* of difference 3, which contradicts the assumption of the lemma.  $\square$ (Lemma 3.7)

Starting from the next lemma to the end of this section, the condition (IV) is assumed.

**Lemma 3.8** *Assume that  $|A| \prec \frac{1}{2}H$  and  $A$  is neither a subset of an a.p. of difference  $> 1$  nor a subset of a *b.p.* Suppose that there is  $x \succ 0$  such that  $A(0, x) \succ 0$  and  $A[0, x]$  is a subset of an a.p. of difference  $> 1$ . Then  $|2A| \succ 3|A|$ .*

**Proof:** Let  $a = \min\{y \in A : \gcd(A[0, y]) = 1\}$  and  $c = \max A[0, a - 1]$ . Let  $d = \gcd(A[0, c])$ . Note that  $d > 1$  is a standard natural number because  $A(0, x) \succ 0$ .

First, we can assume that  $a \prec H$  by the following reason: Suppose  $a \sim H$ . If there is  $b \in A$  such that  $b \not\equiv 0 \pmod{d}$  and  $b \not\equiv a \pmod{d}$ , then

$$|2A| \succeq |A[0, c] + A[0, c]| + |a + A[0, c]| + |b + A[0, c]| \sim 4|A|.$$

If for any  $b \in A$ ,  $b \equiv 0 \pmod{d}$  or  $b \equiv a \pmod{d}$ , then  $A$  is a subset of a *b.p.* unless  $d = 2$ . Assume  $d = 2$ . Hence  $A[0, c]$  is a set of even numbers and  $a$  is odd. If  $A[a, H]$  is a set of odd numbers, then  $A = A[0, c] \cup A[a, H]$  is a subset of a *b.p.* So we can assume that there is an even number  $b \in A$  with  $b > a$ . Clearly  $b \sim H$ . Let  $A_e$  be the set of all even number in  $A$ . Then

$$|2A| \succeq |2A_e| + |a + A_e| \succeq 3|A_e| \sim 3|A|.$$

Hence if  $|2A| \sim 3|A|$ , then  $|2A_e| \sim 2|A_e|$ . By Theorem A.1 and the fact that  $b \sim H$  the set  $A_e$  is full in the set of all even numbers in  $[0, H]$ , which contradicts (IV).

Second, we can assume that  $A(a, H) \succ 0$  by the following reason: Suppose  $A(a, H) \sim 0$ . If  $|A[0, c] + A[0, c]| \succ 2A(0, c)$ , then

$$|2A| \succeq |A[0, c] + A[0, c]| + |a + A[0, c]| \succ 3|A|.$$

So we can assume  $|A[0, c] + A[0, c]| \sim 2A(0, c)$ . By Theorem A.1  $A[0, c]$  is full. This implies  $A(a + c - H, c) \succ 0$ . Hence we have

$$\begin{aligned} |2A| &\succeq |A[0, c] + A[0, c]| \\ &\quad + |a + A[0, c]| + |H + A[a + c - H, c]| \\ &\succeq 3A(0, c) + A(a + c - H, c) \\ &\sim 3|A| + A(a + c - H, c) \succ 3|A|. \end{aligned}$$

Now we are ready to prove the lemma. The proof is divided into five cases.

**Case 3.8.1**  $d = 2$  and  $A(a, H) \preceq \frac{1}{2}(H - a)$ .

Clearly  $a$  is odd. Since  $A$  is not a subset of a *b.p.*, then  $\gcd(A[a, H] - a) = d'$  is not an even number.

Suppose  $d' > 2$ . Let  $c' = \max\{x \in A : \gcd(A[x, H] - x) = 1\}$ . Then  $c' \leq c$  and  $A[c' + 1, H] \subseteq (H - (d'' * \mathbb{N}))$  for some  $d'' > 1$  and  $d'' | d'$ . By a symmetric argument of showing  $A(a, H) \succ 0$  above, we can assume  $A(0, c') \succ 0$ . With a little more effort we can show that

$$(A[a, H] + A[a, H]) \cap (c' + A[a + 1, H]) = \emptyset,$$

$$(A[0, c] + A[0, c]) \cap (c' + A[a + 1, H]) = \emptyset, \text{ and}$$

$$(a + A[0, c]) \cap (c' + A[a + 1, H]) = \emptyset.$$

The second equality above is due to the fact that if  $x_1, x_2 \in A[0, c]$  and  $a' \in A[a + 1, H]$  having  $x_1 + x_2 = c' + a' \geq c' + a + 1$ , then  $x_1, x_2 > c'$ , which implies  $x_1, x_2 \in (H - (d'' * \mathbb{N}))$ . This implies  $c' = x_1 + x_2 - a' \in (H - (d'' * \mathbb{N}))$ , which contradicts the choice of  $c'$ . The reason for the third equality above is similar. Hence

$$\begin{aligned} |2A| &\succeq |A[0, c] + A[0, c]| + |a + A[0, c]| \\ &\quad + |A[a, H] + A[a, H]| + |c' + A[a + 1, H]| \\ &\succeq 3A(0, c) + 3A(a, H) \sim 3|A|. \end{aligned}$$

If  $|2A| \sim 3|A|$ , then  $|A[0, c] + A[0, c]| \sim 2A(0, c)$  and  $|A[a, H] + A[a, H]| \sim 2A(a, H)$ . Hence  $A[0, c]$  is full in the set of all even numbers in  $[0, c]$  and  $A[a, H]$  is full in  $(a + (d' * \mathbb{N})) \cap [a, H]$ . Without loss of generality, we can assume  $c, c - 2, c - 4 \in A$  and  $c' = c$ . Note that  $c + A[a, H]$ ,  $c - 2 + A[a, H]$ , and  $c - 4 + A[a, H]$  are pairwise disjoint because  $d'$  is odd and  $d' > 2$ . Hence we have

$$\begin{aligned} |2A| &\succeq |A[0, c] + A[0, c]| + |a + A[0, c]| \\ &\quad + |\{c, c - 2, c - 4\} + A[a, H]| + |H + A[a, H]| \\ &\succeq 3A(0, c) + 4A(a, H) \succ 3|A|. \end{aligned}$$

So  $|2A| \succ 3|A|$  must be true. This ends the proof of the case for  $d' > 2$ .

Now assume that  $\gcd(A[a, H] - a) = d' = 1$ . This implies  $|A[a, H] + A[a, H]| \succeq 3A(a, H)$ . Hence

$$|2A| \succeq (2A)(0, 2c) + |a + A[0, c]| + (2A)(2a, 2H) \succeq 3|A|.$$

We now derive a contradiction by assuming  $|2A| \sim 3|A|$ . By the inequality above we have that  $A[0, c]$  is full in the set of all even numbers in  $[0, c]$ . Suppose  $c \prec a$ . If there is  $x \succ a$  in  $A$  such that  $x - a < a - c$ , then we have

$$\begin{aligned} |2A| &\succeq 2A(0, c) + |a + A[0, c]| \\ &\quad + |x + A[a + c - x, c]| + |A[a, H] + A[a, H]| \\ &\succeq 3A(0, c) + 3A(a, H) + A(a + c - x, c) \succ 3|A|, \end{aligned}$$

which contradicts  $|2A| \sim 3|A|$ . Otherwise we can find  $x \succ a$  in  $A$  such that  $A(a, x) \sim 0$ . Let  $F \subseteq A[a, H]$  be finite such that  $a \in F$  and  $\gcd((F \cup A[x, H]) - a) = 1$ . Then

$$(2A)(x + a, 2H) \succeq |(F \cup A[x, H]) + (F \cup A[x, H])| \succeq 3A(x, H) \sim 3A(a, H).$$

Hence we have

$$|2A| \succeq 3A(0, c) + 3A(a, H) + |x + A[2c - x, c]| \succ 3|A|.$$

So we can assume  $c \sim a$ . Recall that we have  $A(0, c) \succ 0$ ,  $A(a, H) \succ 0$ ,  $\gcd(A[0, c]) = 2$ ,  $A[0, c]$  is full,  $\gcd(A[a, H] - a) = 1$ , and  $A(a, H) \preceq \frac{1}{2}(H - a)$ . Note that since  $A(0, c) \sim \frac{1}{2}(c + 1)$ , then  $|A| \prec \frac{1}{2}H$  implies  $A(a, H) \prec \frac{1}{2}(H - a)$ . Since  $A[0, c]$  is full, we can, without loss of generality, assume that  $c, c - 2, c - 4 \in A$ .

**Subcase 3.8.1.1**  $\underline{d}_{a+U}(A) = 0$ .

Choose  $x \in A$  with  $x \succ a$  such that  $A(a, x) < \frac{1}{8}(x - a + 1)$ . Let  $F \subseteq A[a, H]$  be finite such that  $a \in F$  and  $\gcd(F \cup A[x, H]) = 1$ . Then

$$\begin{aligned} |2A| &\succeq |A[0, c] + A[0, c]| + |a + A[0, c]| + |x + A[a + c - x, c]| \\ &\quad + |(F \cup A[x, H]) + (F \cup A[x, H])| \\ &\succeq 3A(0, c) + \frac{1}{2}(x - a + 1) + 3A(x, H) \\ &\succeq 3|A| + \frac{1}{2}(x - a + 1) - 3A(a, x) \succ 3|A|, \end{aligned}$$

which is again a contradiction.  $\square$ (Subcase 3.8.1.1)

**Subcase 3.8.1.2**  $\underline{d}_{a+U}(A) > \frac{1}{2}$ .

By (3) of Lemma 2.6 there exists  $b \succ a$  such that  $A[a, b]$  is a forward triangle from  $a$  to  $b$ . Since  $A(a, H) \prec \frac{1}{2}(H - a)$ , then  $A(b, H) \prec \frac{1}{2}(H - b)$  and  $b \prec H$ . If  $|2A| \sim 3|A|$ , then  $|A[c - 4, H] + A[c - 4, H]| \sim 3A(c - 4, H)$ . Note that  $A[c - 4, H]$  is not a subset of a  $b.p.$  of difference 3 because  $c, c - 2, c - 4 \in A$ . Hence by Lemma 3.5,  $A[c - 4, H]$  is a full subset of a  $b.p.$   $[c - 4, a'] \cup [b', H]$  for some  $a', b' \in A$ . If  $b' \sim H$ , then by the fact that  $2a' \prec a + H$  we have

$$|2A| \succeq 3A(0, c) + 2a' - 2a + |H + A[2a' - H, H]| \succeq 3|A| + A(2a' - H, a) \succ 3|A|.$$

If  $b' \prec H$ , then the lemma follows from Lemma 3.4.  $\square$ (Subcase 3.8.1.2)

**Subcase 3.8.1.3**  $0 < \underline{d}_{a+U}(A) \leq \frac{1}{2}$ .

Suppose for any  $x \succ a$  in  $A$  we have  $\gcd(A[x, H] - x) > 1$ . Choose  $x \sim a$  in  $A$  such that  $\gcd(A[x, H] - x) = d' > 1$ . Since  $\gcd(A[a, H] - a) = 1$ , then  $|A[a, H] + A[a, H]| \sim 3A(a, H)$  implies that  $A[x, H]$  is full.

If  $d' = 2$ , then  $|A| \sim \frac{1}{2}H$ , which contradicts the condition (IV).

Suppose  $d' = 4$ . Let  $c' = c$  and  $c'' = c - 2$  when  $x$  is odd, or let  $c' \in \{c, c - 2\}$  such that  $c' + x \equiv 2x + 2 \pmod{d'}$  and  $c'' = a$  when  $x$  is even. Then  $c' + A[x, H]$ ,  $c'' + A[x, H]$ , and  $A[x, H] + A[x, H]$  are pairwise disjoint. Hence

$$\begin{aligned} |2A| &\succeq |A[0, a] + A[0, a]| + |c' + A[x, H]| \\ &\quad + |c'' + A[x, H]| + |A[x, H] + A[x, H]| \\ &\succeq 3|A| + A(x, H) \succ 3|A|. \end{aligned}$$

Suppose  $d' = 3$  or  $d' > 4$ . Then there are  $c', c''$  in  $\{c, c-2, c-4\}$  such that  $c' + A[x, H]$ ,  $c'' + A[x, H]$ , and  $A[x, H] + A[x, H]$  are pairwise disjoint. Hence  $|2A| \succ 3|A|$  by the same reason above.

Therefore, we can now assume that there is  $x \succ a$  in  $A$  such that  $\gcd(A[x, H] - x) = 1$ . Since  $\gcd((A[c-4, H] - c+4) \cap U) = 1$  and  $(A[c-4, H] - c+4) \cap U$  is not a subset of a  $U$ -unbounded  $b.p.$  of difference  $d > 1$  because  $a, c, c-2, c-4 \in A$ , then by Lemma 2.2 there exists  $y \succ a$  in  $A$  with  $c \prec y \leq x$  and  $A(y, H) \preceq \frac{1}{2}(H - y + 1)$  such that  $(2A)(2(c-4), 2y) \succ 3A(c-4, y)$ . Hence by Lemma 2.3  $|A[c-4, H] + A[c-4, H]| \succ 3A(c-4, H)$ , which implies  $|2A| \succ 3|A|$ . This ends the proof.  $\square$ (Case 3.8.1)

**Case 3.8.2**  $d = 2$  and  $A(a, H) \succ \frac{1}{2}(H - a)$ .

By Lemma 2.6 we can find  $0 \leq a' \prec a \prec a'' \leq H$  such that  $A[a', a'']$  is a backward triangle from  $a'$  to  $a''$  and  $A(a'', H) \sim \frac{1}{2}(H - a'')$ . Without loss of generality we can assume  $\gcd(A[a'', H] - a'') = 1$ . Then by Lemma 3.1 we have that  $|A[0, a''] + A[0, a'']| \succ 3A(0, a'')$  or  $A[0, a'']$  is a full subset of a  $b.p.$  of difference 3. However, the former implies  $|2A| \succ 3|A|$  by Lemma 2.3 and the latter is impossible because  $d = 2$ .  $\square$ (Case 3.8.2)

**Case 3.8.3**  $d = 3$  and  $A(a, H) \preceq \frac{1}{2}(H - a)$ .

(Note that this case does not occur when  $|A| \sim \frac{1}{2}H$ .) Since  $A$  is not a subset of a  $b.p.$ , we can define

$$b = \min\{x \in A : x \notin \{0, a\} \pmod{3}\}.$$

Let  $A_0 = A \cap (3 * \mathbb{N})$ ,  $A_a = A \cap (a + (3 * \mathbb{N}))$ , and  $A_b = A \cap (b + (3 * \mathbb{N}))$ . Let  $l_0, l_a, l_b$  be the least element of  $A_0, A_a, A_b$ , respectively. Let  $u_0, u_a, u_b$  be the largest element of  $A_0, A_a, A_b$ , respectively. Note that the rest of the proof does not use the fact that  $a \succ 0$ .

**Subcase 3.8.3.1**  $b \sim H$ .

We have  $|A| \sim |A_0| + |A_a|$ . We can also assume  $|A_a| \succ 0$  because otherwise

$$|2A| \succeq |2A_0| + |a + A_0| + |b + A_0| = 4|A_0| = 4|A|.$$

Since  $A_0 \cup A_a$  is a subset of a  $b.p.$ , then by Theorem A.1,  $A_0$  is full and  $A_a$  is full. This implies  $u_a \prec H$  or  $u_0 \prec H$  because  $A(a, H) \preceq \frac{1}{2}(H - a)$ . Suppose  $u_a \prec H$  and  $u_a \leq u_0$ . Then

$$|2A| \succeq |2A_0| + |2A_a|$$



$$\begin{aligned}
& +|A_0 + A_a| + |b + A_0[u_a + u_0 - b, u_0]| \\
& \succeq 3|A| + A_0(u_a + u_0 - b, u_0) \succ 3|A|.
\end{aligned}$$

By the same reason, if  $u_0 \prec H$  and  $u_0 \leq u_a$ , then  $|2A| \succ 3|A|$ . Note that if both  $u_0 \prec H$  and  $u_a \prec H$  are true, then either  $u_0 \leq u_a$  or  $u_a \leq u_0$ .  $\square$ (Subcase 3.8.3.1)

**Subcase 3.8.3.2**  $b \prec H$ .

We can assume  $A(b, H) \succ 0$  because otherwise  $|2A| \succeq (2A)(0, 2b) + |H + A[2b - H, b]| \succ 3|A|$ .

Suppose  $d' = \gcd(A[b, H] - b) > 1$ . If  $d' = 2$ , then the proof of this case is same as the proof in Case 3.8.1 and Case 3.8.2 by considering  $H - A$  in the place of  $A$ . So we can assume that  $d' > 2$ .

If  $d' = 3$ , then  $u_0, u_a < b$ . Note that  $b \notin \{0, a\} \pmod{3}$ . We can assume  $|A_a| \succ 0$  because if  $|A_a| \sim 0$ , then

$$|2A| \succeq |2A_0| + |2A_b| + |A_0 + A_b| + |u_a + A_0| \succ 3|A|.$$

We can also assume  $|A_b| \succ 0$  because otherwise let  $c \leq b$  such that  $A(c, b) \sim 0$  and for every  $x \prec c$ ,  $A(x, c) \succ 0$ . Then

$$|2A| \succeq 3|A_0| + 3|A_a| + |H + A[c + b - H, c]| \succ 3|A|.$$

Let  $u = \max\{u_0, u_a\}$ . If  $|2A| \sim 3|A|$ , then

$$\begin{aligned}
|2A| & \succeq |2A_0| + |2A_a| \\
& + |A_0 + A_a| + |2A_b| + |u + A_b| \\
& \succeq 3|A_0| + 3|A_a| + 3|A_b| = 3|A|
\end{aligned}$$

implies that  $A_0$ ,  $A_a$ , and  $A_b$  are full. If  $u_0 \prec b$  and  $u_0 < u_a$ , then

$$\begin{aligned}
|2A| & \succeq |2A_0| + |2A_a| + |A_0 + A_a| + |2A_b| \\
& + |b + A_a[u_a + u_0 - b, u_a]| + |u_a + A_b| \\
& \succeq 3|A_0| + 3|A_a| + 3|A_b| + A_a(u_a + u_0 - b, u_a) \succ 3|A|.
\end{aligned}$$

So we can assume  $u_0 \sim b$ . By a similar argument we can also assume  $u_a \sim b$ . However, above assumptions imply that

$$\begin{aligned}
|2A| & \succeq |2A_0| + |2A_a| + |A_0 + A_a| \\
& + |2A_b| + |u_a + A_b| + |u_0 + A_b| \\
& \succeq 3|A_0| + 3|A_a| + 4|A_b| \succ 3|A|.
\end{aligned}$$

Suppose  $d' \geq 4$ . We re-define  $A_0$  to be  $A_0[0, b-1]$ ,  $A_a$  to be  $A_a[0, b-1]$ ,  $u_0 = \max A_0$ , and  $u_a = \max A_a$ . Let  $u = \max\{u_0, u_a\}$ . Then

$$|2A| \succeq |2A_0| + |2A_a| + |A_0 + A_a| + |2A_b| + |u + A_b| \succeq 3|A|$$

together with  $|2A| \sim 3|A|$  imply that  $A_0$ ,  $A_a$ , and  $A_b$  are all full. Note that  $|A_0| \succ 0$  is always true. We can also assume  $|A_a| \succ 0$  because otherwise we have

$$|2A| \succeq |2A_0| + |a + A_0| + |b + A_0| + |u + A_b| + |2A_b| \succeq 4|A_0| + 3|A_b| \succ 3|A|.$$

Hence we can assume  $u, u-3, u-6 \in A_0 \cup A_a$ . Since there are  $u', u'' \in \{u, u-3, u-6\}$  such that  $u' + A_b$ ,  $u'' + A_b$ , and  $2A_b$  are pairwise disjoint, we have

$$\begin{aligned} |2A| &\succeq |2A_0| + |2A_a| + |A_0 + A_a| \\ &\quad + |u' + A_b| + |u'' + A_b| + |2A_b| \\ &\succeq 3|A_0| + 3|A_a| + 4|A_b| \succ 3|A|. \end{aligned}$$

Therefore, we can now assume that  $d' = 1$ . If  $A(b, H) \succ \frac{1}{2}(H - b)$ , then by Lemma 2.6 there exist  $b' \prec b \prec b'' \leq H$  such that  $A(b'', H) \sim \frac{1}{2}(H - b'')$  and  $A[b', b'']$  is a backward triangle. Since  $0, a, b \in A[0, b'']$ , then  $A[0, b'']$  is not a subset of a *b.p.* of difference 3. Clearly  $A[0, b'']$  is not a subset of a *b.p.* of difference 1 because  $d > 1$ . Hence we have  $|2A| \succ 3|A|$  by Lemma 3.5 and Lemma 2.3. So we can now assume that  $A(b, H) \preceq \frac{1}{2}(H - b)$ . Let's re-define  $A_0$  to be  $A_0[0, b-1]$ ,  $A_a$  to be  $A_a[0, b-1]$ ,  $u_0 = \max A_0$ , and  $u_a = \max A_a$ . Then by Lemma 2.10 we have that  $A_0$  and  $A_a$  are full and  $u_0, u_a \sim b$ . We can also assume  $A_a(l_a, u_a) \succ 0$  because otherwise  $(2A)(0, 2b) \succeq 4A(0, b)$ , which implies  $|2A| \succ 3|A|$ .

If  $A(b, H) \sim \frac{1}{2}(H - b)$ , then  $A(0, b) \prec \frac{1}{2}b$ . Since  $u_0 \sim b$ ,  $u_a \sim b$ , and  $d = 3$ , then  $u_a - l_a \prec \frac{1}{2}b$ , which implies  $l_a \succ \frac{1}{2}b$ . Hence

$$\begin{aligned} |2A| &\succeq |2A_0| + |2A_a| + |A_0 + A_a| \\ &\quad + |A[b, H] + A[b, H]| + |b + A_0[0, 2l_a - b]| \\ &\succeq 3A(0, b) + 3A(b, H) + A_0(0, 2l_a - b) \succ 3|A|. \end{aligned}$$

So we can assume  $A(b, H) \prec \frac{1}{2}(H - b)$ .

If  $\underline{d}_{b+U}(A) = 0$ , then there is  $x \in A$ ,  $x \succ b$  such that either  $x - b < u_0$  and  $A(b, x) \preceq \frac{1}{10}(x - b + 1)$ , or  $A(b, x) \sim 0$ . Let  $F \subseteq A[b, H]$  be a finite set such that

$b \in F$  and  $\gcd((F \cup A[x, H]) - b) = 1$ . Then

$$\begin{aligned}
|2A| &\succeq |2A_0| + |2A_a| + |A_0 + A_a| \\
&\quad + |(F \cup A[x, H]) + (F \cup A[x, H])| + |x + A_0[2u_0 - x, u_0]| \\
&\succeq 3A(0, b) + 3A(x, H) + A_0(2u_0 - x, u_0) \\
&\succeq 3A(0, b) + 3A(x, H) + \frac{1}{3}(x - u_0 + 1) \\
&\succeq 3|A| + \frac{1}{3}(x - u_0 + 1) - \frac{3}{10}(x - b + 1) \succ 3|A|.
\end{aligned}$$

If  $\underline{d}_{b+U}(A) > \frac{1}{2}$ , then there is  $x \succ b$  such that  $A[b, x]$  is a forward triangle. Clearly  $x \prec H$  and  $A(x, H) \prec \frac{1}{2}(H - x)$ . Let  $u' = \min\{u_0, u_a\}$ . Note that  $u' \sim b$ . By Lemma 3.5 and Lemma 2.3,  $|2A| \sim 3|A|$  implies that  $A[u', H]$  is either a full subset of a  $b.p.$  of difference 3 or a full subset of a  $b.p.$  of difference 1. Since  $u_0, u_a, b \in A[u', H]$ ,  $A[u', H]$  cannot be a subset of a  $b.p.$  of difference 3. Let  $A[b, H]$  be a full subset of the  $b.p.$   $[b, z] \cup [z', H]$  for some  $z < z'$  in  $A[b, H]$ . Note that  $2z \prec b + z'$ . Then

$$\begin{aligned}
|2A| &\succeq |2A_0| + |2A_a| + |A_0 + A_a| + |A[b, z] + A[b, z]| \\
&\quad + |A[b, z] + A[z', H]| + |A[z', H] + A[z', H]| + |z' + A[2z - z', b]| \\
&\succeq 3A(0, b) + 3A(b, H) + A(2z - z', b) \succ 3|A|.
\end{aligned}$$

Now we can assume  $0 < \underline{d}_{b+U}(A) \leq \frac{1}{2}$ . Suppose there is  $b' \sim b$  in  $A$  such that  $\gcd(A[b', H] - b') = d'' > 1$ . If  $d'' = 2$ , then there is  $b'' \sim b$  such that  $b'' - b'$  is odd. Hence  $|A[b'', H] + A[b', H]| \sim 3A(b'', H)$  implies that  $A[b', H]$  is full, which contradicts  $A(b, H) \prec \frac{1}{2}(H - b)$ . If  $d'' \geq 3$ , then  $|A[u', H] + A[u', H]| \succeq 4A(u', H) \succ 3A(u', H)$ , which contradicts  $|2A| \sim 3|A|$  by Lemma 2.3. So we can assume that there is an  $x \succ b$  in  $A$  such that  $\gcd(A[x, H] - x) = 1$ . Since  $A_0$  and  $A_a$  are full, we can assume  $u' - 3 \in A$ . Hence  $A \cap (u' - 3 + U)$  is neither a subset of an  $a.p.$  of difference  $> 1$  nor a subset of a  $(u' - 3 + U)$ -unbounded  $b.p.$  By Lemma 2.2 there exists  $y \in A$  with  $b \prec y < x$  such that  $A(y, H) \preceq \frac{1}{2}(H - y)$  and  $(2A)(2(u' - 2), 2y) \succ 3A(u' - 2, y)$ . By Lemma 2.3 we have  $|A[b, H] + A[b, H]| \succ 3A(b, H)$ , which implies  $|2A| \succ 3|A|$  again by Lemma 2.3.  $\square$ (Case 3.8.3)

**Case 3.8.4**  $d = 3$  and  $A(a, H) \succ \frac{1}{2}(H - a)$ .

By Lemma 2.8 and (IV) we can find  $0 \prec a' \prec a \prec a'' \leq H$  such that  $A[a', a'']$  is a backward triangle and  $A(a'', H) \sim \frac{1}{2}(H - a'')$ . By (IV) we have  $A(0, a'') \prec \frac{1}{2}a''$ . If  $a'' \sim H$ , then the lemma follows from Lemma 3.5. So we can assume  $a'' \prec H$ . Now

the lemma follows from Lemma 3.6 unless  $A[0, a'']$  is a subset of a *b.p.* of difference 3. Without loss of generality, let  $A[0, a'']$  be a subset of a *b.p.*  $I_0 \cup I_1$  of difference 3, where  $I_i = i + (3 * \mathbb{N})$  for  $i \in [0, 2]$ . Let  $A_i = A \cap I_i$  and  $b = \min(A \cap I_2)$ . Then  $b \succeq a''$ . Let  $l_1 = \min A_1[0, a'']$ . Then  $2l_1 \succ a''$ . If  $b \sim a''$ , then

$$(2A)(0, 2a'') \succeq 3A(0, a'') + |b + A_0[0, 2l_1 - b]| \succ 3A(0, a''),$$

which implies  $|2A| \succ 3|A|$  by Lemma 2.3. So we can assume  $b \succ a''$ .

**Subcase 3.8.4.1**  $A(a'', b) \prec \frac{1}{2}(b - a'')$ .

Then we have  $A(b, H) \succ \frac{1}{2}(H - b)$ . Hence we can find  $a'' \leq c \prec b \prec c' \leq H$  such that  $A[c, c']$  is a backward triangle from  $c$  to  $c'$  and  $A(c', H) \sim \frac{1}{2}(H - c')$ . Since  $A[0, c']$  contains two backward triangles, it cannot be a subset of a *b.p.* of difference  $d'$  for  $d' = 1$  or  $d' = 3$ . Hence by Lemma 3.5 we have  $|A[0, c'] + A[0, c']| \succ 3A(0, c')$ , which implies  $|2A| \succ 3|A|$ .  $\square$ (Subcase 3.8.4.1)

**Subcase 3.8.4.2**  $A(a'', b) \succeq \frac{1}{2}(b - a'')$ .

Let  $c = \max\{x \in [a'', b - 1] : x, x - 1 \in A\}$ . It is easy to see that  $c \succ a''$  and  $A(c, b) \preceq \frac{1}{3}(b - c)$ . Since  $|2A| \sim 3|A|$  implies  $|A[0, c] + A[0, c]| \sim 3A(0, c)$ , then we can assume that  $A[0, c]$  is full in the *b.p.*  $I_0 \cup I_1$ , which implies  $A(a'', c) \sim \frac{2}{3}(c - a'')$ . Hence  $A(c, H) \prec \frac{1}{2}(H - c)$ . So we can find  $m$  with  $a'' \prec m \prec c$  such that  $A(m, H) \prec \frac{1}{2}(H - m)$ . It is easy to show that there is a  $m' \prec H$  such that  $A[m, m']$  is a forward triangle. Since  $A[m, H]$  cannot be a full subset of a *b.p.* of difference 3, because it contains  $b$ , or 1 because  $A[m, b - 1] \prec b - m$ , then by Lemma 3.5 we have  $|A[m, H] + A[m, H]| \succ 3A(m, H)$ . Since  $A[0, m]$  is a subset of a *b.p.* of difference 3 we have  $|A[0, m] + A[0, m]| \succeq 3A(0, m)$ . By Lemma 2.3 we have  $|2A| \succ 3|A|$ .  $\square$ (Case 3.8.4)

**Case 3.8.5**  $d \geq 4$ .

Since  $A$  is not a subset of a *b.p.*, the number  $b = \min\{x \in A : x \notin \{0, a\} \pmod{d}\}$  is well defined. Let  $I_i = i + (d * \mathbb{N})$  and  $A_i = A \cap I_i$ . Let  $u_i = \max A_i[0, b - 1]$  and  $l_i = \min A_i[0, b - 1]$  for  $i = 0, a$ .

If  $b \equiv 2a \pmod{d}$ , then  $a + b \not\equiv 0 \pmod{d}$  because otherwise  $A[0, b]$  is a subset of an *a.p.* with difference  $\frac{d}{3} > 1$ . Hence we have either  $b \not\equiv 2a \pmod{d}$  or  $a + b \not\equiv 0 \pmod{d}$ . This implies

$$(b + A_0[0, u_0]) \cap (A[0, b - 1] + A[0, b - 1]) = \emptyset$$

or

$$(b + A_a[l_a, u_a]) \cap (A[0, b - 1] + A[0, b - 1]) = \emptyset.$$

If  $A(b, H) \succ \frac{1}{2}(H - b)$ , then there are  $0 \prec b' \prec b \prec b'' \leq H$  such that  $A(b'', H) \sim \frac{1}{2}(H - b'')$  and  $A[b', b'']$  is a backward triangle. If  $|2A| \sim 3|A|$ , then  $A[0, b'']$  is a full subset of either a *b.p.* of difference 3 or a *b.p.* of difference 1. But both contradict  $d \geq 4$ . So we can assume  $A(b, H) \leq \frac{1}{2}(H - b)$ .

Suppose  $\gcd(A[b, H] - b) = 1$ . By Lemma 2.10 we can show that  $u_a \sim b$ ,  $u_0 \sim b$ ,  $A_a(l_a, u_a) \succ 0$ ,  $A_0(0, u_0) \succ 0$ ,  $A_0[0, u_0]$  is full in  $[0, u_0]$ , and  $A_a[l_a, u_a]$  is full in  $[l_a, u_a]$ . Hence

$$\begin{aligned} |2A| &\geq |A_0[0, u_0] + A_0[0, u_0]| + |A_a[l_a, u_a] + A_a[l_a, u_a]| \\ &\quad + |A_0[0, u_0] + A_a[l_a, u_a]| + |A[b, H] + A[b, H]| \\ &\quad + \min\{|b + A_0[0, u_0]|, |b + A_a[l_a, u_a]|\} \\ &\geq 3A(0, b) + 3A(b, H) + \min\{A_0(0, u_0), A_a(l_a, u_a)\} \succ 3|A|. \end{aligned}$$

Suppose  $\gcd(A[b, H] - b) = d' > 1$ . Let  $c' = \max\{x \in A : \gcd(A[x, H] - x) < d'\}$ . Then we have

$$\begin{aligned} (c' + A[b, H]) \cap (A[b, H] + A[b, H]) &= \emptyset, \\ (c' + A[b, H]) \cap (A[0, b - 1] + A[0, b - 1]) &= \emptyset, \quad \text{and} \\ (c' + A[b + 1, H]) \cap (b + A[0, b - 1]) &= \emptyset. \end{aligned}$$

If  $A_a(l_a, u_a) \succ 0$ , then we have

$$\begin{aligned} |2A| &\geq |A_0[0, u_0] + A_0[0, u_0]| + |A_0[0, u_0] + A_a[l_a, u_a]| \\ &\quad + |A_a[l_a, u_a] + A_a[l_a, u_a]| + \min\{|b + A_0[0, u_0]|, |b + A_a[l_a, u_a]|\} \\ &\quad + |A[b, H] + A[b, H]| + |c' + A[b, H]| \succ 3|A|. \end{aligned}$$

If  $A_a(l_a, u_a) \sim 0$ , then we have

$$\begin{aligned} |2A| &\geq |A_0[0, u_0] + A_0[0, u_0]| + |a + A_0[0, u_0]| \\ &\quad + |b + A_0[0, u_0]| + |A[b, H] + A[b, H]| + |c' + A[b, H]| \\ &\geq 4A_0(0, u_0) + 3A(b, H) \succ 3|A|. \end{aligned}$$

This ends the proof of Case 3.8.5 as well as the lemma.  $\square$ (Lemma 3.8)

**Lemma 3.9** *Assume  $|A| \prec \frac{1}{2}H$ ,  $0 < \underline{d}_U(A) \leq \frac{1}{2}$ , and  $A \cap U$  is a subset of a  $U$ -unbounded b.p. of difference  $d$ . If  $A$  is not a subset of a b.p., then  $|2A| \succ 3|A|$ .*

**Proof:** Suppose  $A \cap U$  is a subset of the  $U$ -unbounded b.p.  $(d * U) \cup (a + (d * U))$  for some  $a \in A \cap U$ . Clearly  $d \neq 2$ . Let  $b = \min\{x \in A : x \notin \{0, a\} \pmod{d}\}$ . Note that  $A(0, b-1) \succ 0$ . By Lemma 3.8 we can assume  $\gcd(A[0, b-1]) = 1$ . If  $A(b, H) \succ \frac{1}{2}(H-b)$ , then we can find  $0 \prec b' \prec b \prec b'' \leq H$  such that  $A(b'', H) \sim \frac{1}{2}(H-b'')$  and  $A[b', b'']$  is a backward triangle from  $b'$  to  $b''$ . Note that  $A[0, b'']$  cannot be a subset of a b.p. of difference 3 because otherwise  $A \cap U$  is a subset of an a.p. of difference 3. By Lemma 2.3 we can assume  $|A[0, b''] + A[0, b'']| \sim 3A(0, b'')$ . By Lemma 3.5  $A[0, b'']$  is a full subset of a b.p.  $[0, c] \cup [c', b'']$ , which contradicts the assumption  $0 < \underline{d}_U(A) \leq \frac{1}{2}$ . So we can assume  $A[b, H] \preceq \frac{1}{2}(H-b)$ . If  $d > 3$ , then the proof of the lemma is the same as the proof of Case 3.8.5. Suppose  $d = 3$ . If  $b \prec H$  and  $\gcd(A[b, H] - b) > 1$ , then the lemma follows from Lemma 3.8. If  $\gcd(A[b, H] - b) = 1$  or  $b \sim H$ , then  $|2A| \sim 3|A|$  implies  $|A[0, b-1] + A[0, b-1]| \sim 3A(0, b-1)$ , which implies  $A[0, b-1]$  is a full subset of a b.p. of difference 3. Since  $A \cap U$  is already a subset of the b.p., then  $\underline{d}_U(A) = \frac{2}{3}$ , which contradicts  $\underline{d}_U(A) \leq \frac{1}{2}$ .  $\square$ (Lemma 3.9)

Now we summarize all the proofs in this section into a theorem, which takes care of the case in Theorem 1.7 under the condition (IV).

**Theorem 3.10** *Assume  $A \subseteq [0, H]$  and  $0, H \in A$ . Suppose  $\gcd(A) = 1$  and  $0 \prec |A| \prec \frac{1}{2}H$ . If  $A$  is not a subset of a b.p., then  $|2A| \succ 3|A|$ .*

**Proof:** By Lemma 3.8 we can assume that for every  $x \succ 0$ , if  $A(0, x) \succ 0$ , then  $\gcd(A[0, x]) = 1$  for every  $x \succ 0$ . If there is  $x \prec H$  in  $A$  such that  $A(x, H) \succ 0$  and  $\gcd(A[x, H] - x) > 1$ , then the theorem is true again by Lemma 3.8 with  $A$  replaced by  $H - A$ . So we can assume that for every  $x \prec H$  in  $A$ , if  $A(x, H) \succ 0$ , then  $\gcd(A[x, H] - x) = 1$ . We now divide the proof into four cases according to the value of  $\underline{d}_U(A)$ .

**Case 3.10.1**  $\underline{d}_U(A) > \frac{1}{2}$ .

Then there is  $c \succ 0$  such that  $A[0, c]$  is a forward triangle from 0 to  $c$ . Since  $|A| \prec \frac{1}{2}(H+1)$ , then  $c \prec H$ . Now the theorem follows from Lemma 3.5.

**Case 3.10.2**  $0 < \underline{d}_U(A) \leq \frac{1}{2}$ .

If  $A \cap U$  is a subset of an *a.p.* of difference  $> 1$ , then the theorem follows from Lemma 3.8. If  $A \cap U$  is a subset of a  $U$ -unbounded *b.p.*, then the theorem follows from Lemma 3.9. Otherwise by Lemma 2.2 we can find  $y \in A$  with  $0 \prec y \prec H$  such that  $A(y, H) \preceq \frac{1}{2}(H - y)$  and  $(2A)(0, 2y) \succ 3A(0, y)$ . If  $A(y, H) \sim 0$ , then the theorem is already true because  $|A| \sim A(0, y)$ . So we can assume  $A(y, H) \succ 0$ . If  $\gcd(A[y, H] - y) > 1$ , then the theorem follows from Lemma 3.8. If  $\gcd(A[y, H] - y) = 1$ , then the theorem follows from Lemma 2.3.  $\square$ (Case 3.10.2)

**Case 3.10.3**  $\underline{d}_U(A) = 0$  and there is  $x \succ 0$  such that  $A(0, x) \sim 0$ .

By Lemma 2.6 we can find such  $x \in A$  such that for any  $y \succ x$ ,  $A(x, y) \succ 0$ .

If  $A(x, H) \succ \frac{1}{2}(H - x)$ , then we can find  $0 \prec c' \prec x \prec c \leq H$  such that,  $A(c, H) \sim \frac{1}{2}(H - c)$ , and  $A[c', c]$  is a backward triangle. If  $c \sim H$ , then the theorem follows from Lemma 3.5. Suppose  $c \prec H$ . Note that  $A[0, c]$  cannot be a full subset of a *b.p.* of difference 3 by the condition of the case. Hence the lemma follows from Lemma 3.6.

If  $A(x, H) \preceq \frac{1}{2}(H - x)$  and  $A[x, H]$  is a subset of an *a.p.* of difference  $> 1$ , then the theorem follows from Lemma 3.8. If  $A(x, H) \preceq \frac{1}{2}(H - x)$  and  $A[x, H]$  is not a subset of an *a.p.* of difference  $> 1$ , then

$$|2A| \succeq A(x, 2x) + |A[x, H] + A[x, H]| \succeq 3|A| + A(x, 2x) \succ 3|A|.$$

$\square$ (Case 3.10.3)

**Case 3.10.4**  $\underline{d}_U(A) = 0$  and for every  $x \succ 0$ ,  $A(0, x) \succ 0$ .

By symmetry we can also assume  $\underline{d}_{H-U}(A) = 0$  and for every  $y \prec H$ ,  $A(y, H) \succ 0$ .

Let  $|A| \sim \alpha H$ . Then  $0 < \alpha < \frac{1}{2}$ . By Lemma 2.6 there is  $b \succ 0$  in  $A$  such that  $A(0, b) \sim \alpha b$  and  $A(0, x) \prec \alpha x$  for every  $0 \prec x \prec b$ . By the assumption of this case, we have  $A(0, b) \succ 0$  and  $A(b, H) \succ 0$ . If there is  $0 \prec x \prec H$  such that  $A[0, x]$  or  $A[x, H]$  is a subset of an *a.p.* of difference  $> 1$ , then the theorem follows from Lemma 3.8. Note that  $\underline{d}_{b-U}(A) \geq \alpha$  by the choice of  $b$ . By Lemma 2.3 we can assume  $|A[0, b] + A[0, b]| \sim 3A(0, b)$ . By Case 3.10.1 and Case 3.10.2 for  $A[0, b]$  we can assume that  $A[0, b]$  is a subset of a *b.p.* of difference  $d$ . Clearly  $A[0, b]$  is a full subset of the *b.p.* If  $d = 1$ , then  $A[0, b]$  is a full subset of  $[0, x] \cup [x', b]$ , which implies either  $A(0, x') \sim 0$  or  $\underline{d}_U(A) = 1$ . Each of them contradicts the assumption of the case. If  $d > 1$ , then  $\underline{d}_U(A) = \frac{2}{d}$ , which is again a contradiction to the assumption of the case.  $\square$ (Theorem 3.10)

## 4 Second Step: When $\frac{|A|}{H}$ is almost $\frac{1}{2}$ .

In this section we again assume  $A \subseteq [0, H]$ ,  $0, H \in A$ , and  $\gcd(A) = 1$ . In addition we also assume

$$|A| \sim \frac{1}{2}H, \quad (\text{V})$$

$$A \text{ is not a subset of a } b.p., \quad (\text{VI})$$

and

$$|2A| = 3|A| - 3 + b \quad (\text{VII})$$

for  $0 \leq b \sim 0$ . (VII) implies  $|2A| \sim 3|A|$ . Under the condition above we want to prove

$$H + 1 \leq 2|A| - 1 + 2b. \quad (\text{VIII})$$

Without loss of generality, we can assume that

$$|A| \leq \frac{1}{2}(H + 1) \quad (\text{IX})$$

because otherwise (VIII) is trivially true. In this section the letter  $b$  is reserved only for the purpose of (VII).

**Lemma 4.1** *Let  $z \in [0, H] \setminus A$  and let  $A' = A \cup \{z\}$ . Suppose  $|(2A') \setminus (2A)| \leq 2$  and  $|2A'| = 3|A'| - 3 + b'$ . If*

$$H + 1 > 2|A| - 1 + 2b, \quad (\text{X})$$

*then  $0 \leq b' \leq b - 1$ ,  $|A'| \leq \frac{1}{2}(H + 1)$ , and  $H + 1 > 2|A'| - 1 + 2b'$ .*

**Proof:** If  $b = 0$ , then  $|2A| = 3|A| - 3$ . By Theorem A.2 we have  $H + 1 \leq 2|A| - 1$ , which contradicts  $|A| \leq \frac{1}{2}(H + 1)$ . So we can assume  $b > 0$ . By the assumption of the lemma we have  $H + 1 \geq 2|A| + 2b$ . Hence  $|A'| = |A| + 1 \leq \frac{1}{2}(H + 1) - b + 1 \leq \frac{1}{2}(H + 1)$ . Since

$$\begin{aligned} |2A'| &= 3|A'| - 3 + b' \\ &\leq |2A| + 2 = 3|A| - 1 + b \\ &= 3|A'| - 3 + (b - 1), \end{aligned}$$

then  $b' \leq b - 1$ . If  $b' < 0$ , then by Theorem A.1  $A'$  is a subset of an  $a.p.$  of length  $\leq 2|A'| - 3 = 2|A| - 1$ , which implies  $H + 1 \leq 2|A| - 1$ , a contradiction to (X).



Hence  $b' \geq 0$ . Finally  $H + 1 > 2|A| - 1 + 2b = 2|A'| - 1 + 2(b - 1) \geq 2|A'| - 1 + 2b'$ .  
 $\square$ (Lemma 4.1)

**Lemma 4.2** *If there is a  $\sim 0$  such that  $A[a + 1, H]$  is a subset of a b.p. of difference 3, then  $H + 1 \leq 2|A| - 1 + 2b$ .*

**Proof:** Without loss of generality we can assume  $a \in A$  and  $A[a, H]$  is not a subset of a b.p. of difference 3. Fix  $j \in [-2, 0]$  such that  $A[a + 1, H] \subseteq A_0 \cup A_1$  where  $A_i = A \cap J_i$  and  $J_i = j + i + (3 * \mathbb{N})$  for  $i = 0, 1, 2$ . For  $i = 0, 1, 2$  let  $l_i = \min A_i$  and  $u_i = \max A_i$ . For  $i = 0, 1$  let  $I_i = J_i[0, u_i]$  and let  $I_2 = J_2[l_2, u_2]$ . Clearly  $a = u_2 \sim 0$ . Since

$$|2A| \succeq |2A_0| + |2A_1| + |A_0 + A_1| \succeq 3|A_0| + 3|A_1| \sim 3|A|,$$

then we have  $|2A_i| \sim 2|A_i|$  for  $i = 0, 1$ . By Theorem A.1 we have that  $A_i$  is full for  $i = 0, 1$ . Note that  $|A_i| \succ 0$  for  $i = 0, 1$  by (V). If  $l_0 \succ 0$  and  $l_0 \geq l_1$ , then  $|2A| \succeq 3|A| + |a + A_1[l_1, 2l_0]| \succ 3|A|$ . By symmetry we can also prove that  $l_1 \succ 0$  and  $l_1 \geq l_0$  together are impossible. So we can assume  $l_0 \sim 0$  and  $l_1 \sim 0$ . Without loss of generality we assume  $u_0 = H$ . Then  $|A| \sim \frac{1}{2}(H + 1)$  implies  $u_1 \sim \frac{H}{2}$ .

Suppose the lemma is not true. Then we can assume that (VII), (IX), and (X) are true. Without loss of generality we can assume that  $|A|$  is the maximum among all the sets in  $I_0 \cup I_1 \cup I_2$  containing the original set and satisfying (VII), (IX), and (X).

**Claim 4.2.1:** If  $l_i \prec z \prec u_i$  and  $z \equiv j + i \pmod{3}$  for  $i = 0$  or  $i = 1$ , then  $z \in A$ .

Proof of Claim 4.2.1: Suppose not and let  $A' = A \cup \{z\}$ . By Lemma 4.1 and by the maximality of  $|A|$  we need only to show that  $|(2A') \setminus (2A)| \leq 2$  for a contradiction. First let  $z \equiv j \pmod{3}$ . Let  $y \in A'$ .

If  $y \in A_0 \cup \{z\}$  and  $y \prec u_0$ , then  $A_0[y + 1, y + t] \cap (y + z - A_0[z - t, z - 1]) \neq \emptyset$  for some  $0 \prec t \prec \min\{u_0 - y, z\}$ , which implies  $y + z \in (2A_0) \subseteq (2A)$ , by the pigeonhole principle. If  $y \in A_0 \cup \{z\}$  and  $y \sim u_0$ , then  $A_0[y - t, y - 1] \cap (y + z - A_0[z + 1, z + t]) \neq \emptyset$  for some  $0 \prec t \prec \min\{y, u_0 - z\}$ , which implies  $y + z \in (2A_0) \subseteq (2A)$ . If  $y \in A_1$  and  $y \prec u_1$ , then  $A_1[y + 1, y + t] \cap (y + z - A_0[z - t, z - 1]) \neq \emptyset$  for some  $0 \prec t \prec \min\{u_1 - y, z\}$ , which implies  $y + z \in A_1 + A_0 \subseteq (2A)$ . If  $y \in A_1$  and  $y \sim u_1$ , then  $A_1[y - t, y - 1] \cap (y + z - A_0[z + 1, z + t]) \neq \emptyset$  for some  $0 \prec t \prec \min\{y, u_0 - z\}$ , which implies  $y + z \in A_1 + A_0 \subseteq (2A)$ . If  $y \in A_2$ , then  $0 \prec y + z \prec u_0 \sim 2u_1$ . Since  $A_1$  is

full, then there are  $x \sim 2l_1$  and  $x' \sim 2u_1$  such that  $(J_0 + J_2)[x, x'] \subseteq (2A_1)$ . Hence  $y + z \in (2A_1) \subseteq (2A)$ . By all the arguments above we have  $(2A') = (2A)$ .

For the case that  $z \equiv j + 1 \pmod{3}$  the proof is similar.  $\square$ (Claim 4.2.1)

**Claim 4.2.2:** If  $\frac{u_1}{2} < z < u_1$  and  $z \equiv j + 1 \pmod{3}$ , then  $z \in A$ .

Proof of Claim 4.2.2: Suppose not and let  $z$  be the least number such that the claim is not true. By Claim 4.2.1 we have  $z \sim u_1$ . Let  $A' = A \cup \{z\}$ . It suffices to show  $|(2A') \setminus (2A)| \leq 2$ . Let  $y \in A'$ .

If  $y \in A_0$  and  $y \prec u_0$ , then  $A_0[y + 1, y + t] \cap (y + z - A_1[z - t, z - 1]) \neq \emptyset$  for some  $0 \prec t \prec \min\{u_0 - y, z\}$ , which implies  $y + z \in A_0 + A_1 \subseteq (2A)$ . If  $y \in A_0$  and  $u_0 \sim y < u_0$ , then  $y + z = u_0 + (z - (u_0 - y)) \in A_0 + A_1 \subseteq (2A)$  by the minimality of  $z$ . If  $y \in A_1 \cup \{z\}$  and  $y \prec u_1$ , then  $A_1[y + 1, y + t] \cap (y + z - A_1[z - t, z - 1]) \neq \emptyset$  for some  $0 \prec t \prec \min\{u_1 - y, z\}$ , which implies  $y + z \in (2A_1) \subseteq (2A)$ . If  $y \in A_1 \cup \{z\}$  and  $u_1 \sim y < u_1$ , then  $y + z = u_1 + (z - u_1 + y) \in (2A_1) \subseteq (2A)$ . If  $y \in A_2$ , then  $y + z \in (2A_0)$  by the facts that  $A_0$  is full and  $2l_0 \prec y + z \prec 2u_0$ . Hence  $((2A') \setminus (2A)) \subseteq \{z + u_0, z + u_1\}$ .  $\square$ (Claim 4.2.2)

**Claim 4.2.3:** If  $\frac{u_0}{2} < z < u_0$  and  $z \equiv j \pmod{3}$ , then  $z \in A$ .

Proof of Claim 4.2.3: Suppose not and let  $z$  be the least number such that the claim is not true. By Claim 4.2.1 we have  $z \sim u_0$ . Let  $A' = A \cup \{z\}$ . Again it suffices to show  $|(2A') \setminus (2A)| \leq 2$ . Let  $y \in A'$ .

If  $y \in A_0 \cup \{z\}$  and  $y \prec u_0$ , then  $A_0[y + 1, y + t] \cap (y + z - A_0[z - t, z - 1]) \neq \emptyset$  for some  $0 \prec t \prec \min\{u_0 - y, z\}$ , which implies  $y + z \in (2A_0) \subseteq (2A)$ . If  $y \in A_0 \cup \{z\}$  and  $u_0 \sim y < u_0$ , then  $y + z = u_0 + (z - (u_0 - y)) \in (2A_0) \subseteq (2A)$  by the minimality of  $z$ . If  $y \in A_1$  and  $y \prec u_1$ , then  $A_1[y + 1, y + t] \cap (y + z - A_0[z - t, z - 1]) \neq \emptyset$  for some  $0 \prec t \prec \min\{u_1 - y, z\}$ , which implies  $y + z \in A_1 + A_0 \subseteq (2A)$ . If  $y \in A_1$  and  $u_1 \sim y \leq u_1$ , then  $y + z = (y - (u_0 - z)) + u_0 \in A_1 + A_0 \subseteq (2A)$ . Note that  $y - u_0 + z \in A_1$  by Claim 4.2.1 and Claim 4.2.2. If  $y \in A_2$  and  $y < u_2$ , then  $y + z = u_2 + (z - u_2 + y) \in A_2 + A_0 \subseteq (2A)$ . Hence  $((2A') \setminus (2A)) \subseteq \{u_0 + z, u_2 + z\}$ .  $\square$ (Claim 4.2.3)

**Claim 4.2.4:** There is  $i \in \{0, 1\}$  such that  $l_i < z < \frac{u_i}{2}$  and  $z \equiv j + i \pmod{3}$  imply  $z \in A$ .

Proof of Claim 4.2.4: Suppose not and let

$$z_i = \max\{z \in [0, H] : l_i < z < \frac{u_i}{2}, z \equiv j + i \pmod{3} \text{ and } z \notin A_i\}$$

for  $i = 0, 1$ . By Claim 4.2.1 we have  $z_i \sim 0$ .

**Subclaim 4.2.4.1:**  $z_0 - l_0 = z_1 - l_1$ .

Proof of Subclaim 4.2.4.1: Suppose the subclaim is not true. Without loss of generality we assume  $z_0 - l_0 < z_1 - l_1$ . Let  $A' = A \cup \{z_1\}$ . Since  $z_0 + l_1 < z_1 + l_0$ , then  $z_1 + l_0 = (z_0 + t) + l_1 \in A_0 + A_1 \subseteq (2A)$  for  $t = (z_1 + l_0) - (z_0 + l_1)$  by the maximality of  $z_0$ . By the similar arguments in the last several claims we have  $((2A') \setminus (2A)) \subseteq \{z_1 + l_1, z_1 + l_2\}$ . This contradicts the maximality of  $|A|$  by Lemma 4.1. By a symmetric argument we can show  $z_0 - l_0 > z_1 - l_1$  is also impossible.  $\square$ (Subclaim 4.2.4.1)

**Case 4.2.4.1:**  $z_0 + l_2 < z_1 + l_1$ .

Let  $A' = A \cup \{z_1\}$ . Note that  $z_0 + l_2 \equiv z_1 + l_1 \pmod{3}$ . Then  $z_1 + l_1 = z_0 + t + l_2 \in A_0 + A_2 \subseteq (2A)$  for  $t = (z_1 + l_1) - (z_0 + l_2) > 0$ . Hence by the similar arguments as in Subclaim 4.2.4.1 we can show  $((2A') \setminus (2A)) \subseteq \{z_1 + l_0, z_1 + l_2\}$ .  $\square$ (Case 4.2.4.1)

**Case 4.2.4.2:**  $z_0 + l_2 > z_1 + l_1$ .

Let  $A' = A \cup \{z_0\}$ . Then  $z_0 + l_2 = z_1 + t + l_1 \in (2A_1)$  for  $t = (z_0 + l_2) - (z_1 + l_1) > 0$  by the maximality of  $z_1$ . Hence  $((2A') \setminus (2A)) \subseteq \{z_0 + l_0, z_0 + l_1\}$ .  $\square$ (Case 4.2.4.2)

Following the two cases above we have  $z_0 + l_2 = z_1 + l_1$ . By symmetric arguments we can also show that  $z_1 + l_2 = z_0 + l_0$ . Subtracting the second equality from the first we have  $z_0 - z_1 = z_1 - z_0 + l_1 - l_0$ . This implies  $2(z_0 - z_1) = l_1 - l_0$ . But by Subclaim 4.2.4.1 we have  $z_0 - z_1 = -(l_1 - l_0)$ . Hence  $l_1 - l_0 = 0$ , which is absurd.  $\square$ (Claim 4.2.4)

**Claim 4.2.5:** If  $l_i < z < \frac{u_i}{2}$  and  $z \equiv j + i \pmod{3}$ , then  $z \in A$  for  $i = 0, 1$ .

Proof of Claim 4.2.5: Suppose the claim is not true. By Claim 4.2.4 we can assume, without loss of generality, that  $l_0 < z < \frac{u_0}{2}$  and  $z \equiv j \pmod{3}$  imply  $z \in A$ .

Let  $z_1 = \max\{z \in [0, H] : l_1 < z < \frac{u_1}{2}, z \equiv j + 1 \pmod{3} \text{ and } z \notin A_1\}$ . Let  $A' = A \cup \{z_1\}$ . Then  $z_1 + l_0 = l_1 + (l_0 + z_1 - l_1) \in A_1 + A_0 \subseteq (2A)$ . It is now easy to see that  $((2A') \setminus (2A)) \subseteq \{z_1 + l_1, z_1 + l_2\}$ . This contradicts the maximality of  $|A|$ .  $\square$ (Claim 4.2.5)

**Claim 4.2.6:**  $l_0 + l_2 \geq 2l_1 - 3$  and  $l_1 + l_2 \geq 2l_0 - 3$ .

Proof of Claim 4.2.6: By symmetry we need only to show the first inequality. Assume it is not true and we have  $l_0 + l_2 \leq 2l_1 - 6$ . Then  $l_1 \notin [0, 2]$ . Let  $z = l_1 - 3$  and  $A' = A \cup \{z\}$ . Let  $y \in A'$ .

If  $y \in A_0$  and  $y \neq l_0$ , then  $y + z = (y - 3) + l_1 \in A_0 + A_1 \subseteq (2A)$ . If  $y \in A_1$  and  $y \succ l_1$ , then  $A_1[y - t, y - 1] \cap (y + z - A_1[z + 1, z + t]) \neq \emptyset$  for some  $0 \prec t \prec \min\{y, u_1\}$ , which implies  $y + z \in (2A_1) \subseteq (2A)$ . If  $y \in A_1 \cup \{z\}$  and  $y \sim l_1$ , then  $y + z = (l_0 + t) + l_2 \in A_0 + A_2$  for  $t = (y + z) - (l_0 + l_2) \geq (2l_1 - 6) - (l_0 + l_2) \geq 0$ . If  $y \in A_2$  and  $y > l_2$ , then  $y + z = l_2 + (z + y - l_2) \in A_2 + A_1 \in (2A)$ . Hence  $((2A') \setminus (2A)) \subseteq \{z + l_0, z + l_2\}$ , a contradiction to the maximality of  $|A|$  by Lemma 4.1.  $\square$ (Claim 4.2.6)

**Claim 4.2.7:** Let  $z = u_2 + 3$  and  $A' = A \cup \{z\}$ . Then (VII), (IX), and (X) maintain true with  $A$  and  $b$  being replaced by  $A'$  and  $b' \leq b - 1$ , respectively.

Proof of Claim 4.2.7: By Lemma 4.1 it suffices to prove  $|(2A') \setminus (2A)| \leq 2$ . Suppose not. We derive a contradiction.

**Subclaim 4.2.7.1:**  $l_0 + l_1 - 6 \geq u_2 + l_2$ .

Prove of Subclaim 4.2.7.1: Assume the subclaim is not true. So we have  $u_2 + l_2 \geq l_0 + l_1 - 3$ . Let  $y \in A'$

If  $y \in A_0$  and  $y < u_0$ , then  $y + z = (y + 3) + (z - 3) \in A_0 + A_2 \subseteq (2A)$ . If  $y \in A_1$  and  $y < u_1$ , then  $y + z = (y + 3) + (z - 3) \in A_1 + A_2 \subseteq (2A)$ . If  $y \in A_2 \cup \{z\}$ , then  $y + z = y + u_2 + 3 \geq l_2 + u_2 + 3 \geq l_0 + l_1$ . Hence  $y + z = (l_0 + t) + l_1 \in A_0 + A_1 \subseteq (2A)$  for  $t = (y + z) - (l_0 + l_1) \geq 0$ . Now we have  $((2A') \setminus (2A)) \subseteq \{z + u_0, z + u_1\}$ , which contradicts the assumption that  $|(2A') \setminus (2A)| \leq 2$  is not true.  $\square$ (Subclaim 4.2.7.1)

We now ready to derive a contradiction. By Claim 4.2.6 and Subclaim 4.2.7.1 we have  $2(l_0 + l_1 + l_2) \geq 2l_0 + 2l_1 + u_2 + l_2$ . This implies  $l_2 \geq u_2$ . Hence  $A_2 = \{l_2\}$ . So by Subclaim 4.2.7.1 again we have  $l_0 + l_1 - 6 \geq 2l_2$ . Since  $0 = \min A$ , then  $0 \in \{l_0, l_1, l_2\}$ . We want to show  $l_2 = 0$ . Suppose  $l_0 = 0$ . Then by Claim 4.2.6 and Subclaim 4.2.7.1 we have  $l_2 \geq 2l_1 - 3$  and  $l_1 - 6 \geq 2l_2$ . So  $l_1 - 6 \geq 2(2l_1 - 3) = 4l_1 - 6$  implies  $l_1 \geq 4l_1$ , which is absurd because  $l_0 = 0$  implies  $l_1 > 0$ . By symmetry we also have  $l_1 > 0$ . Hence  $l_2 = 0$ .

By Claim 4.2.6 and Subclaim 4.2.7.1 again we have  $l_0 \geq 2l_1 - 3$  and  $l_1 \geq 2l_0 - 3$ , which imply  $l_0 + l_1 \geq 2(l_0 + l_1) - 6$  or equivalently  $l_0 + l_1 \leq 6$ . Hence by Subclaim 4.2.7.1 we have  $l_0 + l_1 = 6$ . Note that  $l_0 \not\equiv l_1 \pmod{3}$ . So  $(l_0, l_1) \neq (3, 3)$ . Assume  $l_0 < l_1$ . The  $(l_0, l_1) = (2, 4)$  or  $(l_0, l_1) = (1, 5)$ . But each of the two cases contradicts the inequality  $l_0 \geq 2l_1 - 3$  in Claim 4.2.6 with  $l_2 = 0$ .  $\square$ (Claim 4.2.7)

By Claim 4.2.7 we can add  $u_2 + 3, u_2 + 6, u_2 + 9, \dots$  successively to  $A$  to form a set  $A'$  so that (VII), (IX), and (X) maintain true with  $A$  and  $b$  being replaced

by  $A'$  and  $b'$ , respectively. However, (IX) will eventually be violated in this process.  $\square$ (Lemma 4.2)

**Lemma 4.3** *Let  $A_i = \{z \in A : z \equiv i \pmod{3}\}$  for  $i = 0, 1, 2$ . If there is  $i \in [0, 2]$  such that  $\max A_i - \min A_i \sim 0$ , then  $H + 1 \leq 2|A| - 1 + 2b$ .*

**Proof:** The ideas are the same as in the proof of Lemma 4.2. We will describe the steps without too much technical details. Let  $I_i = i + (3 * \mathbb{N})$ ,  $A_i = A \cap I_i$ ,  $l_i = \min A_i$ , and  $u_i = \max A_i$  for  $i = 0, 1, 2$ .

Without loss of generality let  $u_2 \sim l_2$ . By Lemma 4.2 we can assume  $0 \prec l_2 \leq u_2 \prec u_2$ . Since

$$|2A| \succeq |2A_0| + |2A_1| + |A_0 + A_1| \succeq 3|A_0| + 3|A_1| \sim 3|A|,$$

then  $|2A| \sim 3|A|$  implies that  $A_i$  is full for  $i = 0, 1$ . Note that  $|A_0| \succeq \frac{H}{6}$ ,  $|A_1| \succeq \frac{H}{6}$ , and  $|A_0 \cup A_1| \sim \frac{1}{2}H$ .

Suppose the lemma is not true. Without loss of generality we can assume that  $|A|$  is the maximum among all the sets in  $\cup_{i=0}^2 I_i[l_i, u_i]$  containing the original set and satisfying (VII), (IX), and (X). Without loss of generality let's assume  $l_0 = 0$ .

**Case 4.3.1**  $H = u_0$ .

If  $l_2 \prec 2l_1$ , then

$$|2A| \succeq |2A_0| + |2A_1| + |A_0 + A_1| + |l_2 + A_0[0, 2l_1 - l_2]| \succ 3|A|.$$

So we can assume  $2l_1 \preceq l_2$ . By symmetry we can assume  $u_2 + H \preceq 2u_1$ . Since  $u_1 - l_1 \sim \frac{1}{2}H$  by (V), we have  $2l_1 \sim l_2$  and  $u_2 + H \sim 2u_1$ . Hence

$$A_2 + A_1 \subseteq (2A_0)[l_1 + l_2, u_1 + u_2] = (2I_0)[l_1 + l_2, u_1 + u_2].$$

This implies  $A_1 = I_1[l_1, u_1]$  by Lemma 4.1 and the maximality of  $|A|$ . Then we can show  $A_0 = I_0[l_0, u_0]$  again by Lemma 4.1 and the maximality of  $|A|$ . Furthermore, we can show  $A_2 = I_2[l_2, u_2]$  by the fact that  $(2I_2)[2l_2, 2u_2] \subseteq A_0 + A_1$  and by Lemma 4.1. Now we add  $z = u_2 + 3$ ,  $z = u_2 + 6$ ,  $z = u_2 + 9$ , etc. successively to  $A$  so that the set maintains satisfying (VII), (IX), and (X). However, this process will eventually violate (IX).  $\square$ (Case 4.3.1)

**Case 4.3.2**  $H = u_1$ .

We can again show that  $2l_1 \preceq l_2$  and  $u_2 + H \preceq 2u_0$  because otherwise we can show  $|2A| \succ 3|A|$ . Since  $H - l_1 + u_0 \sim \frac{3}{2}H$ , then  $u_0 - l_1 \sim \frac{H}{2}$ , which implies  $2l_1 \sim l_2$  and  $u_2 + H \sim 2u_0$ . Again assume that  $A \subseteq \cup_{i=0}^2 I_i[l_i, u_i]$  has the maximum cardinality among the sets satisfying (VII), (IX), and (X). Then we can show  $A_i = I_i[l_0, u_0]$  for  $i = 0, 1, 2$ . Finally we can again add  $z = u_2 + 3, z = u_2 + 6, z = u_2 + 9$ , etc. successively to  $A$  so that the set maintains satisfying (VII), (IX), and (X). Again this process will eventually violate (IX).  $\square$ (Lemma 4.3)

**Lemma 4.4** *Suppose there are  $0 \prec a \sim c \prec H$  such that  $A[0, a]$  is a backward triangle as well as a subset of a b.p. of difference 3 and  $A[c, H]$  is a forward triangle as well as a subset of a b.p. of difference 3. Then  $H + 1 \leq 2|A| - 1 + 2b$ .*

**Proof:** The ideas are again the same as in the proof of Lemma 4.2. Let  $I_i = (i + (3 * \mathbb{N}))$  for  $i = 0, 1, 2$ . By Lemma 4.3 we can assume that  $A[0, a] \subseteq I_0 \cup I_1$  and  $A[c, H] \subseteq I_0 \cup I_2$ . For  $i = 0, 1, 2$  let  $A_i = A \cap I_i$ ,  $l_i = \min A_i$ ,  $u_i = \max A_i$ , and  $J_i = I_i \cap [l_i, u_i]$ . Then we have  $u_1 \sim l_2 \sim a$ . Suppose the lemma is not true. Then  $A$  satisfies (VII), (IX), and (X). We again assume the maximality of  $|A|$  for  $A \subseteq J_0 \cup J_1 \cup J_2$  satisfying (VII), (IX), and (X). By Lemma 4.1 we can prove that for each  $x$ ,  $l_i \prec x \prec u_i$  implies  $x \in A$ . Then we can prove  $A_i = J_i$  by the same ideas as in the proof of Lemma 4.2. Now we can add  $u_1 + 3, u_1 + 6, u_1 + 9, \dots$  successively to  $A$  such that the conditions (VII), (IX), and (X) maintain true. But this process will eventually violate (IX).  $\square$ (Lemma 4.4)

**Lemma 4.5** *If there is  $x \sim 0$  in  $A$  such that  $\gcd(A[x, H] - x) = d > 1$ , then  $H + 1 \leq 2|A| - 1 + 2b$ .*

**Proof:** Since (V) is true, then  $d = 2$ . Let  $c = \min\{x \in A : \gcd(A[x, H] - x) = 2\}$ . Then  $c \sim 0$ ,  $c > 0$ , and  $A[c, H]$  is full. Let  $E$  be the set of all even numbers and  $O$  be the set of all odd numbers. Let  $A_e = A \cap E$  and  $A_o = A \cap O$ . Let  $l_e = \min A_e$ ,  $l_o = \min A_o$ ,  $u_e = \max A_e$ , and  $u_o = \max A_o$ .

**Case 4.5.1**  $c$  is even.

Then  $l_e = 0$ ,  $u_e = H$ , and  $A_e$  is full. We want to show  $H + 1 \leq 2|A| - 1 + 2b$ .

Let  $|2A_e| = 2|A_e| - 1 + b_e$ . Then  $b_e \sim 0$ . By Theorem A.1 we have  $\frac{H}{2} + 1 \leq |A_e| + b_e$ . On the other hand, by Theorem A.4,

$$|A_e + A_o| \geq \min\{|A_e| + 2|A_o| - 2, \frac{H}{2} + |A_o|\}.$$

If  $\frac{H}{2} + |A_o| \geq |A_e| + 2|A_o| - 2$ , then

$$\begin{aligned} 3|A| - 3 + b &= |2A| \\ &\geq 2|A_e| - 1 + b_e + |A_e| + 2|A_o| - 2 \\ &\geq 3|A| - 3 + b_e - |A_o|. \end{aligned}$$

This implies  $b_e \leq b + |A_o|$ . Hence  $\frac{H}{2} + 1 \leq |A_e| + b_e \leq |A| + b$ , which implies  $H + 1 \leq 2|A| - 1 + 2b$ .

If  $\frac{H}{2} + |A_o| < |A_e| + 2|A_o| - 2 = |A| + |A_o| - 2$ , then  $|A| > \frac{H}{2} + 2$ , which contradicts (IX).  $\square$ (Case 4.5.1)

**Case 4.5.2**  $c$  is odd.

Clearly,  $0 = l_e$  and  $H = u_o$ . If  $l_o > u_e$ , then  $A$  is a subset of a  $b.p$ . Hence we can assume  $l_o < u_e$  and need to show  $H + 1 \leq 2|A| - 1 + 2b$ . Suppose  $H + 1 > 2|A| - 1 + 2b$ . Let

$$S = \{x \in O[l_o, u_o] : \frac{A_o(l_o, x)}{x - l_o + 1} \leq \frac{1}{4}\}.$$

If  $S \neq \emptyset$ , let  $l' = (\max S) + 1$ . Otherwise, let  $l' = l_o$ . Let

$$T = \{x \in O[l_o, u_o] : \frac{A_o(x, u_o)}{u_o - x + 1} \leq \frac{1}{4}\}.$$

If  $T \neq \emptyset$ , let  $u' = (\min T) - 1$ . Otherwise, let  $u' = u_o$ . Note that  $l', u' \in A_o$ . Since  $A_o$  is full, then  $l' \sim l_o$  and  $u' \sim u_o$ . For each  $x \in O[l', u_o]$ , we have  $\frac{A_o(l_o, x)}{x - l_o + 1} > \frac{1}{4}$  and for any  $x \in O[l_o, u']$ , we have  $\frac{A_o(x, u_o)}{u_o - x + 1} > \frac{1}{4}$ . By the pigeonhole principle

$$E[l_o + l', u_o + u'] \subseteq (2A_o).$$

Let  $p = O(l', u') - A_o(l', u')$ ,  $p' = O(l', l' + u_e) - A_o(l', l' + u_e)$ , and  $p'' = p - p'$ .

Let  $\bar{A}_o = A_o \cup O[l', u']$ . Then  $2\bar{A}_o = 2A_o$ . Hence

$$|2\bar{A}_o| = |2A_o| = 2|A_o| - 1 + b_o = 2|\bar{A}_o| - 1 + b_o - 2p.$$

This implies, by Theorem A.1,

$$\frac{1}{2}(H - l_o) + 1 \leq |\bar{A}_o| + b_o - 2p = |A_o| + b_o - p.$$

**Subcase 4.5.2.1**  $p'' \geq 2|A_e|$ .

Since

$$\begin{aligned}
3|A| - 3 + b &= |2A| \\
&\geq |2A_o| + |A_e + A_o| \\
&\geq 2|A_o| - 1 + b_o + |0 + A_o[l_o, l' + u_e - 2]| + |u_e + A_o[l', H]| \\
&\geq 2|A_o| - 1 + b_o + A_o(l_o, l' + u_e - 2) + A_o(l', H) \\
&\geq 3|A| - 3 + b_o - 3|A_e| + A_o(l', l' + u_e) + 1,
\end{aligned}$$

then  $b_o \leq b + 3|A_e| - A_o(l', l' + u_e) - 1$ . Hence

$$\begin{aligned}
\frac{1}{2}(H - l_o) + 1 &\leq |A_o| + b_o - p \\
&\leq |A_o| + b + 3|A_e| - A_o(l', l' + u_e) - 1 - p \\
&\leq |A| + b + (2|A_e| - p'') - O(l', l' + u_e) - 1 \\
&\leq |A| + b - \frac{1}{2}u_e - 2 < |A| + b - \frac{1}{2}u_e.
\end{aligned}$$

This implies  $H - l_o + 2 \leq 2|A| + 2b - u_e$  and

$$H + 1 \leq 2|A| - 1 + 2b - (u_e - l_o) < 2|A| - 1 + 2b.$$

□(Subcase 4.5.2.1)

**Subcase 4.5.2.2**  $p'' < 2|A_e|$ .

Since  $u' - (l' + u_e) \succ 0$  and  $u_e \sim 0$ , then there exist  $t, t' \in O[l' + 2u_e + 1, u' - 2u_e - 1]$  such that  $t' - t > \max\{u_o - u', l' - l_o\}$  and  $|A_e| + A_o(t - u_e, t' + 2u_e) > \frac{3}{2}u_e + \frac{t' - t}{2} + 1$  because otherwise we can find three disjoint intervals of length  $t' - t + 3u_e + 1$  for  $t' - t = 2 + \max\{u_o - u', l' - l_o\}$  in  $[l' + u_e + 2, u']$  such that each contains at least  $|A_e|$  elements from the set  $O \setminus A_o$ . This contradicts the assumption  $p'' < 2|A_e|$ . Suppose  $H + 1 > 2|A| - 1 + 2b$ . We want to derive a contradiction by induction on the size of the counterexamples  $A' \supseteq A$ .

Suppose  $A' = A_e \cup A'_o$  is the set with the maximum cardinality  $|A'|$  such that  $A_o \subseteq A'_o \subseteq O[l_o, u_o]$ ,  $|2A'| = 3|A'| - 3 + b'$  for  $0 \leq b' \leq b$ , and  $H + 1 > 2|A'| - 1 + 2b'$ .

**Claim 4.5.2.2.1**  $A'_o[t, t' + u_e] = O[t, t' + u_e]$ .

Proof of Claim 4.5.2.2.1: Suppose the claim is not true and let  $g \in O[t, t' + u_e] \setminus A'_o$ . Let  $A''_o = A'_o \cup \{g\}$  and  $A'' = A_e \cup A''_o$ . Note that if  $x \in A''_o[l_o, u_o]$ , then



$l_o + l' < x + g < u_o + u'$ . Hence  $x + g \in (2A_o) \subseteq (2A')$ . Let  $x \in A_e[0, u_e]$ . Since

$$\begin{aligned} & |A_e| + A'_o(g + x - u_e, g + x) \\ &= |A_e| + A'_o(t - u_e, t' + 2u_e) \\ &\quad - A'_o(t - u_e, g + x - u_e - 2) - A'_o(g + x + 2, t' + 2u_e) \\ &> \frac{3}{2}u_e + \frac{t' - t}{2} + 1 - u_e - \frac{t' - t}{2} \\ &= \frac{1}{2}u_e + 1, \end{aligned}$$

then  $(g + x - A_e) \cap A'_o[g + x - u_e, g + x] \neq \emptyset$ . Hence  $g + x \in A'_o + A_e \subseteq (2A')$ . Now we have that  $(2A'') = (2A')$ , which contradicts the maximality of  $|A'|$  by Lemma 4.1.  $\square$ (Claim 4.5.2.2.1)

**Claim 4.5.2.2.2**  $A'_o[t' + u_e + 2, H] = O[t' + u_e + 2, H]$ .

Suppose the claim is not true and let  $g = \min(O[t' + u_e + 2, H] \setminus A'_o)$ . Let  $A''_o = A'_o \cup \{g\}$  and  $A'' = A_e \cup A''_o$ . Then  $g > t' + u_e$ . Note that if  $x \in A''_o[l', u']$ , then  $l_o + l' \leq x + g \leq u_o + u'$ . Hence  $x + g \in (2A')$ . If  $x \in A''_o[l_o, l' - 2]$ , then  $y = g - (l' - x) \in A'_o$  by the minimality of  $g$  and Claim 4.5.2.2.1. Hence  $x + g = l' + y \in (2A')$ . If  $x \in A''_o[u' + 2, H - 2]$ , then  $x + g = H + (g - (H - x)) \in (2A')$ . If  $x \in A_e[0, u_e - 2]$ , then  $x + g = u_e + (g - (u_e - x)) \in (2A')$ . From the arguments above, we have  $(2A'') \setminus (2A') \subseteq \{H + g, u_e + g\}$ , which contradicts the maximality of  $|A'|$  by Lemma 4.1.  $\square$ (Claim 4.5.2.2.2)

**Claim 4.5.2.2.3**  $A'_o[l_o, t - 2] = O[l_o, t - 2]$ .

Proof of Claim 4.5.2.2.3: Suppose the claim is not true and let  $g = \max(O[l_o, t - 2] \setminus A'_o)$ . Let  $A'' = A' \cup \{g\}$ . Then  $(2A'') \setminus (2A') \subseteq \{0 + g, l_o + g\}$ , which contradicts the maximality of  $|A'|$ .  $\square$ (Claim 4.5.2.2.3)

By the three claims above we have  $A'_o = O[l_o, u_o]$ .

Without loss of generality, we can assume that our original counterexample  $A$  satisfies  $A_o = O[l_o, u_o]$ . Hence  $|A_o| = \frac{H-l_o}{2} + 1$  or  $2|A_o| - 1 = H - l_o + 1$ . Note also that since  $H + 1 = H - l_o + 1 + l_o = 2|A| - 1 + l_o - 2|A_e|$ , we can assume that  $l_o \geq 2|A_e| + 1$ .

**Claim 4.5.2.2.4**  $2l_o \geq u_e + 4$ .

Proof of Claim 4.5.2.2.4: Suppose  $2l_o \leq u_e + 2$ . For each even number  $z \in E[u_e, H - 1]$  let  $A_z = A \cup E[u_e, z]$ . Let

$$S = \{z \in E[u_e, H - 1] : |2A_z| = 3|A_z| - 3 + b_z \geq 3|A_z| - 3,$$

$$b_z \leq b, \text{ and } H + 1 > 2|A_z| - 1 + 2b_z\}.$$

Clearly,  $u_e \in S$ . For each  $z \succ 0$ ,  $|A_z| = |A| + E(u_e + 2, z) \succ \frac{1}{2}H$ , which implies  $z \notin S$ . Let  $z_0 = \max S \sim 0$ . We now derive a contradiction.

By Theorem A.2 we can assume  $b_{z_0} > 0$ . Note that since  $|A_{z_0+2}| = |A_{z_0}| + 1$ , we have

$$H + 1 > 2|A_{z_0}| - 1 + 2b_{z_0} = 2|A_{z_0+2}| - 1 + 2(b_{z_0} - 1).$$

Since for each  $x \in A_{z_0+2} \cap E$ ,  $x + z_0 + 2 \in E[2l_o, 2H]$ . Hence  $(2A_{z_0+2}) \setminus (2A_{z_0}) \subseteq \{z_0 + 2 + H\}$ , which contradicts the maximality of  $z_0$  by Lemma 4.1.  $\square$ (Claim 4.5.2.2.4)

Let  $d' = \gcd(A_e)$ . Then  $d'$  must be an even number. Let  $q = \min(A_e[l_o + 1, u_e])$  and  $q' = \max(A_e[0, l_o - 1])$ .

**Subsubcase 4.5.2.2.1**  $d' \geq 4$ .

First we can assume that  $u_e = q$  by the following argument.

Let  $A' = A \setminus A_e[q+2, u_e]$ . Then  $|A'| = |A| - A_e(q+2, u_e)$ . Note that  $A_e(q+2, u_e) \leq \frac{u_e - q}{d'} \leq \frac{u_e - q}{4}$ . Since  $(2A') \subseteq (2A) \setminus O[q+2+H, u_e+H]$  and  $O[q+2+H, u_e+H] \subseteq A_e + A_o \subseteq (2A)$ , then there is  $b' \geq 0$  such that

$$\begin{aligned} 3|A'| - 3 + b' &= |2A'| \\ &\leq |2A| - \frac{u_e - q}{2} \\ &= 3|A| - 3 + b - \frac{u_e - q}{2} \\ &= 3|A'| - 3 + b - \frac{u_e - q}{2} + 3A_e(q+2, u_e) \\ &\leq 3|A'| - 3 + b + A_e(q+2, u_e). \end{aligned}$$

This shows  $b' \leq b + A_e(q+2, u_e)$ . So

$$\begin{aligned} H + 1 &> 2|A| - 1 + 2b \\ &= 2|A'| + 2A_e(q+2, u_e) - 1 + 2b \\ &= 2|A'| - 1 + 2(b + A_e(q+2, u_e)) \\ &\geq 2|A'| - 1 + 2b'. \end{aligned}$$

Hence  $A'$  is the desired counterexample. Now identify  $A$  with  $A'$ .

**Subsubsubcase 4.5.2.2.1.1**  $u_e = l_o + 1$ .

Since  $q' \leq u_e - d' < u_e - 2$ , then  $q' + u_e \leq 2l_o - 2$  and  $q' + u_e \notin A_e[0, q'] + A_e[0, q']$ . Hence we have  $(2A_e)(0, 2l_o - 2) \geq 2A_e(0, q')$ . So we have

$$\begin{aligned}
|2A| &= |2A_o| + |A_o + A_e| + (2A_e)(0, 2l_o - 2) \\
&\geq 2|A_o| - 1 + |A_o| + \frac{u_e}{2} + 2A_e(0, q') \\
&\geq 3|A| - 1 - 3|A_e| + \frac{u_e}{2} + 2|A_e| - 2 \\
&\geq 3|A| - 3 - |A_e| + \frac{u_e}{2}.
\end{aligned}$$

This shows  $-|A_e| + \frac{u_e}{2} \leq b$ . On the other hand,

$$\begin{aligned}
H + 1 &= 2|A_o| - 1 + l_o \\
&= 2|A| - 1 + l_o - 2|A_e| \\
&\leq 2|A| - 1 + l_o + 2(b - \frac{u_e}{2}) \\
&= 2|A| - 1 + 2b + (l_o - u_e) < 2|A| - 1 + 2b.
\end{aligned}$$

This contradicts the assumption that  $H + 1 > 2|A| - 1 + 2b$ .  $\square$ (Subsubsubcase 4.5.2.2.1.1)

**Subsubsubcase 4.5.2.2.1.2**  $u_e > l_o + 1$ .

Then we have

$$\begin{aligned}
|2A| &= |2A_o| + |A_o + A_e| + (2A_e)(0, 2l_o - 2) \\
&\geq 2|A_o| - 1 + |A_o| + \frac{u_e}{2} + 2A_e(0, q') - 1 \\
&\geq 3|A| - 1 - 3|A_e| + \frac{u_e}{2} + 2|A_e| - 3 \\
&\geq 3|A| - 3 - |A_e| + \frac{u_e}{2} - 1.
\end{aligned}$$

This shows  $-|A_e| + \frac{u_e}{2} - 1 \leq b$ . On the other hand,

$$\begin{aligned}
H + 1 &= 2|A_o| - 1 + l_o \\
&= 2|A| - 1 + l_o - 2|A_e| \\
&\leq 2|A| - 1 + l_o + 2(b - \frac{u_e}{2} + 1) \\
&= 2|A| - 1 + 2b + (l_o - u_e + 2) \\
&\leq 2|A| - 1 + 2b,
\end{aligned}$$

by the assumption  $u_e > l_o + 1$ . This again contradicts  $H + 1 > 2|A| - 1 + 2b$ .  $\square$ (Subsubcase 4.5.2.2.1)

**Subsubcase 4.5.2.2.2**  $d' = 2$ .

**Subsubsubcase 4.5.2.2.2.1**  $|2A_e| = 2|A_e| - 1 + b_e$  for some  $b_e < |A_e| - 2$ .

Since  $A_e \subseteq E$ , then by Theorem A.1  $\frac{u_e}{2} + 1 \leq |A_e| + b_e$ . We also have

$$\begin{aligned} |2A| &= |2A_o| + |2A_e| + |A_o + A_e| - (2A_e)(2l_o, 2u_e) \\ &\geq 2|A_o| - 1 + 2|A_e| - 1 + b_e + |A_o| + \frac{u_e}{2} - \frac{2u_e - 2l_o}{2} - 1 \\ &\geq 3|A| - 3 - |A_e| + b_e + \frac{u_e}{2} + l_o - u_e, \end{aligned}$$

which implies  $-|A_e| + b_e - \frac{u_e}{2} + l_o \leq b$ . Hence

$$\begin{aligned} \frac{u_e}{2} + 1 + \frac{H - l_o}{2} + 1 \\ &\leq |A_e| + b_e + |A_o| = |A| + b_e \\ &\leq |A| + b + |A_e| + \frac{u_e}{2} - l_o. \end{aligned}$$

This implies  $u_e + 2 + H - l_o + 2 \leq 2|A| + 2b + 2|A_e| + u_e - 2l_o$ . Hence

$$H + 1 \leq 2|A| - 1 + 2b + (2|A_e| - l_o - 2) < 2|A| - 1 + 2b$$

because

$$|A| = |A_e| + |A_o| = |A_e| + \frac{H - l_o}{2} + 1 \leq \frac{1}{2}(H + 1)$$

implies  $l_o > 2|A_e|$ .  $\square$ (Subsubsubcase 4.5.2.2.2.1)

**Subsubsubcase 4.5.2.2.2.2**  $|2A_e| \geq 3|A_e| - 3$  and  $u_e - l_o \leq 2|A_e| - 4$ .

Since

$$\begin{aligned} |2A| &= 2|A_o| - 1 + |A_o| + \frac{u_e}{2} + (2A_e)(0, 2l_o - 2) \\ &= 3|A| - 3 + \frac{u_e}{2} - 3|A_e| + (2A_e)(0, 2l_o - 2) + 2, \end{aligned}$$

then  $\frac{u_e}{2} - 3|A_e| + (2A_e)(0, 2l_o - 2) + 2 \leq b$ . Hence  $\frac{H - l_o}{2} + 1 = |A_o|$  implies

$$\begin{aligned} H + 1 &= 2|A| - 1 + l_o - 2|A_e| \\ &\leq 2|A| - 1 + l_o + 2(b - \frac{u_e}{2} + 2|A_e| - (2A_e)(0, 2l_o - 2) - 2) \\ &\leq 2|A| - 1 + 2b + (4|A_e| - 4 - (u_e - l_o) - 2(2A_e)(0, 2l_o - 2)). \end{aligned}$$

Since

$$\begin{aligned} (2A_e)(0, 2l_o - 2) &= (2A_e)(0, 2u_e) - (2A_e)(2l_o, 2u_e) \\ &\geq 3|A_e| - 3 - (u_e - l_o + 1) = 3|A_e| - 4 - u_e + l_o, \end{aligned}$$

then

$$\begin{aligned}
& 4|A_e| - 4 - (u_e - l_o) - 2(2A_e)(0, 2l_o - 2) \\
& \leq 4|A_e| - 4 - (u_e - l_o) - 2(3|A_e| - 4 - u_e + l_o) \\
& \leq -2|A_e| + 4 + u_e - l_o \leq 0.
\end{aligned}$$

Hence  $H + 1 \leq 2|A| - 1 + 2b$ , a contradiction.  $\square$ (Subsubsubcase 4.5.2.2.2.2)

**Subsubsubcase 4.5.2.2.2.3**  $|2A_e| \geq 3|A_e| - 3$  and  $u_e - l_o > 2|A_e| - 4$ .

This time we use the fact that  $(2A_e)(0, 2l_o - 2) \geq |0 + A_e| = |A_e|$  implied by Claim 4.5.2.2.4. Since

$$\begin{aligned}
|2A| & \geq 3|A| - 3 + \frac{u_e}{2} - 3|A_e| + (2A_e)(0, 2l_o - 2) + 2 \\
& \geq 3|A| - 3 + \frac{u_e}{2} - 2|A_e| + 2,
\end{aligned}$$

then  $\frac{u_e}{2} - 2|A_e| + 2 \leq b$ . Hence

$$\begin{aligned}
H + 1 & = 2|A| - 1 + l_o - 2|A_e| \\
& \leq 2|A| - 1 + l_o + 2(b - \frac{u_e}{2} + |A_e| - 2) \\
& = 2|A| - 1 + 2b + (l_o - u_e + 2|A_e| - 4) \\
& < 2|A| - 1 + 2b.
\end{aligned}$$

This ends the proof of the lemma.  $\square$ (Lemma 4.5)

**Lemma 4.6** *Suppose  $A = A_e \cup A_o$ , where  $A_e = A \cap E$  is the set of all even numbers in  $A$  and  $A_o = A \cap O$  is the set of all odd numbers in  $A$ . Let  $u_i = \max A_i$  for  $i = e, o$  and  $l_o = \min A_o$ . If (a)  $u_e = H$  and  $u_o - l_o \sim 0$  or (b)  $H = u_o$ ,  $0 \prec l_o < u_e \prec H$ , and  $u_e - l_o \sim 0$ , then  $H + 1 \leq 2|A| - 1 + 2b$ .*

**Proof:** The proof of (a) of the lemma is identical to the proof of Case 4.5.1. We sketch the proof of (b) using Lemma 4.1. It is easy to see that  $A_e$  is full in  $E[0, u_e]$  and  $A_o$  is full in  $O[l_o, H]$ . Suppose the lemma is not true. Following the same ideas as in the proof of Claim 4.2.1, Claim 4.2.2, and Claim 4.2.3, we can assume that  $A = E[0, u_e] \cup O[l_o, H]$ . However, this implies

$$|A| = \frac{u_e}{2} + 1 + \frac{H - l_o}{2} + 1 = \frac{H + (u_e - l_o)}{2} + 2 > \frac{H + 1}{2},$$

which contradicts (IX).  $\square$ (Lemma 4.6)

**Lemma 4.7** *If  $A$  is a forward triangle from 0 to  $H$ , then  $H + 1 \leq 2|A| - 1 + 2b$ .*

**Proof:** Assume the lemma is not true and we need to derive a contradiction. Clearly we have  $\underline{d}_U(A) \geq \frac{1}{2}$ . Furthermore we can assume  $\underline{d}_U(A) \geq \frac{2}{3}$  by the following argument: If  $\underline{d}_U(A) < \frac{2}{3}$ , then there exists  $y' \succ 0$  in  $A$  such that for all  $0 \prec y \leq y'$

$$(2A)(0, 2y) \sim 2y = 3 \cdot \frac{2}{3}y \succ 3A(0, y).$$

If for every  $x \succ 0$  in  $A$  we have  $\gcd(A[x, H] - x) > 1$ , then there is  $u \sim 0$  in  $A$  such that  $\gcd(A[u, H] - u) > 1$  by Lemma 2.7. This implies that for each  $x \succ 0$  we have  $A(0, x) \preceq \frac{1}{2}x$ , which contradicts that  $A[0, H]$  is a forward triangle. Hence we can choose  $y$  with  $0 \prec y \leq y'$  in  $A$  such that  $\gcd(A[y, H] - y) = 1$ . By Lemma 2.3 we have  $|2A| \succ 3|A|$ . Let

$$z = \max\{x \in U : A(0, x - 1) \leq \frac{1}{2}x\}.$$

Note that the smallest possible value of  $z$  is 0. Note also that  $z$  is well defined because  $\underline{d}_U(A) \geq \frac{2}{3}$ . It is easy to check that  $z \in A$ ,  $A(0, z - 1) = \frac{1}{2}z$ , and for every  $x \geq z$  in  $U$  we have  $A(z, x) > \frac{1}{2}(x - z + 1)$  by the maximality of  $z$ .

Define  $a$  by

$$a = \min\{x \in [z, H] : A(z, x) \leq \frac{1}{2}(x - z + 1)\}.$$

The number  $a$  is well defined by the fact that  $A(0, z - 1) = \frac{1}{2}z$ ,  $H \in A$ , and  $|A| \leq \frac{1}{2}(H + 1)$ . It is also easy to check that  $a \sim H$ ,  $A[z, a]$  is a forward triangle from  $z$  to  $a$ ,  $a \notin A$ , and  $A(z, a) = \frac{1}{2}(a - z + 1)$ . If  $z \leq x < a$ , then  $A(z, x) > \frac{1}{2}(x - z + 1)$  by the minimality of  $a$ . Let  $a' = \max(A[z, a])$ .

**Claim 4.7.1:**  $[z, a + z - 1] \subseteq (2A)$ .

Proof of Claim 4.7.1: Let  $x \in [z, a + z - 1]$ . If  $x < 2z$ , then  $x \prec H \sim a$ . Hence

$$A(0, x) = A(0, z - 1) + A(z, x) > \frac{1}{2}z + \frac{1}{2}(x - z + 1) = \frac{1}{2}(x + 1).$$

This implies  $A[0, x] \cap (x - A[0, x]) \neq \emptyset$ . Hence  $x \in (2A)$ . If  $x \geq 2z$ , then  $A(z, x - z) > \frac{1}{2}(x - 2z + 1)$ . Hence  $A[z, x - z] \cap (x - A[z, x - z]) \neq \emptyset$ . This again implies  $x \in (2A)$ .  $\square$ (Claim 4.7.1)

Let  $c = \min(A[a + 1, H])$ . If  $2a' < c$ , then  $a' + H \sim a' + c \prec 2c$ . Hence  $A$  is a subset of the *b.p.*  $[0, a'] \cup [c, H]$ . So from now on in this lemma we can assume  $2a' \geq c$ .

**Claim 4.7.2:** Suppose  $2a' \geq a + z$  and  $a + z < x \leq \min\{2a', c + z\}$ . If  $(2A)(a + z, x - 1) < \frac{1}{2}(x - a - z)$ , then  $x \in (2A)$ .

Proof of Claim 4.7.2: Since  $A(z, a) = \frac{1}{2}(a - z + 1)$  and  $A(a' + 1, a) = \emptyset$ , then

$$\begin{aligned} A(z + a - a', a') &= A(z, a) - A(z, z + a - a' - 1) \\ &\geq \frac{1}{2}(a - z + 1) - (a - a') = \frac{1}{2}(2a' - z - a + 1). \end{aligned}$$

Note that  $x - a' \leq a'$ . Since  $a' + A[a + z - a', x - 1 - a'] \subseteq (2A)[a + z, x - 1]$ , then  $A(a + z - a', x - 1 - a') < \frac{1}{2}(x - a - z)$ . Hence

$$\begin{aligned} A(x - a', a') &= A(a + z - a', a') - A(a + z - a', x - a' - 1) \\ &> \frac{1}{2}(2a' - z - a + 1) - \frac{1}{2}(x - a - z) = \frac{1}{2}(2a' - x + 1). \end{aligned}$$

This shows that  $A[x - a', a'] \cap (x - A[x - a', a']) \neq \emptyset$ , which implies  $x \in (2A)$ .  $\square$ (Claim 4.7.2)

Let  $S = (2A)[0, z - 1] \setminus A[0, z - 1]$ .

**Claim 4.7.3:** Suppose  $2a' < c + z$  and  $\max\{a + z, 2a'\} \leq x < c + z - 1$ . If  $(2A)(x + 1, c + z - 1) < \frac{1}{2}(c + z - x - 1)$ , then  $x \in (2A)$  or  $x - c \in S$ .

Proof of Claim 4.7.3: Assume  $(2A)(x + 1, c + z - 1) < \frac{1}{2}(c + z - x - 1)$ . Since  $c + A[x + 1 - c, z - 1] \subseteq (2A)[x + 1, c + z - 1]$ , then  $A(x + 1 - c, z - 1) < \frac{1}{2}(c + z - x - 1)$ . Hence

$$\begin{aligned} A(0, x - c) &= A(0, z - 1) - A(x + 1 - c, z - 1) \\ &> \frac{1}{2}z - \frac{1}{2}(c + z - x - 1) = \frac{1}{2}(x - c + 1). \end{aligned}$$

Hence  $A[0, x - c] \cap (x - c - A[0, x - c]) \neq \emptyset$ . This shows  $x - c \in (2A)[0, z - 1]$ . If  $x - c \in A[0, z - 1]$ , then  $x \in c + A[0, z - 1] \subseteq (2A)$ . If  $x - c \notin A[0, z - 1]$ , then  $x - c \in S$ .  $\square$ (Claim 4.7.3)

**Claim 4.7.4:**  $(2A)(0, c + z) \geq 3A(0, z - 1) + 2A(z, a) - 1 + \frac{1}{2}(c - a + 1)$ .

Proof of Claim 4.7.4: The proof is divided into three cases for  $2a' \geq c + z$ ,  $2a' \leq a + z$ , and  $a + z < 2a' < c + z$ .

**Case 4.7.4.1:**  $2a' \geq c + z$ .

By Claim 4.7.2 we have  $(2A)(a + z, c + z - 1) \geq \frac{1}{2}(c - a - 1)$  (this can be proven by induction on  $x \in [a + z, c + z - 1]$ ). Hence  $(2A)(a + z, c + z) \geq \frac{1}{2}(c - a + 1)$  because

$c + z \in (2A)$ . So we have

$$\begin{aligned}
(2A)(0, c + z) &= (2A)(0, z - 1) + (2A)(z, a + z - 1) + (2A)(a + z, c + z) \\
&\geq A(0, z - 1) + a + \frac{1}{2}(c - a + 1) \\
&= 3A(0, z - 1) + 2A(z, a) - 1 + \frac{1}{2}(c - a + 1).
\end{aligned}$$

□(Case 4.7.4.1)

**Case 4.7.4.2:**  $2a' \leq a + z$ .

By Claim 4.7.3 we have  $(2A)(a + z, c + z) \geq \frac{1}{2}(c - a + 1) - |S|$ . Hence

$$\begin{aligned}
(2A)(0, c + z) &= (2A)(0, z - 1) + a + (2A)(a + z, c + z) \\
&\geq A(0, z - 1) + |S| + 2A(0, z - 1) \\
&\quad + 2A(z, a) - 1 + \frac{1}{2}(c - a + 1) - |S| \\
&= 3A(0, z - 1) + 2A(z, a) - 1 + \frac{1}{2}(c - a + 1).
\end{aligned}$$

□(Case 4.7.4.2)

**Case 4.7.4.3:**  $a + z < 2a' < c + z$ .

By Claim 4.7.2 we have  $(2A)(a + z, 2a') \geq \frac{1}{2}(2a' - a - z + 1)$  and by Claim 4.7.3 we have  $(2A)(2a' + 1, c + z) \geq \frac{1}{2}(c + z - 2a') - |S|$ . Hence

$$\begin{aligned}
(2A)(0, c + z) &= A(0, z - 1) + |S| + 2A(0, z - 1) \\
&\quad + 2A(z, a) - 1 + (2A)(a + z, 2a') + (2A)(2a' + 1, c + z) \\
&\geq 3A(0, z - 1) + 2A(z, a) - 1 + \frac{1}{2}(2a' - a - z + 1) + \frac{1}{2}(c + z - 2a') \\
&\geq 3A(0, z - 1) + 2A(z, a) - 1 + \frac{1}{2}(c - a + 1).
\end{aligned}$$

□(Claim 4.7.4)

We now prove the lemma. The proof is divided into two cases. The first case is easy and the second case is hard.

**Case 4.7.1**  $H - c \leq 2A(c + 1, H) = 2A(c, H) - 2$ .



Since  $c - z + A(c, H) \succ A(z, c) + 2A(c, H) - 2$ , then by Theorem A.4 we have  $|A[z, c] + A[c, H]| \geq A(z, c) + 2A(c, H) - 2$ . Hence

$$\begin{aligned}
3|A| - 3 + b &= |2A| \\
&\geq (2A)(0, c + z) - 1 + |A[z, c] + A[c, H]| + |H + A[c + 1, H]| \\
&\geq 3A(0, z - 1) + 2A(z, a) - 1 + \frac{1}{2}(c - a + 1) - 1 \\
&\quad + A(z, c) + 2A(c, H) - 2 + A(c + 1, H) \\
&= 3A(0, z - 1) + 3A(z, a) + 3A(c, H) - 4 + \frac{1}{2}(c - a + 1) \\
&= 3|A| - 3 + \frac{1}{2}(c - a - 1).
\end{aligned}$$

This shows  $\frac{1}{2}(c - a - 1) \leq b$ . Hence

$$\begin{aligned}
H + 1 &= H - c + c - a - 1 + a - z + z + 2 \\
&\leq 2A(c, H) - 2 + 2b + 2A(z, a) - 1 + 2A(0, z - 1) + 2 \\
&= 2|A| - 1 + 2b.
\end{aligned}$$

□(Case 4.7.1)

**Case 4.7.2**  $H - c \geq 2A(c + 1, H) + 1 = 2A(c, H) - 1$ .

Note that Case 4.7.1 covers the case for  $c = H$ . So we can assume  $c < H$ . First we prove a claim.

**Claim 4.7.2.1** If  $(2A)(c + z, 2H) \geq \frac{1}{2}(H - c) + A(c, H) + A(z, H) - 1$ , then  $H + 1 \leq 2|A| - 1 + 2b$ .

Proof of Claim 4.7.2.1 By the assumption we have

$$\begin{aligned}
3|A| - 3 + b &= |2A| \\
&\geq 3A(0, z - 1) + 2A(z, a) - 1 + \frac{1}{2}(c - a + 1) + (2A)(c + z, 2H) - 1 \\
&\geq 3A(0, z - 1) + 2A(z, a) - 1 + \frac{1}{2}(c - a + 1) \\
&\quad + \frac{1}{2}(H - c) + A(c, H) + A(z, H) - 2 \\
&\geq 3|A| - 3 + \frac{1}{2}(H - c) - A(c, H) + \frac{1}{2}(c - a + 1).
\end{aligned}$$

Hence  $\frac{1}{2}(H - c) - A(c, H) + \frac{1}{2}(c - a + 1) \leq b$ . This implies

$$H + 1 = H - c + c - a + 1 + a - z + z$$

$$\begin{aligned}
&\leq 2(b + A(c, H)) + 2A(z, a) - 1 + 2A(0, z - 1) \\
&= 2|A| - 1 + 2b.
\end{aligned}$$

□(Claim 4.7.2.1)

By Claim 4.7.2.1 we need only to show that  $(2A)(c+z, 2H) \geq \frac{1}{2}(H-c) + A(c, H) + A(z, H) - 1$  is true. We divide the proof into cases according to the structural properties of  $A[c, H]$ .

**Subcase 4.7.2.1**  $\gcd(A[c, H] - c) = 1$ .

Note that  $c \neq H$  and  $c \neq H - 1$  by the condition of Case 4.7.2. Since  $\gcd(A[c, H] - c) = 1$ , then  $A(c, H) \geq 3$  and  $H - c \geq 2A(c + 1, H) + 1 \geq 5$ . Since  $d_U(A) \geq \frac{2}{3}$ , there is  $t \in U$  such that for all  $u \geq t$  in  $U$  we have  $\frac{A(t, u)}{u-t+1} \gtrsim \frac{2}{3}$ . Let  $u = t + H - c - 1$ . Then there is a non-negative infinitesimal  $r$  such that  $\frac{A(t, u)}{u-t+1} = \frac{A(t, u)}{H-c} \geq \frac{2}{3} - r$ . By Theorem A.4 we have

$$\begin{aligned}
(2A)(c+z, 2H) &\geq |A[c, H] + A[z, H]| \\
&\geq |c + A[z, t-1]| + |A[c, H] + A[t, u]| + |H + A[u+1, H]| \\
&\geq A(z, t-1) + \min\{H - c + A(t, u), A(c, H) + 2A(t, u) - 2\} + A(u+1, H).
\end{aligned}$$

Since

$$\begin{aligned}
H - c &= \frac{1}{2}(H - c) + \frac{1}{2}(H - c) \\
&\geq \frac{1}{2}(H - c) + \frac{1}{2}(2A(c, H) - 1) \\
&= \frac{1}{2}(H - c) + A(c, H) - \frac{1}{2}
\end{aligned}$$

and

$$\begin{aligned}
A(t, u) &\geq \left(\frac{2}{3} - r\right)(H - c) \\
&\geq \frac{1}{2}(H - c) + \left(\frac{1}{6} - r\right)(H - c) \geq \frac{1}{2}(H - c) + \frac{5}{6} - 5r,
\end{aligned}$$

by the fact that  $H - c \geq 5$ , then we have

$$\begin{aligned}
&\min\{H - c + A(t, u), A(c, H) + 2A(t, u) - 2\} \\
&\geq \frac{1}{2}(H - c) + A(c, H) + A(t, u) - \frac{7}{6} - 5r.
\end{aligned}$$

Hence

$$\begin{aligned}
(2A)(c+z, 2H) &\geq |A[c, H] + A[z, H]| \\
&\geq A(z, t-1) + \frac{1}{2}(H-c) + A(c, H) + A(t, u) \\
&\quad - \frac{7}{6} - 5r + A(u+1, H) \\
&= A(z, H) + A(c, H) + \frac{1}{2}(H-c) - 1 - \left(\frac{1}{6} + 5r\right).
\end{aligned}$$

Since  $(2A)(c+z, 2H)$  is an integer, then we have

$$(2A)(c+z, 2H) \geq A(z, H) + A(c, H) + \frac{1}{2}(H-c) - 1.$$

Now the lemma follows from Claim 4.7.2.1.  $\square$ (Subcase 4.7.2.1)

**Subcase 4.7.2.2**  $\gcd(A[c, H] - c) = d > 1$  but  $d \neq 3$ .

Again let  $t \in U \cap A$  be such that  $\frac{A(t, u)}{u-t+1} \gtrsim \frac{3}{2}$  for all  $u \geq t$  in  $U$ .

**Claim 4.7.2.2.1** For each  $x \in A[c, H]$ ,  $(2A)(t+c, t+x-1) \geq A(c, x-1) + \frac{1}{2}(x-c)$ .

Proof of Claim 4.7.2.2.1: We prove the claim by induction on  $x \geq c$ .

The case of  $x = c$  is trivially true.

Suppose the claim is true for  $y \in A[c, H-1]$ . Let  $x = \min A[y+1, H]$ . Since  $[y+1, x-1] \cap A = \emptyset$ , and  $x-y = nd$  for some  $n > 0$  implies  $x-y = 2$  or  $x-y \geq 4$ , then

$$\begin{aligned}
(2A)(t+y, t+x-1) &\geq |y + A[t, t+(x-y)-1]| \\
&= A(t, t+(x-y)-1) \geq \left(\frac{2}{3} - r\right)(x-y) \\
&= \frac{1}{2}(x-y) + \left(\frac{1}{6} - r\right)(x-y),
\end{aligned}$$

for some non-negative infinitesimal  $r$ . Since either  $x-y = 2$  or  $x-y > 3$ , and  $(2A)(t+y, t+x-1)$  is an integer, then we have

$$(2A)(t+y, t+x-1) \geq \frac{1}{2}(x-y) + 1 = \frac{1}{2}(x-y) + A(y, x-1).$$

Hence

$$\begin{aligned}
(2A)(t+c, t+x-1) &= (2A)(t+c, t+y-1) + (2A)(t+y, t+x-1) \\
&\geq A(c, y-1) + \frac{1}{2}(y-c) + \frac{1}{2}(x-y) + A(y, x-1) \\
&= A(c, x-1) + \frac{1}{2}(x-c).
\end{aligned}$$

□(Claim 4.7.2.2.1)

By Claim 4.7.2.2.1 we now have

$$\begin{aligned} (2A)(t+c, t+H) &= (2A)(t+c, t+H-1) + 1 \\ &\geq A(c, H-1) + \frac{1}{2}(H-c) + 1 = A(c, H) + \frac{1}{2}(H-c). \end{aligned}$$

This implies

$$\begin{aligned} (2A)(c+z, 2H) &\geq |A[c, H] + A[z, H]| \\ &\geq |c + A[z, t-1]| + (2A)(t+c, t+H) + |H + A[t+1, H]| \\ &\geq A(z, t-1) + \frac{1}{2}(H-c) + A(c, H) + A(t+1, H) \\ &= \frac{1}{2}(H-c) + A(c, H) + A(z, H) - 1. \end{aligned}$$

Now the lemma follows from Claim 4.7.2.1. □(Subcase 4.7.2.2)

**Subcase 4.7.2.3**  $\gcd(A[c, H] - c) = 3$ .

Note that  $t, t+1 \in A$ . Suppose  $\{x \in A[z, a] : x-t \equiv 2 \pmod{3}\} = \emptyset$ . If  $c \in \{t, t+1\} \pmod{3}$ , then  $A[z, H]$  is a subset of the *b.p.*  $(t + (3 * \mathbb{Z})) \cup (t+1 + (3 * \mathbb{Z}))$ . Hence the lemma follows from Lemma 4.2. So we can assume that  $c \notin \{t, t+1\} \pmod{3}$ . This implies  $(A[c, H] + A[c, H]) \cap (A[c, H] + A[z, a]) = \emptyset$  and hence

$$\begin{aligned} |A[c, H] + A[z, H]| &\geq |c + A[z, z+H-c-1]| + |H + A[z, a]| + |A[c, H] + A[c, H]| \\ &\geq A(z, z+H-c-1) + A(z, a) + 2A(c, H) - 1 \\ &\geq \frac{1}{2}(H-c) + A(c, H) + A(z, H) - 1. \end{aligned}$$

Now the lemma follows from Claim 4.7.2.1. So we can assume

$$\{x \in A[z, a] : x-t \equiv 2 \pmod{3}\} \neq \emptyset.$$

Suppose  $\{x \in A[t, u] : x-t \equiv 2 \pmod{3}\} = \emptyset$ , where  $u = t+H-c-1$ . If  $\{x \in A[z, t-1] : x-t \equiv 2 \pmod{3}\} \neq \emptyset$ , let

$$k = \max\{x \in A[z, t-1] : x-t \equiv 2 \pmod{3}\}.$$

Then  $(k + A[c+1, H]) \cap (c + A[z, t-1]) = \emptyset$  and  $(k + A[c+1, H]) \cap (A[c, H] + A[t, u]) = \emptyset$ . Hence

$$\begin{aligned}
(2A)(c + z, 2H) &\geq |c + A[z, t-1]| + |A[c, H] + A[t, u]| + |k + A[c+1, H]| + |H + A[u+1, H]| \\
&\geq A(z, t-1) + |c + A[t, u]| + |H + A[t, u]| + A(c+1, H) + A(u+1, H) \\
&\geq A(z, t-1) + 2A(t, u) + A(c, H) - 1 + A(u+1, H) \\
&\geq A(z, H) + A(c, H) + \frac{1}{2}(u-t+1) - 1 \\
&= \frac{1}{2}(H-c) + A(c, H) + A(z, H) - 1.
\end{aligned}$$

Now the lemma follows from Claim 4.7.2.1.

If  $\{x \in A[z, u] : x-t \equiv 2 \pmod{3}\} = \emptyset$ , let  $k = \min\{x \in A[u+1, a] : x-t \equiv 2 \pmod{3}\}$ . Then  $(k + A[c, H]) \cap (A[c, H] + A[z, k-1]) = \emptyset$ . Hence

$$\begin{aligned}
(2A)(c + z, 2H) &\geq |A[c, H] + A[z, H]| \\
&\geq |c + A[z, t-1]| + |A[c, H] + A[t, u]| + |H + A[u+1, k-1]| \\
&\quad + |k + A[c, H]| + |H + A[k+1, H]| \\
&\geq A(z, t-1) + |c + A[t, u]| + |H + A[t, u]| \\
&\quad + A(u+1, k-1) + A(c, H) + A(k+1, H) \\
&\geq A(z, t-1) + 2A(t, u) + A(u+1, k-1) + A(c, H) + A(k+1, H) \\
&\geq A(z, H) - 1 + A(c, H) + \frac{1}{2}(u-t+1) \\
&= \frac{1}{2}(H-c) + A(c, H) + A(z, H) - 1.
\end{aligned}$$

This implies the lemma.

Now we can assume that  $\{x \in A[t, u] : x-t \equiv 2 \pmod{3}\} \neq \emptyset$ . For  $i = 0, 1, 2$  let

$$A_i = \{x \in A[t, u] : x-t \equiv i \pmod{3}\}.$$

Clearly  $A_i \neq \emptyset$  for  $i = 0, 1, 2$ . By Theorem A.4,

$$|A[c, H] + A_i| \geq \min\left\{\frac{1}{3}(H-c) + |A_i|, A(c, H) + 2|A_i| - 2\right\}.$$

Let  $Q = |A[c, H] + A[t, u]| = \sum_{i=0}^2 |A[c, H] + A_i|$ . Note that each term  $|A[c, H] + A_i|$  in the sum has two possible lower bounds  $\frac{1}{3}(H-c) + |A_i|$  or  $A(c, H) + 2|A_i| - 2$ . We

divide the proof into the cases according to the different combinations of these lower bounds of  $|A[c, H] + A_i|$  for  $i = 0, 1, 2$ .

**Subsubcase 4.7.2.3.1**  $Q \geq \sum_{i=0}^2 (\frac{1}{3}(H-c) + |A_i|)$ .

Together with the assumption of Case 4.7.2, this subsubcase implies

$$Q \geq H - c + A(t, u) \geq \frac{1}{2}(H - c) + A(c, H) - \frac{1}{2} + A(t, u).$$

Hence

$$\begin{aligned} (2A)(c+z, 2H) &\geq |A[c, H] + A[z, H]| \\ &\geq |c + A[z, t-1]| + Q + |H + A[u+1, H]| \\ &\geq A(z, t-1) + \frac{1}{2}(H-c) + A(c, H) - \frac{1}{2} + A(t, u) + A(u+1, H) \\ &\geq \frac{1}{2}(H-c) + A(c, H) + A(z, H) - 1. \end{aligned}$$

Now the lemma follows from Claim 4.7.2.1.  $\square$ (Subsubcase 4.7.2.3.1)

**Subsubcase 4.7.2.3.2**  $Q \geq \sum_{i=0}^1 (\frac{1}{3}(H-c) + |A_i|) + A(c, H) + 2|A_2| - 2$ .

Then  $Q \geq \frac{2}{3}(H-c) + A(t, u) + A(c, H) + |A_2| - 2$ . Hence

$$\begin{aligned} (2A)(c+z, 2H) &\geq |A[c, H] + A[z, H]| \\ &\geq |c + A[z, t-1]| + Q + |H + A[u+1, H]| \\ &\geq A(z, t-1) + \frac{2}{3}(H-c) + A(t, u) + A(c, H) + |A_2| - 2 + A(u+1, H) \\ &> \frac{1}{2}(H-c) + A(c, H) + A(z, H) - 1. \end{aligned}$$

$\square$ (Subsubcase 4.7.2.3.2)

**Subsubcase 4.7.2.3.3**  $Q \geq (\frac{1}{3}(H-c) + |A_0|) + \sum_{i=1}^2 (A(c, H) + 2|A_i| - 2)$ .

If  $H-c = 3$ , then  $A(c, H) = 2$  and  $A[t, u] = \{t, t+1, t+2\}$ . Hence  $|A[c, H] + A_i| = 2 = \frac{1}{3}(H-c) + |A_i|$ , which implies the assumption of Subsubcase 4.7.2.3.1. So we can assume  $H-c \geq 6$  and  $A(c, H) \geq 3$ . Since  $A(t, u) \geq (\frac{2}{3}-r)(H-c)$  and  $|A_0| \leq \frac{1}{3}(H-c)$ , then  $|A_1| + |A_2| \geq (\frac{1}{3}-r)(H-c)$  for some non-negative infinitesimal  $r$ . Hence

$$\begin{aligned} Q &\geq \frac{1}{3}(H-c) + A(t, u) + 2A(c, H) + |A_1| + |A_2| - 4 \\ &\geq \frac{1}{2}(H-c) + (\frac{1}{6}-r)(H-c) + A(t, u) + 2A(c, H) - 4 \\ &\geq \frac{1}{2}(H-c) + (\frac{1}{6}-r)(H-c) + A(t, u) + A(c, H) - 1 \\ &> \frac{1}{2}(H-c) + A(t, u) + A(c, H) - 1. \end{aligned}$$

Hence again

$$(2A)(c + z, 2H) \geq |A[c, H] + A[z, H]| \geq \frac{1}{2}(H - c) + A(c, H) + A(z, H) - 1.$$

□(Subsubcase 4.7.2.3.3)

**Subsubcase 4.7.2.3.4**  $Q \geq \sum_{i=0}^2 (A(c, H) + 2|A_i| - 2)$ .

Again we can assume  $H - c \geq 6$  and  $A(c, H) \geq 3$ . Then

$$\begin{aligned} Q &\geq 3A(c, H) + 2A(t, u) - 6 \\ &\geq A(t, u) + \frac{1}{2}(H - c) + A(c, H), \end{aligned}$$

for some non-negative infinitesimal  $r$ . Hence  $Q > A(t, u) + \frac{1}{2}(H - c) + A(c, H) - 1$ .

This implies

$$(2A)(c + z, 2H) \geq |A[c, H] + A[0, H]| \geq \frac{1}{2}(H - c) + A(c, H) + A(z, H) - 1,$$

which again implies the lemma. The rest of the cases can be proven by symmetry of the proofs above. □(Lemma 4.7)

**Lemma 4.8** *Suppose  $0 \prec s \prec H$  such that  $A[0, s]$  is a backward triangle from 0 to  $s$  and  $A[s + 1, H]$  is a forward triangle from  $s + 1$  to  $H$ . Then  $H + 1 \leq 2|A| - 1 + 2b$ .*

**Proof:** Let

$$u = \min\{x : \frac{s}{2} < x < \frac{s + H}{2} \text{ and } A(0, x) \geq \frac{1}{2}(x + 1)\}.$$

Clearly  $u \succeq s$  because  $A[0, s]$  is a backward triangle. Also  $u \preceq s$  because otherwise

$$A(0, u) \sim A(0, s) + A(s, u) \succ \frac{1}{2}s + \frac{1}{2}(u - s) \sim \frac{1}{2}u.$$

Hence we have  $u \sim s$ . It is easy to see that  $u \in A$  and  $A(0, u) = \frac{1}{2}(u + 1)$  by the minimality of  $u$ . Also by the minimality of  $u$  we have that for any  $0 \prec x \leq u$ ,  $A(x, u) > \frac{1}{2}(u - x + 1)$ . Let

$$X = \{x : u + 1 \leq x < \frac{u + H}{2} \text{ and } A(u + 1, x) \leq \frac{1}{2}(x - u)\}.$$

If  $X \neq \emptyset$ , let  $z = 1 + \max X$ . Otherwise let  $z = u + 1$ . It is also easy to see that  $z \in A$ ,  $z \sim u$ , and  $A(u + 1, z - 1) = \frac{1}{2}(z - u - 1)$ . Since  $A(0, z - 1) = \frac{1}{2}z$ ,  $A(0, H) \leq \frac{1}{2}(H + 1)$ , and  $H \in A$ , then the number  $a$  below is well defined.

$$a = \min\{x : z \leq x < H \text{ and } A(z, x) \leq \frac{1}{2}(x - z + 1)\}.$$

Clearly  $a < H$ ,  $a \sim H$ ,  $a \notin A$ , and  $A(z, a) = \frac{1}{2}(a - z + 1)$ . By the minimality we have that for any  $z \leq x < a$ ,  $A(z, x) > \frac{1}{2}(x - z + 1)$ . Now let  $a' = \max(A[z, a - 1])$  and  $c = \min(A[a, H])$ . Since  $a' \geq \frac{z+H}{2}$  and  $z \succ 0$ , then  $2a' \succ c$ . Let  $S = (2A)[0, z - 1] \setminus A[0, z - 1]$ .

Without loss of generality we can assume that  $A[z, H]$  is not a subset of a *b.p.* of difference 3 by the following reason:

Suppose not. By symmetry, we can also assume that there is a  $z' \sim z$  such that  $A[0, z']$  is a subset of a *b.p.* of difference 3. Now the lemma follows from Lemma 4.4.

The rest of the proofs are almost identical to the proofs of Lemma 4.7. We will refer to the proofs of Lemma 4.7 when the steps are the same and add more proofs when the steps are not the same.

**Claim 4.8.1:**  $[z, a + z - 1] \subseteq (2A)$ .

Proof of Claim 4.8.1: The proof here is slightly different from the proof of Claim 4.7.1.

If  $2z \leq x < a + z$ , then  $z \leq x - z < a$ . Hence  $A(z, x - z) > \frac{1}{2}(x - 2z + 1)$ . This implies  $A[z, x - z] \cap (x - A[z, x - z]) \neq \emptyset$ . Hence  $x \in (2A)$ . Suppose  $z \leq x < 2z$ . Then  $0 \leq x - z < z$ . If  $0 \prec x - z \leq u$ , then  $A(x - z, z) = A(x - z, u) + A(u + 1, z) > \frac{1}{2}(u - x + z + 1) + \frac{1}{2}(z - u - 1) + 1 > \frac{1}{2}(2z - x + 1)$ . Hence  $A[x - z, z] \cap (x - A[x - z, z]) \neq \emptyset$ , which implies  $x \in (2A)$ . If  $u < x - z < z$ , then by choosing  $y \succ 0$  with  $y \prec \min\{a - z, u\}$  we have  $A(x - z - y, x - z) + A(z, z + y) \succ y + 1$ . Hence  $A[x - z - y, x - z] \cap (x - A[z, z + y]) \neq \emptyset$ , which implies  $x \in (2A)$ . Suppose  $x - z \sim 0$ . If  $A(0, x - z) \geq \frac{1}{2}(x - z + 1)$ , then by  $A(z, x) > \frac{1}{2}(x - z + 1)$  we have  $A[0, x - z] \cap (x - A[z, x]) \neq \emptyset$ , which implies  $x \in (2A)$ . If  $A(0, x - z) < \frac{1}{2}(x - z + 1)$ , then  $A(x - z + 1, z - 1) > \frac{1}{2}(2z - x - 1)$ . Hence  $A[x - z + 1, z - 1] \cap (x - A[x - z + 1, z - 1]) \neq \emptyset$ , which again implies  $x \in (2A)$ .  $\square$ (Claim 4.8.1)

**Claim 4.8.2:** Suppose  $2a' > a + z$  and  $a + z < x \leq \min\{2a', c + z\}$ . If  $(2A)(a + z, x - 1) < \frac{1}{2}(x - a - z)$ , then  $x \in 2A$ .

Proof of Claim 4.8.2: The proof is identical to the proof of Claim 4.7.2.  $\square$ (Claim 4.8.2)

**Claim 4.8.3:** Suppose  $2a' < c + z$  and  $\max\{a + z, 2a'\} \leq x < c + z - 1$ . If  $(2A)(x + 1, c + z - 1) < \frac{1}{2}(c + z - x - 1)$ , then  $x \in (2A)$  or  $x - c \in S$ .

Proof of Claim 4.8.3: Identical to the proof of Claim 4.7.3.  $\square$ (Claim 4.8.3)

**Claim 4.8.4:**  $(2A)(0, c + z) \geq 3A(0, z - 1) + 2A(z, a) - 1 + \frac{1}{2}(c - a + 1)$ .



Proof of Claim 4.8.4: The proof is divided into three cases for  $2a' \geq c + z$ ,  $2a' \leq a + z$ , and  $a + z < 2a' < c + z$ .

**Case 4.8.4.1:**  $2a' \geq c + z$ .

Identical to the proof of Case 4.7.4.1.  $\square$ (Case 4.8.4.1)

**Case 4.8.4.2:**  $2a' \leq a + z$ .

Identical to the proof of Case 4.7.4.2.  $\square$ (Case 4.8.4.2)

**Case 4.8.4.3:**  $a + z < 2a' < c + z$ .

Identical to the proof of Case 4.7.4.3.  $\square$ (Case 4.8.4.3)

Now we prove the lemma. The proof is divided into two cases.

**Case 4.8.1**  $H - c \leq 2A(c + 1, H) = 2A(c, H) - 2$ .

Identical to the proof of Case 4.7.1.  $\square$ (Case 4.8.1)

**Case 4.8.2**  $H - c \geq 2A(c + 1, H) + 1 = 2A(c, H) - 1$ .

**Claim 4.8.2.1** If  $(2A)(c + z, 2H) \geq \frac{1}{2}(H - c) + A(c, H) + A(z, H) - 1$ , then  $H + 1 \leq 2|A| - 1 + 2b$ .

Proof of Claim 4.8.2.1 Identical to the proof of Claim 4.7.2.1.  $\square$ (Claim 4.8.2.1)

By Claim 4.8.2.1 we need only to show that  $(2A)(c + z, 2H) \geq \frac{1}{2}(H - c) + A(c, H) + A(z, H) - 1$  is true. We divide the proof into cases according to the structural properties of  $A[c, H]$ .

**Subcase 4.8.2.1**  $\gcd(A[c, H] - c) = 1$ .

The proof is the same as the proof of Subcase 4.7.2.1 except that the term  $U$  needs to be replaced by the term  $z + U$  throughout the remaining of the proof of the lemma. Note that we can also assume that  $\underline{d}_{z+U} \geq \frac{2}{3}$  by the same reason as stated at the beginning of the proof of Lemma 4.7.  $\square$ (Subcase 4.8.2.1)

**Subcase 4.8.2.2**  $\gcd(A[c, H] - c) = d > 1$  but  $d \neq 3$ .

**Claim 4.8.2.2.1** For each  $x \in A[c, H]$ ,  $(2A)(t + c, t + x - 1) \geq A(c, x - 1) + \frac{1}{2}(x - c)$ .

Proof of Claim 4.8.2.2.1: Identical to the proof of Claim 4.7.2.2.1.  $\square$ (Claim 4.8.2.2.1)

Following Claim 4.8.2.2.1 we now have

$$\begin{aligned} (2A)(t + c, t + H) &= (2A)(t + c, t + H - 1) + 1 \\ &\geq A(c, H - 1) + \frac{1}{2}(H - c) + 1 = A(c, H) + \frac{1}{2}(H - c). \end{aligned}$$

This implies

$$\begin{aligned}
(2A)(c+z, 2H) &\geq |A[c, H] + A[z, H]| \\
&\geq |c + A[z, t-1]| + (2A)(t+c, t+H) + |H + A[t+1, H]| \\
&\geq A(z, t-1) + \frac{1}{2}(H-c) + A(c, H) + A(t+1, H) \\
&= \frac{1}{2}(H-c) + A(c, H) + A(z, H) - 1.
\end{aligned}$$

Now the lemma follows from Claim 4.8.2.1.  $\square$ (Subcase 4.8.2.2)

**Subcase 4.8.2.3**  $\gcd(A[c, H] - c) = 3$ .

the proof is identical to the proof of Subcase 4.7.2.3. Note that we assume  $\{x \in A[z, a] : x - t \equiv 2 \pmod{3}\} \neq \emptyset$  in the beginning of the proof of this lemma.  $\square$ (Lemma 4.8)

**Lemma 4.9** *Suppose  $A = A[0, s] \cup A[s+1, H]$  with  $0 \prec s \prec H$  such that  $A[0, s]$  is a backward triangle and  $A[s+1, H]$  is a subset of an a.p. of difference  $d > 1$ . Then  $H+1 \leq 2|A| - 1 + 2b$ .*

**Proof:** Since (V), we have  $A(s+1, H) \sim \frac{1}{2}(H-s)$ , which implies  $d = 2$ . Let  $E$  be the set of all even numbers and let  $c = \min A[s+1, H]$ . Then  $c \sim s$  and  $A[c, H]$  is full. Without loss of generality we can assume that  $s \in A$  and  $c - s$  is odd. By the pigeonhole principle and Lemma 2.7 we can find  $e \sim 2c$  and  $e' \sim 2H$  such that  $E[e, e'] = (A[c, H] + A[c, H])[e, e']$ .

**Claim 4.9.1:** If  $s \prec x \prec s+H$ , then  $x \in 2A$ .

Proof of Claim 4.9.1: If  $s \prec x \prec 2s$ , then  $0 \prec x-s \prec s$ . Hence  $A(x-s, s) \succ \frac{1}{2}(2s-x+1)$ . So  $A[x-s, s] \cap (x - A[x-s, s]) \neq \emptyset$ , which implies  $x \in 2A$ . Now we assume  $2s \preceq x \prec s+H$ . Let  $z_1 = \lceil \frac{x}{2} \rceil$ . Then  $2z_1 \sim x \prec s+H$  implies  $z_1 - s \prec H - z_1$ . Choose  $y$  with  $z_1 - s \prec y \prec \min\{H - z_1, z_1\}$ . Then  $0 \prec z_1 - y \prec s$ . Hence

$$\begin{aligned}
A(z_1 - y, z_1) &\sim A(z_1 - y, s) + A(s+1, z_1) \\
&\succ \frac{1}{2}(s - z_1 + y + 1) + \frac{1}{2}(z_1 - s) = \frac{1}{2}(y + 1)
\end{aligned}$$

and  $A(x - z_1, x - z_1 + y) \sim \frac{1}{2}(y + 1)$ . Hence  $A[z_1 - y, z_1] \cap (x - A[x - z_1, x - z_1 + y]) \neq \emptyset$ , which implies  $x \in 2A$ .  $\square$ (Claim 4.9.1)

Now let's assume that the lemma is not true. Let  $A' \subseteq [0, H]$  be the set with the largest cardinality  $|A'|$  such that  $A[s+1, H] \subseteq A'[s+1, H] \subseteq (s+1 + E)$ ,

$A'[0, s] = A[0, s]$ , and satisfying (VII), (IX), and (X) with  $A$  replaced by  $A'$  and  $b$  replaced by  $b'$ . We will derive a contradiction.

**Claim 4.9.2:**  $A'[s+1, H] = (s+1+E)[s+1, H]$ .

Proof of Claim 4.9.2: The proof is divided into three cases. Let  $x \in (s+1+E)$  be such that  $s+1 \leq x \leq H$ . We want to show that  $x \in A'$ .

**Case 4.9.2.1**  $s \prec x \prec \frac{s+H}{2}$ .

Suppose  $x \notin A'$ . Let  $A'' = A' \cup \{x\}$ . For each  $y \in A'[s+1, H] \cup \{x\}$ ,  $x+y \in E[e, e'] \subseteq 2A \subseteq 2A'$ , and for each  $y \in A'[0, s]$  we have  $s \prec x+y \prec s + \frac{s+H}{2}$ , which implies  $s \prec x+y \prec s+H$ . Hence  $x+y \in 2A \subseteq 2A'$  by Claim 4.9.1. So  $2A'' = 2A'$ , which contradict the maximality of  $|A'|$  by Lemma 4.1.  $\square$ (Case 4.9.2.1)

**Case 4.9.2.2:**  $x \geq s+1$  and  $x \sim s+1$ .

Suppose  $x \notin A'$ . Without loss of generality we can, by Case 4.9.2.1, assume

$$x = \max((s+1+E)[s+1, \frac{3s+H}{4}] \setminus A').$$

Let  $y \in A'' = A' \cup \{x\}$ .

If  $0 \prec y \prec s$ , then  $A(y+1, s) \succ \frac{1}{2}(s-y)$  and  $A(x-(s-y), x-1) \succ \frac{1}{2}(s-y)$ . Hence  $A[y+1, s] \cap (x+y - A[x-(s-y), x-1]) \neq \emptyset$ , which implies  $x+y \in A[y+1, s] + A[x-(s-y), x-1] \subseteq (2A')$ .

If  $y \sim s$  and  $y \leq x$ , then choose  $z \prec y$  such that  $y-z \prec H-s$ . Hence  $A(z, y-1) \succ \frac{1}{2}(y-z)$  and  $A(x+1, x+(y-z)) \sim \frac{1}{2}(y-z)$  imply  $A[z, y-1] \cap (x+y - A[x+1, x+(y-z)]) \neq \emptyset$ , which implies  $x+y \in (2A) \subseteq (2A')$ .

If  $x < y \prec H$ , then choose  $z$  with  $H \geq z \succ y$  such that  $z-y \prec s$ . Hence  $A(y+1, z) \sim \frac{1}{2}(z-y)$  and  $A(x-(z-y), x-1) \succ \frac{1}{2}(z-y)$  imply  $A[y+1, z] \cap (x+y - A[x-(z-y), x-1]) \neq \emptyset$ , which implies  $x+y \in (2A')$ .

If  $y \sim H$ , then  $y \in (s+1+E)$ . Hence  $x+y$  is even and  $2s \prec x+y \prec 2H$ . Now we have  $x+y \in E[e, e'] \subseteq (2A')$ .

If  $y \sim 0$ ,  $y > 0$ , and  $y$  is even, then  $x+y \in A'$  by the maximality of  $x$ . Hence  $x+y = 0 + (x+y) \in (2A')$ .

If  $y \sim 0$ ,  $y$  is odd, and  $y > l = \min\{z \in A' : z \text{ is odd}\}$ , then  $x+(y-l) \in A'$ . Hence  $x+y = l + (x+y-l) \in (2A')$ .

By the arguments above we conclude that  $(2A'') \setminus (2A') \subseteq \{x, x+l\}$ . Hence  $|2A''| \leq |2A'| + 2$ , which contradicts the maximality of  $|A'|$  by Lemma 4.1. So we conclude that  $x \in A'$ .  $\square$ (Case 4.9.2.2)

**Case 4.9.2.3:**  $\frac{s+H}{2} \preceq x \leq H$ .

Suppose again  $x \notin A'$ . Let  $x = \min(A'[\frac{3s+H}{4}, H])$ . By Case 4.9.2.1 we have  $x \succeq \frac{s+H}{2}$ . Let  $y \in A''$ .

If  $s+1 \leq y \prec H$ , then  $x+y \in E[e, e'] \subseteq (2A')$ .

If  $0 \leq y \prec s$ , then  $s \prec x+y \prec s+H$ , which implies, by Claim 4.9.1,  $x+y \in 2A'$ .

If  $y < H$  and  $y \sim H$ , then  $x - (H - y) \in A'$  by the minimality of  $x$ . Hence  $x+y = H + (x - H + y) \in 2A'$ .

If  $y < s$ ,  $y \sim s$ , and  $s - y$  is odd, then  $s+1 - y$  is even and  $x - (s+1 - y) \in A'$  by the minimality of  $x$ . Hence  $x+y = s+1 + (x - s - 1 + y) \in 2A'$ .

If  $y < s$ ,  $y \sim s$ , and  $s - y$  is even, then  $x - (s - y) \in A'$ . Hence  $x+y = s + (x - s + y) \in 2A'$ .

By the arguments above we conclude that  $(2A'') \setminus (2A') \subseteq \{s+x, x+H\}$ , which contradicts the maximality of  $|A'|$  by Lemma 4.1.  $\square$ (Claim 4.9.2)

Now we are ready to prove the lemma. Without loss of generality we can assume that the set  $A$  is already in the form of the set  $A'$  in Claim 4.9.2. Let  $u = \min(\{z \in A[s+1, H] : A[0, z] \text{ is not a subset of a } b.p.\})$ . Note that if there is  $v \succ s$  such that  $A[0, v]$  is a subset of a  $b.p.$  of difference  $d$ , then  $|2A| \sim 3|A|$  implies that  $A[0, v]$  is full in the  $b.p.$  Hence  $d = 1$  or  $d = 3$  because  $A[0, s]$  is a backward triangle. However,  $A[s+1, v]$  is a full subset of an  $a.p.$  of difference 2, which contradicts  $d = 1$  or  $d = 3$ . Hence we have  $u \sim s$ . By (IX) and  $A(u, H) = \frac{1}{2}(H - u) + 1$  we have  $A(0, u) = |A| - A(u, H) + 1 \leq \frac{H+1}{2} - \frac{H-u}{2} = \frac{1}{2}(u+1)$ . Let  $|A[0, u] + A[0, u]| = 3A(0, u) - 3 + \bar{b}$ . If  $\bar{b} < 0$ , then by Theorem A.1 we have  $u+1 \leq 2A(0, u) - 2 + \bar{b} < 2A(0, u) - 2$ , which contradicts  $A(0, u) \leq \frac{1}{2}(u+1)$ . Hence we can assume  $\bar{b} \geq 0$ . Clearly  $\bar{b} \sim 0$  because otherwise we would have  $|2A| \succ 3|A|$ . By Lemma 4.7, we have  $u+1 \leq 2A(0, u) - 1 + 2\bar{b}$ . Hence

$$\begin{aligned} 3|A| - 3 + b &= |2A| \\ &\geq |A[0, u] + A[0, u]| + |s + A[u+2, H]| + |A[u+2, H] + A[u, H]| \\ &\geq 3A(0, u) - 3 + \bar{b} + A(u+2, H) + 2A(u+2, H) \\ &= 3|A| - 3 + \bar{b}. \end{aligned}$$

Above shows  $\bar{b} \leq b$ . Hence

$$H+1 = H - u + u + 1$$

$$\begin{aligned}
&\leq 2A(u+2, H) + 2A(0, u) - 1 + 2\bar{b} \\
&= 2|A| - 1 + 2\bar{b} \leq 2|A| - 1 + 2b,
\end{aligned}$$

which contradicts  $H + 1 > 2|A| - 1 + 2b$ .  $\square$ (Lemma 4.9)

**Lemma 4.10** *If  $\underline{d}_U(A) > \frac{1}{2}$ , then  $H + 1 \leq 2|A| - 1 + 2b$ .*

**Proof:** If  $\underline{d}_U(A) > \frac{1}{2}$ , then there exists  $x \succ 0$  such that  $A[0, x]$  is a forward triangle. If  $x \sim H$ , then the lemma now follows from Lemma 4.7. If  $x \prec H$ , then the lemma follows from Lemma 3.5.  $\square$ (Lemma 4.10)

**Lemma 4.11** *Let  $\underline{d}_U(A) = \frac{1}{2}$ . If there is  $x \succ 0$  in  $A$  such that  $\gcd(A[x, H] - x) = 1$ , then  $A \cap U$  is either a subset of an a.p. of difference  $> 1$  or a subset of a  $U$ -unbounded b.p.*

**Proof:** The lemma follows from Lemma 2.4 and Lemma 2.2.  $\square$ (Lemma 4.11)

**Lemma 4.12** *Let  $\underline{d}_U(A) = \frac{1}{2}$ . If  $A \cap U$  is a subset of a  $U$ -unbounded b.p., then  $H + 1 \leq 2|A| - 1 + 2b$ .*

**Proof:** Suppose  $A \cap U$  is a subset of a  $U$ -unbounded b.p. of difference  $d$ . Since  $\underline{d}_U(A) = \frac{1}{2}$ , then  $d = 3$  or  $d = 4$ . Let  $I_0 = d * \mathbb{N}$  and  $I_1 = c + (d * \mathbb{N})$  where  $c = \min\{z \in A : z \not\equiv 0 \pmod{d}\}$ . Then  $A \cap U \subseteq I_0 \cup I_1$ . Since  $d = 3$  or  $d = 4$ , then  $\gcd(c, d) = 1$ . By Lemma 4.6 we can assume that there is an  $a \succ 0$  in  $A$  such that  $\gcd(A[a, H] - a) = 1$ .

**Case 4.12.1:**  $d = 3$ .

We want to show this case implies  $|2A| \succ 3|A|$ , hence  $d = 3$  is impossible.

By Lemma 2.4 it suffices to show that for  $\gamma = \frac{1}{25}$  and for every  $N \succ 0$  there is  $K \in A$  with  $0 \prec K \leq N$  such that (III) is true. Suppose  $N \succ 0$  is given. Let  $0 < \epsilon < \frac{1}{12}$ . Without loss of generality we can re-choose  $a$  so that  $a \leq N$ ,  $A[0, a] \subseteq I_0 \cup I_1$ , and for every  $0 \prec y \leq a$  we have  $A(0, y) \succeq (\frac{1}{2} - \epsilon)y$ . Choose  $x$  with  $0 \prec x \prec \frac{1}{2}a$  such that  $A(0, x) \preceq (\frac{1}{2} + \epsilon)x$ . Let  $K = \min\{z \geq x : z, z - 1 \in A\}$ . It is easy to see that  $A(x, K) \preceq \frac{1}{3}(K - x)$  because for any two consecutive numbers in  $(I_0 \cup I_1)[x, K - 1]$ , at least one of them is not in  $A$ . Hence we have that  $K \preceq 2x \prec a$  and  $A(0, K) \preceq (\frac{1}{2} + \epsilon)K$ . Note that  $K, K - 1 \in A$ . So the shortest b.p. containing  $A[0, K]$  must have length  $L \sim \frac{2}{3}K$ . Let  $A_i = A[0, K] \cap I_i$  for  $i = 0, 1$ . Since  $|A_0| \preceq \frac{1}{3}K$ ,

then  $|A_1| = A(0, K) - |A_0| \succeq (\frac{1}{6} - \epsilon)K$ . By the same reason we have  $|A_0| \succeq (\frac{1}{6} - \epsilon)K$ . Let  $|A[0, K] + A[0, K]| = 3A(0, K) - 3 + b$  for some integer  $b$ . Since  $A[0, K]$  is a subset of a *b.p.*, then  $b \geq 0$ .

If  $b \geq \frac{1}{3}A(0, K) - 3$ , then

$$(2A)(0, 2K) \succeq 3A(0, K) + \frac{1}{3}A(0, K) \succeq 3A(0, K) + \frac{1}{3}(\frac{1}{2} - \epsilon)K,$$

which implies (III).

If  $b < \frac{1}{3}A(0, K) - 3$ , then by Theorem A.3 we have that

$$\frac{2}{3}K \sim L \leq A(0, K) + b \preceq (\frac{1}{2} + \epsilon)K + b.$$

Hence  $b \succeq (\frac{1}{6} - \epsilon)K$ . So we have

$$(2A)(0, 2K) \succeq 3A(0, K) + b \succ 3A(0, K) + \frac{1}{12}K,$$

which again implies (III). This ends the proof.  $\square$ (Case 4.12.1)

**Case 4.12.2:**  $d = 4$ .

Without loss of generality we assume  $0 \in I_0$  and  $1 \in I_1$ . Suppose  $A$  is not a subset of a *b.p.* We want to derive a contradiction. Let

$$c = \min\{z \in [0, H] : A[0, z] \text{ is not a subset of a } b.p. \text{ of difference } 4\}.$$

Let  $A[0, c-1] = A_0 \cup A_1$  where  $A_i = A[0, c-1] \cap I_i$  for  $i = 0, 1$ . Then  $|A_i| \succ 0$  for  $i = 0, 1$  because otherwise  $\underline{d}_U(A) \leq \frac{1}{4}$ . Note that since  $d = 4$ , then there is  $i = 0$  or  $i = 1$  such that  $(c + A_i) \cap |A[0, c-1] + A[0, c-1]| = \emptyset$ . Hence we can assume  $c \prec H$  because otherwise  $|2A| \succeq 3|A| + |c + A_i| \succ 3|A|$ .

**Subcase 4.12.2.1:**  $A(c, H) \succ \frac{1}{2}(H - c)$ .

By Lemma 2.6 there exist  $x \prec c \prec y \leq H$  such that  $A[x, y]$  is a backward triangle and  $A(y, H) \sim \frac{1}{2}(H - y + 1)$ .

If  $x \succ 0$ , then the lemma follows from either Lemma 3.5 or Lemma 3.7. so we can assume  $x \sim 0$ .

If  $y \sim H$ , then the lemma follows from Lemma 4.7. So we can assume  $y \prec H$ . If for any  $y \prec y' \prec H$  in  $A$ ,  $\gcd(A[y', H] - y') > 1$ , then the lemma follows from Lemma 4.9. So we can assume that there is  $y' \in A$ ,  $y \prec y' \prec H$  such that  $\gcd(A[y', H] - y') = 1$ .

If  $\underline{d}_{y+U}(A) > \frac{1}{2}$ , then by Lemma 2.6 there is a  $y \prec z \leq H$  such that  $A[y, z]$  is a forward triangle. Now the lemma follows from Lemma 4.8 if  $z \sim H$ . If  $z \prec H$ , then by Lemma 3.5  $|2A| \sim 3|A|$  implies that  $A[y, H]$  is a full subset of a *b.p.*  $[y, z'] \cup [z'', H]$ . Hence  $A[y, H]$  is the union of a forward triangle  $A[y, 2z' - y]$  and a backward triangle  $A[2z' - y + 1, H]$ . Now the lemma follows from Lemma 3.4.

If  $\underline{d}_{y+U} < \frac{1}{2}$ , then by Lemma 2.6 there are  $y \leq z \prec z' \leq H$  such that  $A[z, z']$  is a backward triangle and  $A(z', H) \sim \frac{1}{2}(H - z')$ . Now the lemma follows from Lemma 3.5 because  $A[0, z']$  cannot be a full subset of a *b.p.* of difference 1 or 3.

Assume  $\underline{d}_{y+U}(A) = \frac{1}{2}$ . Since  $A[0, y]$  is a backward triangle, then we can assume  $A(y, y + 3) \geq 3$ . Hence  $(A - y) \cap U$  is neither a subset of an *a.p.* of difference  $> 1$  nor a subset of a  $U$ -unbounded *b.p.* of difference  $d \neq 3$ . If  $(A - y) \cap U$  is a subset of a  $U$ -unbounded *b.p.* of difference 3, then by the proof of Case 4.12.1 we have  $|A[y, H] + A[y, H]| \succ 3A(y, H)$ , which implies  $|2A| \succ 3|A|$ . Note that if there is  $y' \sim y$  in  $A$  such that  $\gcd(A[y', H] - y') = d' > 1$ , then  $d' = 2$  and the lemma follows from Lemma 4.9. So we can assume that  $(A - y) \cap U$  is neither a subset of an *a.p.* of difference  $> 1$  nor a subset of a  $U$ -unbounded *b.p.* and there is  $y' \succ y$  in  $A$  such that  $\gcd(A[y', H] - y') = 1$ . Now the lemma follows from Lemma 2.2, Lemma 2.4, and Lemma 2.3.  $\square$ (Subcase 4.12.2.1)

**Subcase 4.12.2.2:**  $A(c, H) \preceq \frac{1}{2}(H - c)$ .

By (V) we have  $A(c, H) \sim \frac{1}{2}(H - c)$ . Since  $|A_0 \cup A_1| = A(0, c - 1) \sim \frac{1}{4}c + \frac{1}{4}c$  and  $\gcd(A_0) = \gcd(A_1 - 1) = 4$ , then  $A_i$  is full for  $i = 0, 1$ . Hence we can find  $c' \sim c$  in  $A$  such that  $\gcd(A[c', H] - c') = 1$ . Since there is  $i \in \{0, 1\}$  such that  $(2A)(0, 2c) \succeq 3A(0, c - 1) + |c + A_i| \succ 3A(0, c)$ , then by Lemma 2.3 we have  $|2A| \succ 3|A|$ .  $\square$ (Lemma 4.12)

**Lemma 4.13** *Let  $\underline{d}_U(A) = \frac{1}{2}$ . If  $A \cap U$  is a subset of an *a.p.* of difference  $> 1$ , then  $H + 1 \leq 2|A| - 1 + 2b$ .*

**Proof:** Let  $l_o = \min\{x \in A : \gcd(A[0, x]) = 1\}$ . Since  $\underline{d}_U(A) = \frac{1}{2}$ , then  $\gcd(A[0, l_o - 1]) = 2$  and  $l_o$  is odd.

**Case 4.13.1:** There are  $0 \prec a \prec c \leq H$  such that  $A[a, c]$  is a backward triangle and  $A(c, H) \sim \frac{1}{2}(H - c)$ .

Note that  $l_o \prec c$ . By Lemma 3.5 we have either  $|A[0, c] + A[0, c]| \succ 3A(0, c)$ , which is impossible because it implies  $|2A| \succ 3|A|$  by Lemma 2.3, or  $A[0, c]$  is a full subset

of a *b.p.*  $[0, x] \cup [x', c]$ , which contradicts  $\underline{d}_U(A) = \frac{1}{2}$ , or  $A[0, c]$  is a full subset of a *b.p.* of difference 3, which again contradicts  $\underline{d}_U(A) = \frac{1}{2}$ .  $\square$ (Case 4.13.1)

**Case 4.13.2:**  $A(l_o, H) \succ \frac{1}{2}(H - l_o)$ .

By lemma 2.6 there are  $0 \leq y \prec l_o \prec y' \leq H$  such that  $A(y', H) \sim \frac{1}{2}(H - y')$  and  $A[y, y']$  is a backward triangle. Without loss of generality we can assume that  $A(y', y' + 3) \geq 3$ . By Case 4.13.1 we can assume  $y \sim 0$  and by Lemma 4.7 we can assume  $y' \prec H$ .

**Subcase 4.13.2.1:**  $\underline{d}_{y'+U}(A) > \frac{1}{2}$ .

By Lemma 2.6 there is  $z \succ y'$  such that  $A[y', z]$  is a forward triangle. If  $z \sim H$ , then the lemma follows from Lemma 4.8. So we can assume  $z \prec H$ . By Lemma 2.3 we can assume  $|A[y', H] + A[y', H]| \sim 3A(y', H)$ . By Lemma 3.5 this implies that  $A[y', H]$  is either a full subset of a *b.p.* of difference 1 or a full subset of a *b.p.* of difference 3. Note that  $A(z, H) \sim \frac{1}{2}(H - z)$ . So  $A[y', H]$  cannot be a full subset of a *b.p.* of difference 3 because that would imply  $A[z, H] \sim \frac{1}{3}(H - z) \prec \frac{1}{2}(H - z)$ . If  $A[y', H]$  is a full subset of the *b.p.*  $[y', c'] \cup [c, H]$ , then  $c \prec H$  because otherwise  $A[y', H]$  is a forward triangle, which contradicts  $z \prec H$ . However,  $c \prec H$  implies that  $A[y', H]$  is the union of a forward triangle  $A[y', 2c' - y]$  and a backward triangle  $A[2c' - y + 1, H]$ , which implies  $|2A| \succ 3|A|$  by Lemma 3.4.  $\square$ (Subcase 4.13.2.1)

**Subcase 4.13.2.2:**  $\underline{d}_{y'+U}(A) < \frac{1}{2}$ .

By Lemma 2.6 there are  $y' \leq z \prec z' \leq H$  such that  $A[z, z']$  is a backward triangle and  $A(z', H) \sim \frac{1}{2}(H - z' + 1)$ . By Lemma 3.5 we have  $|A[0, z'] + A[0, z']| \succ 3A(0, z')$  because  $A[0, z']$  cannot be a full subset of a *b.p.* of difference 1 or 3. Hence  $|2A| \succ 3|A|$  by Lemma 2.3.  $\square$ (Subcase 4.13.2.2)

**Subcase 4.13.2.3:**  $\underline{d}_{y'+U}(A) = \frac{1}{2}$ .

If for every  $x \succ y'$  in  $A$ ,  $\gcd(A[x, H] - x) > 1$ , then there is an  $x \sim y'$  such that  $A[x, H]$  is a subset of an *a.p.* of difference 2. Hence the lemma follows from Lemma 4.9. So we can assume that there is  $x \succ H$  in  $A$  such that  $\gcd(A[x, H] - x) = 1$ .

Note that  $A(y', y' + 3) \geq 3$ . Hence  $(A - y') \cap U$  is neither a subset of an *a.p.* of difference  $> 1$  nor a subset of a  $U$ -unbounded *b.p.* of difference  $d \neq 3$ . If  $(A - y') \cap U$  is not a subset of a  $U$ -unbounded *b.p.* of difference 3, then the lemma follows from Lemma 2.4 and Lemma 2.3. If  $(A - y') \cap U$  is a subset of a *b.p.* of difference 3, then again  $|A[y', H] + A[y', H]| \succ 3A(y', H)$  by the proof of Case 4.12.1, which again implies  $|2A| \succ 3|A|$  by Lemma 2.3.  $\square$ (Case 4.13.2)



**Case 4.13.3:**  $A(l_o, H) \preceq \frac{1}{2}(H - l_o)$ .

Since  $A(0, l_o) \preceq \frac{1}{2}l_o$ , then we have  $A(0, l_o) \sim \frac{1}{2}l_o$  and  $A(l_o, H) \sim \frac{1}{2}(H - l_o)$ . Let  $u_e = \max(A[0, l_o - 1])$ . Then  $u_e \sim l_o$  and  $A[0, u_e]$  is full.

**Subcase 4.13.3.1:**  $\underline{d}_{l_o+U}(A) > \frac{1}{2}$ .

By Lemma 2.6 there is  $y \succ l_o$  such that  $A[l_o, y]$  is a forward triangle. If  $y \sim H$ , then the lemma follows from Lemma 4.9. So we can assume  $y \prec H$ , which implies that  $A[l_o, H]$  cannot be a full subset of a *b.p.* of difference 3. By Lemma 2.3 we can assume that  $|A[l_o, H] + A[l_o, H]| \sim 3A(l_o, H)$ . Hence by Lemma 3.5  $A[l_o, H]$  is a full subset of a *b.p.*  $[l_o, z] \cup [z', H]$  for some  $l_o \prec z \prec z' \leq H$ . If  $z' \sim H$ , then  $A[l_o, H]$  is a forward triangle, which contradicts  $y \prec H$ . If  $z' \prec H$ , then  $A[2z' - H, H]$  is a backward triangle. Now the lemma follows from Lemma 3.4.  $\square$ (Subcase 4.13.3.1)

**Subcase 4.13.3.2:**  $\underline{d}_{l_o+U}(A) < \frac{1}{2}$ .

By Lemma 2.6 there are  $l_o \leq z \prec z' \leq H$  such that  $A[z, z']$  is a backward triangle and  $A(z', H) \sim \frac{1}{2}(H - z' + 1)$ . Now the lemma follows from Lemma 3.5 or Lemma 3.7.  $\square$ (Subcase 4.13.3.2)

**Subcase 4.13.3.3:**  $\underline{d}_{l_o+U}(A) = \frac{1}{2}$ .

If there exists  $x \sim l_o$  in  $A$  such that  $\gcd(A[x, H] - x) = d > 1$ , then  $d = 2$  and the lemma follows from Lemma 4.6. So we can assume that there is  $x \succ l_o$  in  $A$  such that  $\gcd(A[x, H] - x) = 1$ . Since  $|2A| \sim 3|A|$ , then  $|A[0, l_o] + A[0, l_o]| \sim |A[0, u_e] + A[0, u_e]| + |l_o + A[0, u_e]| \sim 3A(0, l_o)$  implies that  $A[0, u_e]$  is full. Without loss of generality we can assume  $u_e, u_e - 2, u_e - 4 \in A$ . Hence  $(A - (u_e - 4)) \cap U$  is neither a subset of an *a.p.* of difference  $> 1$  nor a subset of a  $U$ -unbounded *b.p.* Now the lemma follows from Lemma 2.2, Lemma 2.4, and Lemma 2.3.  $\square$ (Lemma 4.13)

**Lemma 4.14** Suppose  $\underline{d}_U(A) < \frac{1}{2}$ . Then  $H + 1 \leq 2|A| - 1 + 2b$ .

**Proof:** Since  $\underline{d}_U(A) < \frac{1}{2}$ , by (4) of Lemma 2.6 there are  $0 \leq x \prec a \leq H$  such that  $A[x, a]$  is a backward triangle from  $x$  to  $a$  and  $A(a, H) \sim \frac{1}{2}(H - a)$ . If  $x \sim 0$  and  $a \sim H$ , then the lemma follows from Lemma 4.7. If  $a \sim H$  and  $x \succ 0$ , then the lemma follows from Lemma 3.5. If  $x \succ 0$  and  $a \prec H$ , then  $A(0, x) \sim \frac{1}{2}x$ . Hence the lemma follows from Lemma 3.6 because  $A[0, a]$  cannot be a full subset of a *b.p.* of difference 3.

So we can now assume  $x \sim 0$  and  $a \prec H$ .

If  $\underline{d}_{a+U}(A) > \frac{1}{2}$ , then there is  $z \succ a$  such that  $A[a, z]$  is a forward triangle. If  $z \sim H$ , then the lemma follows from Lemma 4.8. If  $z \prec H$ , then  $A[a, H]$  cannot be a full subset of a *b.p.* of difference 3 because  $A(z, H) \sim \frac{1}{2}(H - z)$ . Hence the lemma follows from Lemma 3.6.

If  $\underline{d}_{a+U}(A) < \frac{1}{2}$ , then there are  $a < z \prec z' \leq H$  such that  $A(z', H) \sim \frac{1}{2}(H - z')$  and  $A[z, z']$  is a backward triangle. Note that  $A[0, z']$  contains two backward triangle, hence is not a full subset of a *b.p.* By Lemma 3.5 we have  $|A[0, z'] + A[0, z']| \succ 3A(0, z')$ . Hence  $|2A| \succ 3|A|$  by Lemma 2.3.

Assume  $\underline{d}_{a+U}(A) = \frac{1}{2}$ . Since  $A[0, a]$  is a backward triangle, we can assume there is  $c \sim a$  in  $A$  such that  $A(c, c + 3) \geq 3$ . This implies (1)  $(A - c) \cap U$  is not a subset of an *a.p.* of difference  $> 1$  and (2) if  $(A - c) \cap U$  is a subset of a  $U$ -unbounded *b.p.* of difference  $d$ , then  $d = 3$ .

Suppose  $(A - c) \cap U$  is a subset of a  $U$ -unbounded *b.p.* of difference 3. By Case 4.12.1 we have  $|A[c, H] + A[c, H]| \succ 3A(c, H)$ , which imply  $|2A| \succ 3|A|$ .

Suppose  $(A - c) \cap U$  is not a subset of a  $U$ -unbounded *b.p.* of difference 3. If for each  $x' \succ c$  in  $A$ ,  $\gcd(A[x', H] - x') > 1$ , then the lemma follows from Lemma 4.9. So we can assume that there is  $x' \succ c$  such that  $\gcd(A[x', H] - x') = 1$ . Now the lemma follows from Lemma 2.2, Lemma 2.4, and Lemma 2.3.  $\square$ (Lemma 4.14)

**Theorem 4.15** *Let  $A \subseteq [0, H]$  be such that  $0, H \in A$ ,  $\gcd(A) = 1$ ,  $A$  is not a subset of a *b.p.*,  $|A| \leq \frac{1}{2}(H + 1)$ ,  $|A| \sim \frac{1}{2}H$ , and  $|2A| = 3|A| - 3 + b$  for  $0 \leq b \sim 0$ . Then  $H + 1 \leq 2|A| - 1 + 2b$ .*

**Proof:** If  $\underline{d}_U(A) > \frac{1}{2}$ , then the theorem follows from Lemma 4.10. If  $\underline{d}_U(A) < \frac{1}{2}$ , then the theorem follows from Lemma 4.14. Suppose  $\underline{d}_U(A) = \frac{1}{2}$ .

If  $A \cap U$  is a subset of an *a.p.* of difference  $> 1$ , then the theorem follows from Lemma 4.13. If  $A \cap U$  is a subset of a  $U$ -unbounded *b.p.*, then the theorem follows from Lemma 4.12. So we can assume that  $A \cap U$  is neither a subset of an *a.p.* of difference  $> 1$  nor a subset of a  $U$ -unbounded *b.p.* Hence by Lemma 2.2 for every  $x \succ 0$  there exists  $y \in A$  with  $0 \prec y \prec x$  such that  $(2A)(0, 2y) \succ 3A(0, y)$ .

If for any  $x \succ 0$  in  $A$  we have  $\gcd(A[x, H] - x) > 1$ , then the theorem follows from Lemma 4.5. So we can assume that there is  $x \succ 0$  in  $A$  such that  $\gcd(A[x, H] - x) = 1$ . But by Lemma 2.4, we have  $|2A| \succ 3|A|$ , which contradicts the assumption of the theorem. Now we finish the proof.  $\square$ (Theorem 4.15).

**Remark 4.16** *We already proved that Theorem 1.4 follows from Theorem 1.7. Now Theorem 1.7 follows from Theorem 1.3, Theorem 3.10, and Theorem 4.15.*

## 5 A Corollary for Upper Asymptotic Density

In this section we improve the most important part of the main theorem in [Ji2] using Theorem 1.4.

For an infinite set  $A \subseteq \mathbb{N}$  the upper asymptotic density of  $A$  is defined by

$$\bar{d}(A) = \limsup_{n \rightarrow \infty} \frac{|A(0, n-1)|}{n}.$$

In this section we assume that  $0 \in A$  and  $\gcd(A) = 1$ . By Theorem A.1 it is not hard to prove that  $\bar{d}(A) < \frac{1}{2}$  implies  $\bar{d}(2A) \geq \frac{3}{2}\bar{d}(A)$ . In [Ji2] the structure of  $A$  was characterized when  $0 < \bar{d}(A) < \frac{1}{2}$  and  $\bar{d}(2A) = \frac{3}{2}\bar{d}(A)$ . Next we improve this result by substituting the condition  $\bar{d}(2A) = \frac{3}{2}\bar{d}(A)$  with the condition  $\frac{3}{2}\bar{d}(A) \leq \bar{d}(2A) < \frac{3+\epsilon}{2}\bar{d}(A)$  for some positive real number  $\epsilon$ .

**Corollary 5.1** *Let  $0 < \epsilon \leq \frac{1}{3}$  be the real number in Theorem 1.4. For every real number  $\delta$  with  $0 \leq \delta < \epsilon$ , if  $0 < \bar{d}(A) = \alpha < \frac{1}{2(1+\delta)}$  and  $\bar{d}(2A) = \frac{3+\delta}{2}\alpha$ , then either*

(a) *there exist  $d \geq 4$  and  $c \in [1, d-1]$  such that  $A \subseteq (d * \mathbb{N}) \cup (c + (d * \mathbb{N}))$  and  $\frac{6}{(2\delta+3)d} \leq \alpha \leq \frac{2}{d}$ , or*

(b) *for every increasing sequence  $\langle h_n : n \in \mathbb{N} \rangle$  with  $\lim_{n \rightarrow \infty} \frac{|A(0, h_n)|}{h_n+1} = \alpha$ , there exist two sequences  $0 \leq c_n \leq b_n \leq h_n$  such that  $A \cap [c_n + 1, b_n - 1] = \emptyset$  for each  $n \in \mathbb{N}$ ,*

$$\limsup_{n \rightarrow \infty} \frac{c_n + h_n - b_n}{h_n} \leq \alpha(1 + \delta), \text{ and}$$

$$\limsup_{n \rightarrow \infty} \frac{c_n}{h_n - b_n} \leq \frac{\delta}{1 - \alpha(1 + \delta)}.$$

**Proof:** Let  $N$  be any hyperfinite integer and  $H = h_N$  be the term in the internal sequence  $\langle h_n : n \in {}^*\mathbb{N} \rangle$  from (b). Without loss of generality we can assume  $H \in {}^*A$ . Let  $B = {}^*A[0, H]$ . Then  $|B| \sim \alpha H$  and  $|2B| \preceq (3 + \delta)|B| \sim 3|B| - 3 + \delta|B|$ . If  $B$  is a subset of an *a.p.* of length  $l \sim 2|B| - 1 + 2\delta|B|$ , then  $H + 1 \leq l \sim 2(1 + \delta)\alpha H \prec H$ , which is absurd. So by Theorem 1.4 we conclude that  $B$  is a subset of a *b.p.* of difference  $d$  of length at most  $L \preceq (1 + \delta)|B|$ . Now the proofs can be found in [Bo]

(with  $\delta = 2\sigma - 3$ ) that if  $d > 1$ , then  $B \subseteq (d * \mathbb{N}) \cup (c + (d * \mathbb{N}))$  and  $\frac{6}{(2\delta+3)d} \leq \alpha \leq \frac{2}{d}$ , and if  $d = 1$ , then there are  $0 \leq c \leq b \leq H$  such that  $A \subseteq [0, c] \cup [b, H]$ ,

$$c + H - b \leq \alpha(1 + \delta)H,$$

$$\text{and } c \leq \frac{\delta}{1 - \alpha(1 + \delta)}(H - b).$$

Note that the first inequality is a trivial consequence of Theorem 1.4 and the second inequality indicates that the interval  $[0, c]$  is much shorter than  $[b, H]$ . Since the arguments above are true for every hyperfinite integer  $N$ , then the corollary follows from the transfer principle.  $\square$ (Corollary 5.1)

**Remark 5.2** (1) The result in [Ji2] mentioned above is a special case of Corollary 5.1 with  $\delta = 0$ .

(2) Corollary 5.1 is very similar to the main theorem in [Bo]. The main theorem in [Bo] allows all  $\delta < \frac{1}{3}$  instead of  $\delta < \epsilon$  in Corollary 5.1. However, Corollary 5.1 allows, for example  $\bar{d}(A) = \alpha \leq \frac{3}{8}$  (note that  $\frac{3}{8} \leq \frac{1}{2(1+\epsilon)} < \frac{1}{2(1+\delta)}$ ), instead of  $\alpha < \alpha_0$  for a small  $\alpha_0 > 0$  in the main theorem in [Bo]. The reason for the difference is that the main theorem in [Bo] is a corollary of Theorem A.5 while Corollary 5.1 is a corollary of Theorem 1.4. It should be interesting to see whether one can prove Corollary 5.1 with the condition  $\bar{d}(2A) < \frac{3+\epsilon}{2}\bar{d}(A)$  replaced by  $\bar{d}(2A) < \frac{5}{3}\bar{d}(A)$ . In fact, this is a corollary of Conjecture 6.1.

## 6 Comments and a Weak Conjecture

The reader might notice that the proof of the case when  $|A| \prec \frac{1}{2}H$  is much more “nonstandard” than the proof of the case when  $|A| \sim \frac{1}{2}H$ , which is combinatorial. However, the proof of the latter is significantly simplified after the possibility of  $|A| \prec \frac{1}{2}H$  is eliminated.

After reading all the proofs above, the reader should be able to see the crucial role that Lemma 2.2 plays. In order to violate the condition  $|2A| \sim 3|A|$  one needs only to find a small segment  $A[a, b]$  of the set  $A$ , which already violates  $(2A)(2a, 2b) \sim 3A(a, b)$ , as long as the rest of  $A$  at each side of the segment is not too dense and is not a subset of an *a.p.* of difference  $> 1$  (see the conditions of Lemma 2.3). So if  $A \cap U$  does not have expected structural properties such as  $\underline{d}_U(A) > \frac{1}{2}$ ,  $\underline{d}_U(A) = 0$ ,

$A \cap U$  is a subset of an *a.p.* of difference  $> 1$ , or  $A \cap U$  is a subset of a  $U$ -unbounded *b.p.*, then the segment mentioned above is guaranteed by Lemma 2.2. Otherwise, by some nonstandard arguments, there is an  $x > U$  such that  $A[0, x]$  has one of a few desired structural properties, similar to those mentioned above for  $A \cap U$ . This gives us a breaking point to reach our final goal of determine the structure of the whole set  $A$ . As  $A[0, x]$  has these structural properties, the remain part of the proof of the main theorem can be clearly divided into a few possible cases although the verification of these cases are tedious.

Lemma 2.2 is inspired by Kneser's Theorem (cf. [HR]) and uses the fact that  $U$  is an additive semigroup. This tool is not available in the standard setting, i.e. an initial segment of a finite interval cannot be closed under usual addition. This and other arguments in the proofs indicate that the use of nonstandard analysis in this paper is non-trivial.

Now people who may not be willing to learn nonstandard analysis might ask whether the proof of Theorem 1.4 can be done without nonstandard methods. Theoretically, every nonstandard proof of a standard theorem can be translated into a standard proof because a nonstandard model itself can be constructed in ZFC (Zermelo-Fraenkel set theory with the Axiom of Choice), which is considered as an axiomatic system of standard mathematics. However, the literal translation would be unbearably long and tedious. I believe that there must be a reason why the standard proof has not been found. It is interesting to see whether a clever standard proof (not a literal translation of the nonstandard proof) can be found.

Although Theorem 1.4 is a significant advancement of the current results, it confirms only a weak version of Conjecture 1.1. It is interesting to see whether the ideas from nonstandard analysis can play a major role in the ultimate solution of Conjecture 1.1. Many lemmas including Lemma 2.2 in this paper may be generalized. If these generalizations are achieved, then one can first generalize Theorem 1.4 by allowing  $|2A| \leq \alpha|A|$  for any fixed  $\alpha < \frac{10}{3}$ . We would like to state that as a conjecture weaker than Conjecture 1.1. The conjecture stated below should be much easier to prove than proving Conjecture 1.1. However, the solution of the following conjecture is useful for improving Corollary 5.1 and could be the last stepping stone to the ultimate solution of Conjecture 1.1.

**Conjecture 6.1** *For any real number  $\alpha$  with  $3 < \alpha < 3 + \frac{1}{3}$  there exists  $K \in \mathbb{N}$  such*

that for every finite set  $A \subseteq \mathbb{N}$  with  $|A| > K$ , if  $3|A| - 3 \leq |2A| = 3|A| - 3 + b \leq \alpha|A|$ , then  $A$  is either a subset of an a.p. of length at most  $2|A| - 1 + 2b$  or a subset of a b.p. of length at most  $|A| + b$ .

## A Appendix

The following theorem is in [Fr1] and in [Na, p.28].

**Theorem A.1 (G. A. Freiman)** *Let  $A$  be a finite set of integers and  $|A| = k$ . If  $|2A| = 2k - 1 + b < 3k - 3$ , then  $A$  is a subset of an a.p. of length at most  $k + b$ .*

The following theorem is in [Fr1, Bi]

**Theorem A.2 (G. A. Freiman)** *Let  $A$  be a finite set of integers and  $|A| = k$ . If  $|2A| = 3k - 3$ , then  $A$  is either a subset of an a.p. of length at most  $2k - 1$  or a b.p.*

The following theorem is in [Fr1].

**Theorem A.3 (G. A. Freiman)** *Let  $A \subseteq \mathbb{Z}^2$  be such that  $|A| = k > 10$ . If  $|2A| = 3k - 3 + b$  for  $0 \leq b < \frac{1}{3}k - 2$  and  $A$  is not a subset of a straight line, then  $A$  is  $F_2$ -isomorphic to a subset of  $\{(0, 0), (1, 0), \dots, (l_1 - 1, 0)\} \cup \{(0, 1), (1, 1), \dots, (l_2 - 1, 1)\}$  where  $l_1 + l_2 \leq k + b$ .*

The following theorem is in [Na, p.118] and in [LS].

**Theorem A.4 (V. Lev & P. Y. Smeliansky)** *Let  $A$  and  $B$  be two finite set of non-negative integers such that  $0 \in A \cap B$ ,  $|A|, |B| > 1$ ,  $\gcd(A) = 1$ ,  $m = \max A$ , and  $n = \max B \leq m$ . If  $m = n$ , then  $|A + B| \geq \min\{m + |B|, |A| + 2|B| - 3\}$ . If  $m > n$ , then  $|A + B| \geq \min\{m + |B|, |A| + 2|B| - 2\}$ .*

Note that Theorem A.4 is trivially true when  $|B| = 1$ . In this paper Theorem A.4 is used in different variations. For example, we can replace the conditions of the theorem by  $|A|, |B| > 1$ ,  $\gcd(A - \min A) = 1$ , and  $m = \max A - \min A \geq n = \max B - \min B$ . We can also consider that both  $A$  and  $B$  are subsets of a.p.'s of difference  $d$  with the conditions  $|A|, |B| > 1$ ,  $\gcd(A - \min A) = d$ , and  $m = \frac{1}{d}(\max A - \min A) \geq n = \frac{1}{d}(\max B - \min B)$ .

The next theorem is Bilu’s version of Freiman’s famous theorem for the inverse problems about finite sets with “large” doubling property [Bi, Theorem 1.2 and Theorem 1.3]. we state only this weak version enough for the purposes of this paper in order to make the paper a little shorter.

**Theorem A.5 (Y. Bilu & G. A. Freiman)** *Let  $\sigma < 4$ ,  $A$  be a finite subset of integers such that  $k = |A| > 6$ , and  $|2A| \leq \sigma k$ . Then  $A$  is a subset of an  $F_2$ -progression  $P = P(x_0; x_1, x_2; b_1, b_2)$  such that  $|P| \leq c_1 k$  for some constant  $c_1$  and  $b_2 < c_2$  for some constant  $c_2$ . The constants  $c_1, c_2$  are independent of  $k$ .*

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